Logic and Social Choice Theory

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Plan for Today

Preferences are not the only thing we may wish to aggregate.

Today's lecture will be an introduction to *judgment aggregation*, a framework where the views to be aggregated concern the truth or falsehood of formulas expressed in propositional logic. We will cover:

- Motivation: *doctrinal paradox* and *discursive dilemma*
- Definition of the *formal framework* and of basic *axioms*
- Embedding of *preference aggregation* into JA
- Basic impossibility result: *List-Pettit Theorem*
- Discussion of a few specific *aggregation procedures*

The Doctrinal Paradox

Suppose a court with three judges is considering a case in contract law. Legal doctrine stipulates that the defendant is *liable* (r) *iff* the contract was *valid* (p) and it has been *breached* (q): $r \leftrightarrow p \wedge q$.

	p	q	r
Judge 1:	Yes	Yes	Yes
Judge 2:	No	Yes	No
Judge 3:	Yes	No	No
Majority:	Yes	Yes	No

Paradox: Taking majority decisions on the *premises* (p and q) and then inferring the conclusion (r) yields a different result from taking a majority decision on the *conclusion* (r) directly.

L.A. Kornhauser and L.G. Sager. The One and the Many: Adjudication in Collegial Courts. *California Law Review*, 81(1):1–59, 1993.

The Discursive Dilemma

Our judges were expressing judgements on *atoms* (p, q, r) and consistency of a judgement set was evaluated wrt. an *integrity constraint* ($r \leftrightarrow p \wedge q$).

Alternatively, we could allow judgements on *compound formulas*. Examples:

	p	q	$p \wedge q$
Judge 1:	Yes	Yes	Yes
Judge 2:	No	Yes	No
Judge 3:	Yes	No	No
Majority:	Yes	Yes	No

	p	q	$r \leftrightarrow p \wedge q$	r
Judge 1:	Yes	Yes	Yes	Yes
Judge 2:	No	Yes	Yes	No
Judge 3:	Yes	No	Yes	No
Majority:	Yes	Yes	Yes	No

From now on we will work with a framework without integrity constraints ("legal doctrines"), where all inter-relations between propositions stem from the logical structure of those propositions themselves.

In the philosophical literature, the term *doctrinal paradox* is reserved for the first version of our paradox, and the more general term *discursive dilemma* is used when there is no external "doctrine" that is responsible for the problem.

Why Paradox?

Again, what's paradoxical about our example?

	p	q	$p \wedge q$
Judge 1:	Yes	Yes	Yes
Judge 2:	No	Yes	No
Judge 3:	Yes	No	No
Majority:	Yes	Yes	No

Explanation 1: Two seemingly reasonable methods of aggregation, the *premise-based procedure* and the *conclusion-based procedure*, produce different outcomes.

Explanation 2: Each individual judgment set is logically consistent, but applying the seemingly reasonable *majority rule* to all propositions yields a collective judgment set that is inconsistent.

Formal Framework

Notation: Let $\sim\varphi := \varphi'$ if $\varphi = \neg\varphi'$ and let $\sim\varphi := \neg\varphi$ otherwise.

An *agenda* Φ is a finite nonempty set of propositional formulas (w/o double negation) closed under complementation: $\varphi \in \Phi \;\Rightarrow\; \sim\varphi \in \Phi$.

A *judgment set* J on an agenda Φ is a subset of Φ. We call J:

- *complete* if $\varphi \in J$ or $\sim\varphi \in J$ for all $\varphi \in \Phi$
- *complement-free* if $\varphi \not\in J$ or $\sim\varphi \not\in J$ for all $\varphi \in \Phi$
- *consistent* if there exists an assignment satisfying all $\varphi \in J$

Let $\mathcal{J}(\Phi)$ be the set of all complete and consistent subsets of Φ.

Now a finite set of *individuals* $\mathcal{N} = \{1, \ldots, n\}$, with $n \geqslant 2$, express judgments on the formulas in Φ, producing a *profile* $\boldsymbol{J} = (J_1, \ldots, J_n)$.

An *aggregation procedure* for an agenda Φ and a set of n individuals is a function mapping a profile of complete and consistent individual judgment sets to a single collective judgment set: $F : \mathcal{J}(\Phi)^n \to 2^\Phi$.

Example: Majority Rule

Suppose three agents ($\mathcal{N} = \{1, 2, 3\}$) express judgments on the propositions in the agenda $\Phi = \{p, \neg p, q, \neg q, p \vee q, \neg(p \vee q)\}$.

