Game Theory 2022

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Plan for Today

The coming four lectures are about cooperative game theory, where we study so-called coalitional games and the formation of coalitions.

The first two of these lectures are about transferable-utility games. Today we focus on stability for such games:

- definition of transferable-utility games
- examples for transferable-utility games
- the core: set of surplus divisions that are stable

Part of this is also covered in Chapter 8 of the Essentials.

Coalitional Games

A *transferable-utility coalitional game* in characteristic-function form (or simply: a *TU game*) is a tuple \( \langle N, v \rangle \), where

- \( N = \{1, \ldots, n\} \) is a finite set of *players* and
- \( v : 2^N \to \mathbb{R}_{\geq 0} \), with \( v(\emptyset) = 0 \), is a *characteristic function*, mapping every possible *coalition* \( C \subseteq N \) to its *surplus* \( v(C) \).

**Note:** The surplus \( v(C) \) is also known as the *value* or the *worth* of \( C \).

The players are assumed to form coalitions (thereby partitioning \( N \)). Each coalition \( C \) receives its surplus \( v(C) \) and—*somehow*—divides it amongst its members (possible due to utility being transferable).

**Remark:** We’ll see *nontransferable-utility games* later on in the course.
Example: A Network Flow Game

Each pipeline is owned by a different player (1, 2, \ldots). Each pipeline is annotated as [owner: capacity]. The surplus \( v(C) \) for coalition \( C \) is the amount of oil it can pump through the part of the network it owns.

We obtain \( v(12) = 2, \ v(45) = 3, \ v(15) = 0, \ v(134) = 0, \ v(135) = 2, \ v(1345) = 5, \ v(12345) = 7 \), and so forth.

Exercise: What coalition(s) will form? How to divide the surplus?
Simple Games

A *simple game* is a TU game $\langle N, v \rangle$ for which it is the case that $v(C) \in \{0, 1\}$ for every possible coalition $C \subseteq N$, and $v(N) = 1$.

Thus: every coalition is either *winning* or *losing*.
Voting Games

A (weighted) voting game is a tuple \( \langle N, w, q \rangle \), where

- \( N = \{1, \ldots, n\} \) is a finite set of players;
- \( w = (w_1, \ldots, w_n) \in \mathbb{R}^n_{\geq 0} \) is a vector of weights; and
- \( q \in \mathbb{R}_{>0} \) is a quota with \( q \leq w_1 + \cdots + w_n \).

Coalition \( C \subseteq N \) is winning, if the sum of the weights of its members meets or exceeds the quota. Otherwise it is losing.

Thus, a voting game \( \langle N, w, q \rangle \) is in fact a simple game \( \langle N, v \rangle \) with:

\[
v(C) = \begin{cases} 
1 & \text{if } \sum_{i \in C} w_i \geq q \\
0 & \text{otherwise}
\end{cases}
\]

Exercise: Explain why we indeed get \( v(N) = 1 \), as required.
Example: Council of the European Commission

In the *Treaty of Rome* (1957) the founding countries of the EU fixed the voting rule to be used in the Council of the European Commission:

- To pass, a proposal had to get at least 12 votes in favour.
- France, Germany, and Italy each had 4 votes. Belgium and the Netherlands each had 2 votes. Luxembourg had 1 vote.

This is a weighted voting game $\langle N, w, q \rangle$ with

- $N = \{\text{BE, DE, FR, IT, NL, LU}\}$
- $w_{\text{DE}} = w_{\text{FR}} = w_{\text{IT}} = 4$, $w_{\text{BE}} = w_{\text{NL}} = 2$, and $w_{\text{LU}} = 1$
- $q = 12$

**Exercise:** Is this fair? What about Luxembourg in particular?
Properties of Coalitional Games

Some TU games $\langle N, v \rangle$ have certain properties (for all $C, C' \subseteq N$):

- **additive:** $C \cap C' = \emptyset$ implies $v(C \cup C') = v(C) + v(C')$
- **superadditive:** $C \cap C' = \emptyset$ implies $v(C \cup C') \geq v(C) + v(C')$
- **convex:** $v(C \cup C') \geq v(C) + v(C') - v(C \cap C')$
- **cohesive:** $N = C_1 \cup \cdots \cup C_K$ implies $v(N) \geq v(C_1) + \cdots + v(C_K)$
- **monotonic:** $C \subseteq C'$ implies $v(C) \leq v(C')$

Remark: Additive games are not interesting. No synergies between players: every coalition structure is equally good for everyone.

