Submit your solutions for (up to) four of the following five exercises. If you solve all five, we will consult a random number generator to decide which four to look at and grade. Please note that Question 5 concerns material to be covered in the final lecture of the course, two days after publication of this assignment.

**Question 1** (10 marks)

In class, we have seen two results about the nonemptiness of the core that together imply that every TU game that is balanced must also be cohesive. Prove this fact directly, without appealing to the Bondareva-Shapley Theorem.

**Question 2** (10 marks)

Recall the four axioms characterising the Shapley value for TU games. For each of them, either show that it is also satisfied by the Banzhaf value or give a counterexample.

**Question 3** (10 marks)

Write a program to compute the nucleolus of a given bankruptcy game. A bankruptcy game is specified by a vector \( d = (d_1, \ldots, d_n) \) of positive numbers denoting the amount of debt owed to each of the \( n \) creditors as well as a positive number \( E \), with \( E < d_1 + \cdots + d_n \), denoting the size of the estate to be divided. The surplus of any coalition \( C \) is defined to be \( v(C) = \max(0, E - \sum_{i \in N \setminus C} d_i) \), i.e., the part of the estate that \( C \) can guarantee for itself, even in case all creditors not in \( C \) are paid out their claim in full.

As usual, submit (a) your documented code, (b) instructions for running your program, and (c) a short report that details how you have designed your algorithm and that illustrates your program when run on a few insightful examples. This should include the example for bankruptcy games shown in class. You may collaborate in groups of up to three people. Please submit only one solution per group. Submit everything as a single compressed archive.

**Question 4** (10 marks)

We have seen that for any hedonic game with a symmetric profile of additively separable preferences there exists a coalition structure that is Nash stable. Now focus on hedonic games with just two players with additively separable preferences with profiles of preferences that need not be symmetric. Does every such game have a coalition structure that satisfies the property of Nash stability? How about individual stability? How about contractual stability? Justify your answers. Then repeat the exercise for games with three players.
Question 5 (10 marks)

The Gale-Shapley algorithm for finding a stable set of marriages, in its classical formulation with men proposing to women, is unfair in the sense that it will return a solution that is worst for the women amongst all stable matchings, whilst being optimal for the men. Of course, we could inverse the roles and compute solutions that are optimal for the women and worst for the men, but this does not actually address the fairness issue. It is tempting to try and get around this problem by using something like the following approach:

As usual, there are \( n \) men and \( n \) women with strict preferences over each other and we are looking for a stable matching. We proceed in rounds.

At the start of each round, an agent may or may not be engaged to a member of the opposite sex (initially, nobody is engaged). We say that agent \( A \) is eligible to propose to agent \( B \) (a member of the opposite sex), if \( A \) has never been engaged to \( B \), if \( A \) has never proposed to \( B \), and if \( A \) has never rejected a proposal by \( B \). We say that agent \( A \) is willing to propose to \( B \), if \( A \) is eligible to do so and if \( A \) prefers \( B \) to the partner \( A \) currently is engaged to (if any).

In every round, we randomly select one of the agents who are willing to propose to some other agent. That agent then proposes to their most preferred agent amongst those they are eligible to propose to. Suppose \( A \) proposes to \( B \). Then \( B \) accepts the proposal, if \( B \) prefers \( A \) to the partner \( B \) currently is engaged to (if any). Otherwise \( B \) rejects the proposal. In case of acceptance, both \( A \) and \( B \) break up with their current partners (if any) and get engaged to each other.

We stop once no agent is willing to propose anymore.

Will this work? That is, will this protocol always terminate after a finite number of rounds and result in a matching of all agents that is stable? Justify your answer.