For simplicity, we only show the positive formulas in our tables:

	p	q	$p \vee q$	
Agent 1:	Yes	No	Yes	$J_1 = \{p, \neg q, p \vee q\}$
Agent 2:	Yes	Yes	Yes	$J_2 = \{p, q, p \vee q\}$
Agent 3:	No	No	No	$J_3 = \{\neg p, \neg q, \neg(p \vee q)\}$

The (strict) *majority rule* F_{maj} takes a (complete and consistent) profile and returns the set of propositions accepted by $> \frac{n}{2}$ agents.

In our example: $F_{\text{maj}}(\boldsymbol{J}) = \{p, \neg q, p \vee q\}$ [complete and consistent!]

In general, F_{maj} only ensures completeness and complement-freeness [and completeness only in case n is odd].

Example: Embedding Preference Aggregation

In *preference aggregation*, individuals express preferences (linear orders) over a set of alternatives $\mathcal{X}$ and we need to find a collective preference.

We can embed this into JA (suppose $\mathcal{X} = \{A, B, C\}$):

- Take atomic propositions to be $[A \succ A]$, $[A \succ B]$, ...
- Suppose all individuals accept these propositions:
  - Irreflexivity: $\neg[A \succ A]$, $\neg[B \succ B]$, $\neg[C \succ C]$
  - Completeness: $[A \succ B] \vee [B \succ A]$ etc.
  - Transitivity: $[A \succ B] \wedge [B \succ C] \rightarrow [A \succ C]$, etc.

Then the *Condorcet paradox* corresponds to this example in JA:

	$[A \succ B]$	$[A \succ C]$	$[B \succ C]$	*corresponding order*
Agent 1:	Yes	Yes	Yes	$A \succ B \succ C$
Agent 2:	No	No	Yes	$B \succ C \succ A$
Agent 3:	Yes	No	No	$C \succ A \succ B$
Majority:	Yes	No	Yes	*not a linear order*

Axioms

What makes for a "good" aggregation procedure F? The following *axioms* all express intuitively appealing (yet, debatable) properties:

- *Anonymity:* Treat all individuals symmetrically!

  Formally: for any profile $\boldsymbol{J}$ and any permutation $\pi : \mathcal{N} \to \mathcal{N}$ we have $F(J_1, \ldots, J_n) = F(J_{\pi(1)}, \ldots, J_{\pi(n)})$.

- *Neutrality:* Treat all propositions symmetrically!

  Formally: for any φ, ψ in the agenda Φ and any profile $\boldsymbol{J}$, if for all $i \in \mathcal{N}$ we have $\varphi \in J_i \Leftrightarrow \psi \in J_i$, then $\varphi \in F(\boldsymbol{J}) \Leftrightarrow \psi \in F(\boldsymbol{J})$.

- *Independence:* Only the "pattern of acceptance" should matter!

  Formally: for any φ in the agenda Φ and any profiles $\boldsymbol{J}$ and $\boldsymbol{J'}$, if $\varphi \in J_i \Leftrightarrow \varphi \in J'_i$ for all $i \in \mathcal{N}$, then $\varphi \in F(\boldsymbol{J}) \Leftrightarrow \varphi \in F(\boldsymbol{J'})$.

Observe that the *majority rule* satisfies all of these axioms.

(But so do some other procedures! Can you think of some examples?)

Impossibility Theorem

We have seen that the majority rule is *not consistent*. Is there another "reasonable" aggregation procedure that does not have this problem? *Surprisingly, no!* (at least not for certain agendas)

Theorem 1 (List and Pettit, 2002) *No judgment aggregation procedure for an agenda Φ with $\{p, q, p \wedge q\} \subseteq \Phi$ that satisfies the axioms of anonymity, neutrality, and independence will always return a collective judgment set that is complete and consistent.*

Remark 1: Note that the theorem requires $|\mathcal{N}| > 1$.

Remark 2: Similar impossibilities arise for other agendas with some minimal structural richness. To be discussed tomorrow.

C. List and P. Pettit. Aggregating Sets of Judgments: An Impossibility Result. *Economics and Philosophy*, 18(1):89–110, 2002.

Proof: Part 1

Let $N_\varphi^{\boldsymbol{J}}$ be the set of individuals who accept formula φ in profile $\boldsymbol{J}$.

Let F be any aggregator that is independent, anonymous, and neutral.

We observe:

- Due to *independence*, whether $\varphi \in F(\boldsymbol{J})$ only depends on $N_\varphi^{\boldsymbol{J}}$.
- Then, by *anonymity*, whether $\varphi \in F(\boldsymbol{J})$ only depends on $|N_\varphi^{\boldsymbol{J}}|$.
- Finally, due to *neutrality*, the manner in which $\varphi \in F(\boldsymbol{J})$ depends on $|N_\varphi^{\boldsymbol{J}}|$ must itself *not* depend on φ.