Exercise: Show that convexity can equivalently be expressed as $v(S' \cup \{i\}) - v(S') \geq v(S \cup \{i\}) - v(S)$ for $S \subseteq S' \subseteq N \setminus \{i\}$.

Exercise: Show that additive $\Rightarrow$ convex $\Rightarrow$ superadditive $\Rightarrow$ cohesive, and also that superadditive $\Rightarrow$ monotonic.
Examples

What are the properties of the special types of games we have seen?

- **Network flow games** are easily seen to be *monotonic* as well as *superadditive*, but they usually are not *convex*.

- **Voting games** are *monotonic*, but *not* necessarily *cohesive*. 
Issues

The central questions in coalitional game theory are:

- Which coalitions will form?
- How should the members of coalition $C$ divide their surplus $v(C)$?
  - What would be a division that ensures stability? (this lecture)
  - What division would be fair? (next lecture)

Often, the forming of the grand coalition $N$ is considered the goal. This is particularly reasonable for games that are superadditive.
Example

Consider the following 3-player TU game \( \langle N, v \rangle \), with \( N = \{1, 2, 3\} \), in which no single player can generate any surplus on her own:

\[
\begin{align*}
v(\{1\}) &= 0 & v(\{1, 2\}) &= 7 & v(N) &= 10 \\
v(\{2\}) &= 0 & v(\{1, 3\}) &= 6 \\
v(\{3\}) &= 0 & v(\{2, 3\}) &= 5
\end{align*}
\]

Exercise: What coalition(s) will form? How to divide the surplus?
Payoff Vectors and Imputations

Suppose the grand coalition has formed. *How to divide its surplus?*

Recall: \(v(N)\) is the surplus of the grand coalition \(N\), and \(n = |N|\).

A payoff vector is a vector \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0}\). Properties:

- \(x\) is feasible if \(\sum_{i \in N} x_i \leq v(N)\): do no allocate more than there is

- \(x\) is efficient if \(\sum_{i \in N} x_i = v(N)\): allocate all there is

- \(x\) is individually rational if \(x_i \geq v(\{i\})\) for all players \(i \in N\): nobody should be able to do better on her own

An imputation is a payoff vector that is both individually rational and efficient (and thus also feasible). Reasonable to focus on imputations.
The Core

Which imputations incentivise players to form the grand coalition?

Probably the most important solution concept for coalitional games, formalising this kind of stability notion, is the so-called ‘core’ . . .

An imputation \( x = (x_1, \ldots, x_n) \) is in the core of the game \( \langle N, v \rangle \) if no coalition \( C \subseteq N \) can benefit by breaking away from the grand coalition:

\[
\sum_{i \in C} x_i \geq v(C)
\]

Remark: Individual rationality is a special case of this (with \( C = \{i^*\} \)).

Example: Game with an Empty Core

Consider the following 3-player TU game \( \langle N, v \rangle \), with \( N = \{1, 2, 3\} \), in which no single player can generate any surplus on her own:

\[
\begin{align*}
v(\{1\}) &= 0 & v(\{1, 2\}) &= 7 & v(N) &= 8 \\
v(\{2\}) &= 0 & v(\{1, 3\}) &= 6 \\
v(\{3\}) &= 0 & v(\{2, 3\}) &= 5
\end{align*}
\]

For an imputation \( x = (x_1, x_2, x_3) \) to be in the core, we must have:

- for stability: \( x_1 + x_2 \geq 7 \) and \( x_1 + x_3 \geq 6 \) and \( x_2 + x_3 \geq 5 \)
- for efficiency: \( x_1 + x_2 + x_3 = 8 \)

But this clearly is impossible. So the core is empty.

Question: What games have a nonempty core? Characterisation?

Remark: The above game happens to be superadditive but not convex.
Characterisation for Simple Games

Recall: $\langle N, v \rangle$ is a simple game if $v(C) \in \{0, 1\}$ for all $C \subseteq N$.

Player $i \in N$ is said to be a veto player in the simple game $\langle N, v \rangle$, if for all $C \subseteq N$ it is the case that $i \not\in C$ implies $v(C) = 0$.

**Proposition 1** A simple game (and thus also a voting game) has a nonempty core iff it has at least one veto player.

($\Leftarrow$) Suppose there are $k \geq 1$ veto players. Choose imputation $x$ s.t. $x_i = \frac{1}{k}$ if $i$ is a veto player, and $x_i = 0$ otherwise. Then $x$ is in the core:

- for every winning coalition $C$: $\sum_{i \in C} x_i = k \cdot \frac{1}{k} = 1 = v(C)$
- for every losing coalition $C$: $\sum_{i \in C} x_i \geq 0 = v(C)$ holds vacuously

($\Rightarrow$) Suppose the core is nonempty and $x$ is in it. Let $i^* \in \text{argmax}_{i \in N} x_i$.

