

# Game Theory 2023

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## Plan for Today

When players do not have access to a common currency, they cannot necessarily split the utility derived from the joint effort of a coalition.

Today we introduce coalitional games *without* transferable utility:

- general model of *nontransferable-utility games*
- *translation* of transferable-utility into nontransferable-utility games

We then focus on questions of *stability* in what probably are the simplest kinds of NTU games, the *hedonic games*, where a player's preferences depend only on the members of the coalitions she may join.

Hedonic games were introduced by Bogomolnaia and Jackson (2002) and much of this lecture is based on their original paper.

A. Bogomolnaia and M.O. Jackson. The Stability of Hedonic Coalition Structures. *Games and Economic Behavior*, 38(2):201–230, 2002.

## Coalitional Games

A *nontransferable-utility coalitional game* (or simply: an *NTU game*) is a tuple  $\langle N, \Omega, V, \succeq \rangle$ , where

- $N = \{1, \dots, n\}$  is a finite set of *players*,
- $\Omega$  is a nonempty set of *outcomes*,
- $V : 2^N \setminus \{\emptyset\} \rightarrow 2^\Omega \setminus \{\emptyset\}$  is a function mapping any coalition to the set of outcomes it *can bring about*, and
- $\succeq = (\succeq_1, \dots, \succeq_n)$  is a profile of weak *preference orders*  $\succeq_i$  on  $\Omega$  (i.e., binary relations on  $\Omega$  that are transitive and complete).

When coalition  $C \subseteq N$  forms, it has the opportunity of implementing any one of the outcomes in  $V(C) \subseteq \Omega$ . Every member of  $C$  has her own preferences over which outcome she'd like to see implemented.

A *solution concept* now must (i) choose a partition into coalitions and (ii) fix how to select the outcome in  $V(C)$  to be implemented by  $C$ .

## Embedding TU Games into NTU Games

NTU games generalise TU games. That is, we can translate any given TU game  $\langle N, v \rangle$  into an NTU game  $\langle N, \Omega, V, \succeq \rangle$ . Here's how:

- Outcomes are payoff vectors for coalitions:  $\Omega = \bigcup_{k \leq n} \mathbb{R}_{\geq 0}^k$
- Outcomes a specific coalition  $C$  with  $k = |C|$  can bring about are the feasible payoff vectors to divide the surplus  $v(C)$ :

$$V(C) = \left\{ (x_1, \dots, x_k) \in \mathbb{R}_{\geq 0}^k \mid \sum_{i \in C} x_i \leq v(C) \right\}$$

- Preferences of player  $i$  over outcomes are defined by reference to the payoffs she receives in them:  $x \succeq_i x'$  iff  $x_i \geq x'_i$

## Hedonic Games

In a *hedonic game*, a player's preferences depend only on the coalition she joins. Formally, a hedonic game is a tuple  $\langle N, \succeq \rangle$ , where

- $N = \{1, \dots, n\}$  is a finite set of *players* and
- $\succeq = (\succeq_1, \dots, \succeq_n)$  is a profile of weak *preference orders*, where  $\succeq_i$  for player  $i \in N$  is defined on  $\{C \cup \{i\} \mid C \subseteq N \setminus \{i\}\}$ .

Thus, in a hedonic game, the players in a coalition need not agree on one of the outcomes they can bring about (e.g., division of surplus).

A *solution concept* now only has to specify which coalitions will form.

Exercise: How to translate a HG into an NTU game  $\langle N, \Omega, V, \succeq \rangle$ ?

## Embedding Hedonic Games into NTU Games

Hedonic games  $\langle N, \succeq \rangle$  are NTU games  $\langle N, \Omega, V, \succeq \rangle$  where the only outcome coalition  $C$  can bring about is 'itself':

- $\Omega = 2^N \setminus \{\emptyset\}$
- $V(C) = \{C\}$

Remark: So hedonic games are the 'most basic' of all NTU games.

## Coalition Structures

A partition of the set  $N$  of players into coalitions is also known as a *coalition structure*  $\mathcal{C} = \{C_1, \dots, C_K\}$ , with  $C_1 \uplus \dots \uplus C_K = N$ .

Write  $\mathcal{C}(i)$  for the coalition to which player  $i \in N$  belongs in  $\mathcal{C}$ :

$$\mathcal{C}(i) \quad := \quad C \text{ such that } C \in \mathcal{C} \text{ and } i \in C$$

## The Core for Hedonic Games

In analogy to the core for TU games, the core is defined as the set of coalition structures where no coalition has an incentive to break away.

Coalition structure  $\mathcal{C}$  is *in the core* of the game  $\langle N, \succeq \rangle$ , if for no  $C \subseteq N$  all  $i \in C$  are better off in  $C$  than in their assigned coalition:

$\mathcal{C}(i) \succeq_i C$  must hold for all  $C \subseteq N$  for at least one  $i \in C$



## Exercise: “Two’s Company, Three’s a Crowd”

Consider this hedonic game with three players, in which each player prefers a coalition of two, over the grand coalition, over being alone:

$$\{1, 2\} \succ_1 \{1, 3\} \succ_1 \{1, 2, 3\} \succ_1 \{1\}$$

$$\{2, 3\} \succ_2 \{1, 2\} \succ_2 \{1, 2, 3\} \succ_2 \{2\}$$

$$\{1, 3\} \succ_3 \{2, 3\} \succ_3 \{1, 2, 3\} \succ_3 \{3\}$$

*Show that this game has an empty core!*

## Nonemptiness of the Core

Banerjee et al. (2001) and Bogomolnaia and Jackson (2002) discuss several sufficient conditions for nonemptiness of the core.

But we are not going to explore this topic any further today ...