Thus: if φ and ψ are accepted by the same number of individuals, then we must either accept both of them or reject both of them.

Proof: Part 2

Recall: For all $\varphi, \psi \in \Phi$, if $|N_\varphi^{\boldsymbol{J}}| = |N_\psi^{\boldsymbol{J}}|$, then $\varphi \in F(\boldsymbol{J}) \Leftrightarrow \psi \in F(\boldsymbol{J})$.

First, suppose the number n of individuals is *odd* (and $n > 1$).

Consider a profile $\boldsymbol{J}$ where $\frac{n-1}{2}$ individuals accept p and q; one each accept exactly one of p and q; and $\frac{n-3}{2}$ accept neither p nor q.
That is: $|N_p^{\boldsymbol{J}}| = |N_q^{\boldsymbol{J}}| = |N_{\neg(p \wedge q)}^{\boldsymbol{J}}| = \frac{n+1}{2}$. Then:

- Accepting all three formulas contradicts consistency. ✓
- But if we accept none, completeness forces us to accept their complements, which also contradicts consistency. ✓

If n is *even*, we can get our impossibility even without having to make any assumptions regarding the structure of the agenda:

Consider a profile $\boldsymbol{J}$ with $|N_p^{\boldsymbol{J}}| = |N_{\neg p}^{\boldsymbol{J}}|$. Then:

- Accepting both contradicts consistency. ✓
- Accepting neither contradicts completeness. ✓

Change of Perspective

Does the impossibility theorem mean that all hope is lost? *No*.

- We could analyse in more depth for *what agendas* this problem can actually occur. And if it can, we could analyse *how to detect* such a situation. We will follow this route tomorrow.
- We could argue that it is ok to *weaken those axioms:*
  - *Anonymity:* maybe some agents are smarter than others?
  - *Neutrality:* maybe it is actually ok to treat, say, atomic propositions differently from conjunctions?
  - *Independence:* there *are* logical dependencies between propositions; so why not allow them to affect aggregation?

<u>Next</u> we look at some practical aggregators that circumvent the noted impossibility (i.e., they all must violate at least one of the axioms).

Quota Rules

A *quota rule* F_q is defined by a function $q : \Phi \to \{0, 1, \ldots, n{+}1\}$:

$$F_q(\boldsymbol{J}) \quad = \quad \{\varphi \in \Phi \mid |N_\varphi^{\boldsymbol{J}}| \geqslant q(\varphi)\}$$

A quota rule F_q is called *uniform* if q maps any given formula to the same number k. Examples:

- The *unanimous rule* F_n accepts φ *iff* everyone does.
- The *constant rule* F_0 (F_{n+1}) accepts all (no) formulas.
- The *(strict) majority rule* F_{maj} is the quota rule with $q = \lceil \frac{n+1}{2} \rceil$.
- The *weak majority rule* is the quota rule with $q = \lceil \frac{n}{2} \rceil$.

Observe that for *odd* n the majority rule and the weak majority rule coincide. For *even* n they differ (and only the weak one is complete).

Quota Rules with a High Quota

Clearly, a (uniform) quota rule with a sufficiently high quota will be consistent. Dietrich and List (2007) give lower bounds for the quota to ensure consistency as a function of n and the size of the largest *minimally inconsistent subset* of the agenda Φ. Example:

> Let $\Phi = \{p, \neg p, q, \neg q, p \wedge q, \neg(p \wedge q)\}$. The largest mi-subset is $\{p, q, \neg(p \wedge q)\}$. Any quota $> \frac{2}{3}$ will ensure consistency.

But: We (may) lose completeness of the collective judgment set.

F. Dietrich and C. List. Judgment Aggregation by Quota Rules: Majority Voting Generalized. *Journal of Theoretical Politics*, 19(4):391–424, 2007.

Characterisation of Quota Rules

Quota rules are nice to demonstrate the axiomatic method ...

One more axiom:

- *Monotonicity:* If an accepted proposition gets additional support, then we should continue to accept it!

  Formally: for any $\varphi \in \Phi$, profile $\boldsymbol{J}$, agent i, and judgment set J_i', $\varphi \in J_i' \setminus J_i$ entails $\varphi \in F(\boldsymbol{J}) \Rightarrow \varphi \in F(\boldsymbol{J}_{-i}, J_i')$.

We can now *characterise* the class of quota rules:

Proposition 2 (Dietrich and List, 2007) *An aggregation procedure is anonymous, independent and monotonic iff it is a quota rule.*

F. Dietrich and C. List. Judgment Aggregation by Quota Rules: Majority Voting Generalized. *Journal of Theoretical Politics*, 19(4)391–424, 2007.

