Game Theory: Spring 2017

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Plan for Today

Today we are going to review solution concepts for coalitional games with transferable utility that encode some notion of fairness:

- **Banzhaf value**: payoffs should reflect marginal contributions
- **Shapley value**: more sophisticated variant of the same idea
- **Nucleolus**: minimise possible complaints by coalitions

The most important of these is the Shapley value and we are going to use it to exemplify the *axiomatic method* in economic theory.

Much of this is also covered in Chapter 8 of the *Essentials* and in the survey by Airiau (2013).

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Reminder: TU Games and Payoff Vectors

A transferable-utility coalitional game in characteristic-function form (or simply: a TU game) is a tuple $\langle N, v \rangle$, where

- $N = \{1, \ldots, n\}$ is a finite set of players and
- $v : 2^N \to \mathbb{R}_{\geq 0}$, with $v(\emptyset) = 0$, is a characteristic function, mapping every possible coalition $C \subseteq N$ to its surplus $v(C)$.

Suppose the grand coalition $N$ forms. Then we require a payoff vector $x = (x_1, \ldots, x_n) \in \mathbb{R}_{\geq 0}^n$ to fix what payoff each player should get.

More generally, for fixed $N$, we may look for a vector $x$ of functions $x_i$ mapping any given game $\langle N, v \rangle$ to the payoff to be given to player $i$. Any such vector of functions constitutes a solution concept.
Marginal Contribution

Focus on TU games that are *monotonic* (weakest property we've seen).

Player $i$ increases the surplus of coalition $C \subseteq N \setminus \{i\}$ by the amount $v(C \cup \{i\}) - v(C)$ if she joins. This is her *marginal contribution*.

There are (at least) two ways one can define the “average” marginal contribution player $i$ makes to coalitions in game $\langle N, v \rangle$ . . .
The Banzhaf Value

The Banzhaf value gives equal importance to all coalitions in \( \langle N, v \rangle \):

\[
\beta_i(N, v) = \frac{1}{2^{n-1}} \cdot \sum_{C \subseteq N \setminus \{i\}} v(C \cup \{i\}) - v(C)
\]

Note that \( 2^{n-1} \) is the number of subsets \( C \) of \( N \setminus \{i\} \) (normalisation).

This is a solution concept: pick payoff vector \((\beta_1(N, v), \ldots, \beta_n(N, v))\).

Remark: Banzhaf (1965) defined this for the case of voting games.

Example: Computing the Banzhaf Value

Consider the following 3-player TU game \( \langle N, v \rangle \), with \( N = \{1, 2, 3\} \), in which no single player can generate any surplus on her own:

\[
egin{align*}
v(\{1\}) &= 0 & v(\{1, 2\}) &= 7 & v(N) &= 10 \\
v(\{2\}) &= 0 & v(\{1, 3\}) &= 6 \\
v(\{3\}) &= 0 & v(\{2, 3\}) &= 5
\end{align*}
\]

Write \( \Delta_i(C) \) for the marginal contribution \( v(C \cup \{i\}) - v(C) \).

\[
\beta_1(N,v) = \frac{1}{4} \cdot (\Delta_1(\emptyset) + \Delta_1(\{2\}) + \Delta_1(\{3\}) + \Delta_1(\{2, 3\}))
\]
\[
= \frac{1}{4} \cdot (0 + 7 + 6 + 5) = \frac{18}{4}
\]

\[
\beta_2(N,v) = \frac{1}{4} \cdot (0 + 7 + 5 + 4) = \frac{16}{4}
\]

\[
\beta_3(N,v) = \frac{1}{4} \cdot (0 + 6 + 5 + 3) = \frac{14}{4}
\]

Exercise: Arguably, that’s fair. But do you see the (other) problem?
The Shapley Value

The \textit{Shapley value} considers all \textit{sequences} in which the grand coalition may assemble and gives equal importance to every such sequence:

\[ \varphi_i(N, v) = \frac{1}{n!} \cdot \sum_{C \subseteq N \setminus \{i\}} |C|! \cdot |N \setminus (C \cup \{i\})|! \cdot [v(C \cup \{i\}) - v(C)] \]

Here \(|C|\) players join before \(i\) and \(|N \setminus (C \cup \{i\})|\) join after her.

