Submit your solutions for (up to) three of the following four exercises. If you solve all four, we will consult a random number generator to decide which three to look at and grade.

**Question 1** (10 marks)

In class, we had sketched the definition of the *price of anarchy*, \( \text{PoA}(G) \), of a given game \( G = \langle N, A, u \rangle \) in the context of our discussion of congestion games. In general, it is defined as the ratio between the expected utilitarian social welfare (i.e., the sum of individual expected utilities) we obtain under the worst Nash equilibrium and the maximal expected utilitarian social welfare across all action profiles. In fact, the idea is even more general than that, as other objective functions and other solution concepts have also been considered. However, here we are going to restrict ourselves to utilitarian social welfare and Nash equilibria. For congestion games, all social welfares involved are negative numbers. For games where all social welfares involved are positive, the PoA is the inverse of the number just defined. Thus, the PoA is always a number that is greater than or equal to 1 and the best possible scenario is a PoA of 1, which corresponds to the case where the Nash equilibrium is also socially optimal. Answer the following questions:

(a) Calculate the PoA for the congestion game below, with \( n = 20 \) players, \( m = 2 \) resources, and action sets \( A_1 = A_2 = \cdots = A_{20} = \{\{1\}, \{2\}\} \). Show your working.

\[
\begin{align*}
\text{d}_1 : x &\mapsto x + 5 \\
\text{d}_2 : x &\mapsto x^2
\end{align*}
\]

(b) Define a normal-form game \( G \) for which \( \text{PoA}(G) = 1 \). This should be a nontrivial game, where not all action profiles have the same social welfare.

(c) Show that for any \( K \in \mathbb{N} \) there exists a normal-form game \( G \) such that \( \text{PoA}(G) > K \). That is, show that the PoA can become arbitrarily large. Show that this is possible even in case the utility functions of all players are strictly positive.

**Question 2** (10 marks)

We have seen that iterated elimination of strictly dominated strategies is order-independent. Show that the same is *not* true for the iterated elimination of weakly dominated strategies.
Question 3 (10 marks)

We have seen examples where a correlated equilibrium of a normal-form game intuitively seems “better” than any of its Nash equilibria. Indeed, for the Nash equilibria we had to choose between either an equal distribution of expected utility between the players or a high sum of the expected utilities, while for the correlated equilibrium we could get both. In particular, the sum of the expected utilities in the correlated equilibrium was at least as high as it was in any of the Nash equilibria, while at the same time also achieving perfect equity between the players.

This exercise is about showing that we can do even better. Find a game and a correlated equilibrium for that game where the sum of the expected utilities is strictly higher than for any of the Nash equilibria of the same game.

Question 4 (10 marks)

Write a program to compute the set of all (mixed and pure) Nash equilibria of a given normal-form game with two players and two actions per player. Keep in mind that there may be infinitely many equilibria. The exact specification of the task and the choice of programming language are up to you.