Proof

<u>Claim:</u> *anonymous* + *independent* + *monotonic* $\Leftrightarrow$ *quota rule*

Clearly, any quota rule has these properties (right-to-left). ✓

For the other direction (using the same technique as before):

- Independence means that acceptance of φ only depends on $N_\varphi^{\boldsymbol{J}}$.
- Anonymity means that, in fact, it only depends on $|N_\varphi^{\boldsymbol{J}}|$.
- Monotonicity means that acceptance of φ cannot turn to rejection as additional individuals accept φ.

Hence, it must be a quota rule. ✓

<u>Reminder:</u> $N_\varphi^{\boldsymbol{J}}$ is the set of individuals who accept φ in profile $\boldsymbol{J}$.

More Characterisations

Clearly, a quota rule F_q is uniform *iff* it is neutral. Thus:

Corollary 3 *An aggregation procedure is anonymous, neutral, independent and monotonic (= ANIM) iff it is a uniform quota rule.*

Now consider a uniform quota rule F_q with quota q. Two observations:

- For F_q to be *complete*, we need $q \leqslant \max_{0 \leqslant x \leqslant n}(x, n-x) \Rightarrow q \leqslant \lceil \frac{n}{2} \rceil$.
- For F_q to be *compl.-free*, we need $q > \min_{0 \leqslant x \leqslant n}(x, n-x) \Rightarrow q > \lfloor \frac{n}{2} \rfloor$.

For n *even*, no such q exists. Thus:

Proposition 4 *For n even, no aggregation procedure is ANIM, complete and complement-free.*

For n *odd*, such a q does exist, namely $q = \lceil \frac{n}{2} \rceil = \frac{n+1}{2}$. Thus:

Proposition 5 *For n odd, an aggregation procedure is ANIM, complete and complement-free iff it is the (strict) majority rule.*

The Premise-Based Procedure

Suppose we *can* divide the agenda into *premises* and *conclusions* (i.e., we are willing to give up *neutrality*):

$$\Phi \quad = \quad \Phi_p \uplus \Phi_c$$

The *premise-based procedure* PBP for Φ_p and Φ_c is this function:

$$\begin{aligned} \mathrm{PBP}(\boldsymbol{J}) \quad &= \quad \Delta \cup \{\varphi \in \Phi_c \mid \Delta \models \varphi\}, \\ & \quad\ \text{where } \Delta = \{\varphi \in \Phi_p \mid |\{i \mid \varphi \in J_i\}| > \frac{n}{2}\} \end{aligned}$$

If we assume that

- the set of *premises* is the set of *literals* in the agenda,
- the agenda Φ is *closed under propositional letters*, and
- the number n of individuals is *odd*,

then $\mathrm{PBP}(\boldsymbol{J})$ will always be *consistent* and *complete*.

Example: Violation of Propositionwise Unanimity

Consider the following basic axiom:

- *Propositionwise Unanimity:* $\varphi \in J_i$ for all $i \in \mathcal{N} \Rightarrow \varphi \in F(\boldsymbol{J})$.

  Unanimous acceptance implies collective acceptance!

Curiously, the premise-based procedure violates this form of unanimity:

	p	q	r	$p \vee q \vee r$
Agent 1:	Yes	No	No	Yes
Agent 2:	No	Yes	No	Yes
Agent 3:	No	No	Yes	Yes
PBP:	No	No	No	No

Complexity of Winner Determination

How hard is it to compute the collective judgment set for the aggregators we have seen? (This is the *winner determination problem*.)

Fact 6 *Winner determination for any quota rule F_q is polynomial.*

Proposition 7 *Winner determination for the* PBP *is polynomial.*

Proof: counting (for premises) + model checking (for conclusions) ✓

We will see more complex aggregators tomorrow.

U. Endriss, U. Grandi, and D. Porello. Complexity of Judgment Aggregation. *Journal of Artificial Intelligence Research*, 45:481–514, 2012.

Summary

This has been an introduction to judgment aggregation. Main topics:

- *axioms:* independence, neutrality, monotonicity, . . .
- *List-Pettit Theorem:* no consistent aggregator is independent, neutral, and anonymous for the 'conjunctive agenda'
- *quota rules:* characterisation results
- *premise-based procedure:* instances ensuring consistency

What next?

Tomorrow we will continue our discussion of judgment aggregation and cover the following topics:

- more concrete aggregation procedures
- agenda characterisation results
- strategic behaviour in judgment aggregation