Again, \((\varphi_1(N, v), \ldots, \varphi_n(N, v))\) can be considered a payoff vector.

Remark: The special case of simple (and voting) games is particularly interesting. For every sequence, there will be exactly one player with marginal contribution 1 (assuming the grand coalition is winning).

\[ \text{L.S. Shapley. A Value for } n\text{-Person Games. In: H.W. Kuhn and A.W. Tucker (eds.), } Contributions to the Theory of Games, 1953. \]
Example: Computing the Shapley Value

Consider the following 3-player TU game \( \langle N, v \rangle \), with \( N = \{1, 2, 3\} \), in which no single player can generate any surplus on her own:

\[
\begin{align*}
v(\{1\}) &= 0 & v(\{1, 2\}) &= 7 & v(N) &= 10 \\
v(\{2\}) &= 0 & v(\{1, 3\}) &= 6 \\
v(\{3\}) &= 0 & v(\{2, 3\}) &= 5
\end{align*}
\]

Let \( \vec{\Delta}_i(\sigma) \) denote the marginal contribution made by player \( i \) when she joins at the point indicated during the sequence \( \sigma \).

\[
\begin{align*}
\varphi_1(N, v) &= \frac{1}{6} \cdot (\vec{\Delta}_1(123) + \vec{\Delta}_1(132) + \vec{\Delta}_1(213) + \cdots + \vec{\Delta}_1(321)) \\
&= \frac{1}{6} \cdot (0 + 0 + 7 + 5 + 6 + 5) = \frac{23}{6} \\
\varphi_2(N, v) &= \frac{1}{6} \cdot (7 + 4 + 0 + 0 + 4 + 5) = \frac{20}{6} \\
\varphi_3(N, v) &= \frac{1}{6} \cdot (3 + 6 + 3 + 5 + 0 + 0) = \frac{17}{6}
\end{align*}
\]

Observe that \( \frac{23}{6} + \frac{20}{6} + \frac{17}{6} = 10 \) (so this payment vector is efficient).
The Axiomatic Method

Both Banzhaf and Shapley look ok. So which solution concept is fair?

An approach to settle such questions is the axiomatic method:

- Formulate some fundamental normative properties (“axioms”).
- Show that your favourite solution concept satisfies those axioms, and preferably also that it is the only solution concept to do so.

We are going to do this exercise for the Shapley value . . .
Axioms

What is a good vector of functions \((x_1, \ldots, x_n)\), with \(x_i\) mapping any given game \(\langle N, v \rangle\) to the payoff \(x_i(N, v)\) to be paid to player \(i\)?

- **Efficiency**: we should have \(\sum_{i \in N} x_i(N, v) = v(N)\).

- **Symmetry**: if \(v(C \cup \{i\}) = v(C \cup \{j\})\) for all \(C \subseteq N \setminus \{i, j\}\), then \(x_i(N, v) = x_j(N, v)\) (interchangeable players get equal payoffs).

- **Dummy player**: if \(i \in N\) is a “dummy player” in the sense that \(v(C \cup \{i\}) - v(C) = v(\{i\})\) for all coalitions \(C \subseteq N \setminus \{i\}\), then we should have \(x_i(N, v) = v(\{i\})\).

- **Additivity**: we should have \(x_i(N, v_1 + v_2) = x_i(N, v_1) + x_i(N, v_2)\) for the characteristic function \([v_1 + v_2] : C \mapsto v_1(C) + v_2(C)\).

The normative justifications for the first three axioms are convincing. With the additivity axiom some may disagree.