Will show that $i^*$ can veto. By efficiency, $\sum_{i \in N} x_i = 1$. Thus, $x_{i^*} > 0$.

Take any $C \subseteq N$ s.t. $i^* \not\in C$. Then $\sum_{i \in C} x_i < 1$. So $v(C) = 0$. $\checkmark$
Convexity is a Sufficient Condition

Recall: \( \langle N, v \rangle \) is convex if \( v(C \cup C') \geq v(C') + v(C') - v(C \cap C') \).

**Theorem 2 (Shapley, 1971)** Every TU game that is convex has got a nonempty core.

**Proof:** Let \([i] := \{1, \ldots, i\}\) for any \(i \in N\). Define an imputation \(x\):

\[
x_i = v([i]) - v([i-1])
\]

By monotonicity, \(x_i \geq 0\). And: \(\sum x_i = v(N)\), i.e., \(x\) is efficient.

Now take any \(C \subset N\). Pick \(i^* \in N\) s.t. \([i^*-1] \subseteq C\) but \(i^* \notin C\).

By convexity: \(v(C \cup \{i^*\}) \geq v(C') + v([i^*]) - v([i^*-1])\). Thus:

\[
-v(C') \geq x_{i^*} - v(C \cup \{i^*\})
\]

\[
\sum_{i \in C} x_i - v(C) \geq \sum_{i \in C \cup \{i^*\}} x_i - v(C \cup \{i^*\})
\]

Repeat: \(\sum_{i \in C} x_i - v(C) \geq \cdots \geq \sum_{i \in N} x_i - v(N) = 0\). \(\checkmark\)

Cohesiveness is a Necessary Condition

Recall: \( \langle N, v \rangle \) is cohesive if \( v(N) \geq v(C_1) + \cdots + v(C_K) \) for every possible partition \( C_1 \cup \cdots \cup C_K = N \).

**Proposition 3** Every TU game with a nonempty core is cohesive.

**Proof:** Consider any game \( \langle N, v \rangle \) that is not cohesive. So there exists a partition \( C_1 \cup \cdots \cup C_K = N \) with \( v(C_1) + \cdots + v(C_K) > v(N) \).

For the sake of contradiction, suppose the core is nonempty. So there’s an imputation \( x = (x_1, \ldots, x_n) \) with \( \sum_{i \in C_k} x_i \geq v(C_k) \) for all \( k \leq K \).

Putting everything together, we get:

\[
\sum_{i \in N} x_i = \sum_{i \in C_1} x_i + \cdots + \sum_{i \in C_K} x_i \geq v(C_1) + \cdots + v(C_K) > v(N)
\]

But this contradicts feasibility of \( x \), i.e., it cannot be an imputation. \( \checkmark \)
The Bondareva-Shapley Theorem

Thus: convex ⇒ has nonempty core ⇒ cohesive. Getting close.

We can do better and give a complete characterisation (w/o proof):

**Theorem 4 (Bondareva, 1962; Shapley 1967)** A TU game has a nonempty core iff that game is balanced.

A collection of weights $\lambda_C \in [0,1]$, one for each coalition $C \subseteq N$, is called balanced if, for all players $i \in N$, we have $\sum_{C \ni i} \lambda_C = 1$.

A TU game $\langle N, v \rangle$ is called balanced if, for all balanced collections of weights $\lambda_C$, we have $\sum_{C \subseteq N} \lambda_C \cdot v(C) \leq v(N)$.

Interpretation: “Grand coalition beats dividing time over coalitions.”


Summary

This has been an introduction to coalitional games, which include:

- **network flow games**: surplus = capacity of coalition’s network
- **voting games**: surplus = coalition’s ability to win vote

We’ve modelled them as *transferable-utility games*, i.e., the members of a coalition can freely distribute the surplus amongst themselves.

We’ve then asked: when will the grand coalition form and be stable? Arguably, if we can find a *payoff vector* that is *in the core*.

So *nonemptiness of the core* is important. **Results:**

- for *simple (and voting) games*: possible iff there is a veto player
- for *general TU games*: possible iff the game is balanced

**What next?** Shifting attention from stability to fairness issues.