S. Banerjee, H. Konishi, and T. Sönmez. Core in a Simple Coalition Formation Game. *Social Choice and Welfare*, 18(1):135–153, 2001.

A. Bogomolnaia and M.O. Jackson. The Stability of Hedonic Coalition Structures. *Games and Economic Behavior*, 38(2):201–230, 2002.

## Three Further Notions of Stability

Consider a *coalition structure*  $\mathcal{C} = \{C_1, \dots, C_K\}$  for the game  $\langle N, \succeq \rangle$ .

- $\mathcal{C}$  is called *Nash stable* if no player  $i$  wants to switch:  
 $\mathcal{C}(i) \succeq_i C \cup \{i\}$  for all players  $i \in N$  and all coalitions  $C \in \mathcal{C} \cup \{\emptyset\}$ .
- $\mathcal{C}$  is called *individually stable* if no player  $i$  wants to switch and the receiving coalition  $C$  would be happy to have her:  
 $\mathcal{C}(i) \succeq_i C \cup \{i\}$  for all players  $i \in N$  and all coalitions  $C \in \mathcal{C} \cup \{\emptyset\}$  for which it is the case that  $C \cup \{i\} \succeq_j C$  for all  $j \in C$ .
- $\mathcal{C}$  is called *contractually stable* if no player  $i$  wants to switch and both the releasing and the receiving coalition would agree:  
 $\mathcal{C}(i) \succeq_i C \cup \{i\}$  for all  $i \in N$  with  $\mathcal{C}(i) \setminus \{i\} \succeq_j \mathcal{C}(i)$  for all  $j \in \mathcal{C}(i) \setminus \{i\}$  and all  $C \in \mathcal{C} \cup \{\emptyset\}$  with  $C \cup \{i\} \succeq_j C$  for all  $j \in C$ .

The following relationships follow immediately from the definitions:

Nash stable  $\Rightarrow$  individually stable  $\Rightarrow$  contractually stable

## Example: “An Unwelcome Guest”

Consider the following hedonic game with three players:

$$\{1, 2\} \succ_1 \{1\} \succ_1 \{1, 2, 3\} \succ_1 \{1, 3\}$$

$$\{1, 2\} \succ_2 \{2\} \succ_2 \{1, 2, 3\} \succ_2 \{2, 3\}$$

$$\{1, 2, 3\} \succ_3 \{2, 3\} \succ_3 \{1, 3\} \succ_3 \{3\}$$

Let us analyse the coalition structure  $\mathcal{C} = \{\{1, 2\}, \{3\}\}$ :

- $\mathcal{C}$  is *individually stable* (and thus also *contractually stable*):
  - neither 1 or 2 want to join  $\{3\}$  or the empty coalition  $\emptyset$
  - 3 would want to join  $\{1, 2\}$ , but they don't want to admit her
- But  $\mathcal{C}$  is *not Nash stable*: 3 wants to join  $\{1, 2\}$ .
- Note that  $\mathcal{C}$  also is *in the core*: no coalition wants to break off.

## Sufficient Conditions for Stability

The core may be empty. Nash stability also seems quite demanding.

*So when can we be certain that a stable coalitional structure exists?*

Bogomolnaia and Jackson (2002) establish several sufficient conditions.

Here we present and prove just one of their results . . .

A. Bogomolnaia and M.O. Jackson. The Stability of Hedonic Coalition Structures. *Games and Economic Behavior*, 38(2):201–230, 2002.

## Additively Separable Preferences

The preference order  $\succeq_i$  of player  $i$  is *additively separable* if there exists a function  $v_i : N \rightarrow \mathbb{R}$  such that for all  $C, C' \subseteq N$  with  $C, C' \ni i$ :

$$C \succeq_i C' \iff \sum_{j \in C} v_i(j) \geq \sum_{j \in C'} v_i(j)$$

A profile  $\succeq = (\succeq_1, \dots, \succeq_n)$  of additively separable preference orders is *symmetric* if  $v_i(j) = v_j(i)$  for all players  $i, j \in N$ . Abbreviate as  $v_{ij}$ .

## A Sufficient Condition for Nash Stability

**Theorem 1 (Bogomolnaia and Jackson, 2002)** *For any hedonic game with a **symmetric** profile of **additively separable preferences** there exists a coalition structure that is **Nash stable**.*

Proof: For any coalition structure, consider its **social welfare**:

$$sw(\mathcal{C}) = \sum_{i \in N} \sum_{j \in \mathcal{C}(i) \setminus \{i\}} v_{ij}$$

If  $i$  switches from  $\mathcal{C}(i)$  to  $C$ , the players in  $\mathcal{C}(i) \setminus \{i\}$  **lose**  $\sum_{j \in \mathcal{C}(i) \setminus \{i\}} v_{ij}$  (as does  $i$ ), while those in  $C$  **gain**  $\sum_{j \in C} v_{ij}$  (as does  $i$ ).

Hence, **social welfare increases** with every **Nash defection**. But some  $\mathcal{C}$  with maximal social welfare must exist, which is also Nash stable. ✓

Remark: As an immediate corollary, we get the same result for both individual and contractual stability (they are implied by Nash stability).

A. Bogomolnaia and M.O. Jackson. The Stability of Hedonic Coalition Structures. *Games and Economic Behavior*, 38(2):201–230, 2002.

## Summary

We've discussed coalitional games where utility cannot be transferred between players, so it does not make sense to speak of a surplus to be divided or of preferences being expressed in terms of numbers. Instead:

- *NTU games* with ordinal preferences over feasible outcomes
- *hedonic games*, where players have preferences over coalitions only
- *stability concepts* for hedonic games: the core, Nash stability, individual stability, contractual stability

Recall: TU games and hedonic games are special kinds of NTU games.

**What next?** Matching players up in pairs in a stable manner, which may be considered a specific class of hedonic games.