**Exercise**: Show that the Shapley value satisfies all four axioms.
Characterisation Result

Surprisingly, our four axioms fully determine how to divide the surplus:

**Theorem 1 (Shapley, 1953)** *The Shapley value is the only way of satisfying efficiency, symmetry, dummy player axiom, and additivity.*

Proof sketch: \(\Rightarrow\) We’ve seen already that \(\varphi\) satisfies the axioms.

\(\Leftarrow\) Need show axioms uniquely determine some vector of functions \(x\).

Define \(v_S\) for \(S \in 2^N \setminus \{\emptyset\}\) as \(v_S(C) = 1\) if \(S \subseteq C\) and \(= 0\) otherwise. Observe that every \(v : 2^N \to \mathbb{R}_{\geq 0}\) has a unique representation of the form \(v = \sum_{S \in 2^N \setminus \{\emptyset\}} \alpha_S \cdot v_S\) with \(\alpha_S \in \mathbb{R}\) (to see this, note that defining \(v\) requires \(2^n - 1\) numbers, and there are \(2^n - 1\) functions \(v_S\)).

By symmetry, dummy, efficiency: must have \(x_i(N, \alpha_S \cdot v_S) = \frac{\alpha_S}{|S|}\) if \(i \in S\) and \(= 0\) otherwise. Note that this agrees with the Shapley value.

The claim then follows from additivity. \(\checkmark\)
Shapley Value and Stability

How does the Shapley value relate to our stability concepts?

Recall: An imputation is a payoff vector that is efficient and indiv. rat.

**Proposition 2** For superadd. games, the Shapley value is an imputation.

Proof: Efficiency follows from our axiomatic characterisation. ✓

By superadditivity, \( v(C \cup \{i\}) - v(C) \geq v(\{i\}) \), i.e., all marginal contributions of \( i \) are no less than the surplus she can generate alone. The Shapley value is an average over such marginal contributions, so we must have \( \varphi_i(N, v) \geq v(\{i\}) \) (individual rationality). ✓

Recall: The core is the set of efficient payoff vectors for which no coalition has an incentive to break out of the grand coalition.

**Proposition 3** For convex games, the Shapley value is in the core.

We omit the proof. It uses a similar idea as the proof we had given to show that the core of a convex game is always nonempty.
The Nucleolus

A solution concept combining *stability* and *fairness* considerations . . .

Given imputation $x = (x_1, \ldots, x_n)$, we may think of $v(C) - \sum_{i \in C} x_i$ as the strength of $C$'s *complaint*. ($x \in \text{core} \iff \text{no complaints} > 0$)

We now want to *minimise complaints* (as we cannot fully avoid them). Let $\bar{c}(x)$ be the $2^n$-vector of complaints, ordered from high to low.

The *nucleolus* is defined as the set of imputations $x$ for which $\bar{c}(x)$ is *lexicographically minimal*. Thus, you first try to avoid the strongest complaint, then the second strongest, and so forth.

Nice properties of the nucleolus (proofs immediate):

- always *nonempty* (unless the set of imputations is empty)
- subset of the *core* (unless the core is empty)

Also: the nucleolus has *at most one* element (difficult proof omitted).

Computational Considerations

Coalitional games give rise to a number of interesting research challenges of a computational nature:

- **Compact representation:** Assuming that $v$ is simply “given” is unrealistic. We require a compact form of representation. (Related to *preference representation languages*, discussed earlier.)

- **Computing solutions:** Some of the solution concepts discussed (e.g., the nucleolus) require sophisticated algorithm design.

- **Coalition formation:** For superadditive games we can assume that the grand coalition will form, and if it does, this is socially optimal. But for general games, computing a partition $C_1 \cup \cdots \cup C_K = N$ that maximises $\sum_{k \leq K} v(C_k)$ is a nontrivial algorithmic problem.

Summary

This concludes our review of transferable-utility games. We’ve seen:

- solution concepts to reflect stability and fairness considerations: core, nucleolus, Banzhaf value, Shapley value
- questions of existence (nonemptiness of the core)
- questions of relationships between concepts
- axiomatic method: normative characterisation of solutions
- computational considerations

What next? We switch to coalitional games w/o transferable utility.