Natural Deduction

- Natural deduction is a logical deduction system that may be used to prove the validity of logical arguments (including tautologies). It is considered reasonably close to the way humans reason naturally (about logic). Hence the name.
- Nevertheless, it is a *formal system*, which means there are strict rules what constitutes a valid ND proof (and what doesn't).
- It is sufficient to show once that a given set of ND rules forms a system that is *sound* and *complete*. Then the validity of any ND proof can be checked easily (e.g. by a machine).
- The *notation* used for ND proofs is that of nested boxes.

CS2LAP: Logic and Prolog 2000/2001 Natural Deduction Review

Rules for Conjunction and Disjunction

$\wedge$-introduction ($\wedge$I)

$$\frac{A, B}{A \wedge B}$$

$\wedge$-elimination ($\wedge$E)

$$\frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

$\vee$-introduction ($\vee$I)

$$\frac{A}{A \vee B} \qquad \frac{B}{A \vee B}$$

$\vee$-elimination ($\vee$E)

$$\frac{A \vee B, A \Rightarrow C, B \Rightarrow C}{C}$$

Rules for Implication and Negation

⇒-introduction (⇒I)

if $\dfrac{C_1, \ldots, C_k, A}{B}$ then $\dfrac{C_1, \ldots, C_k}{A \Rightarrow B}$

modus ponens (⇒E)

$$\frac{A, A \Rightarrow B}{B}$$

¬-introduction (¬I)

$$\frac{A \Rightarrow B, A \Rightarrow \neg B}{\neg A}$$

¬-elimination (¬E)

$$\frac{\neg A \Rightarrow B, \neg A \Rightarrow \neg B}{A}$$

Boxes

Natural deduction proofs are sequences of single proof steps organised as nested boxes. Example:

Show $P \wedge Q \vdash Q \wedge P$.

		$\underline{Q \wedge P}$
(1)	$P \wedge Q$	data
(2)	P	($\wedge$E) with (1)
(3)	Q	($\wedge$E) with (1)
(4)	$Q \wedge P$	($\wedge$I) with (3) and (2)

Subcomputations

The (⇒I)-rule is applied using a subcomputation:

C_1

⋮

C_k

$A \Rightarrow B$ from subcomputation:

$\underline{B}$

A assumption

⋮

B

This is the *only* case where subcomputations are *necessary*.

Example

Show $\neg\neg A \vdash A$.

$\underline{A}$

(1) $\neg\neg A$ data

(2) $\neg A \Rightarrow \neg A$ from subcomputation

$\underline{\neg A}$

(2.1) $\neg A$ assumption

(2.2) $\neg A$ from (2.1)

(3) $\neg A \Rightarrow \neg\neg A$ from subcomputation

$\underline{\neg\neg A}$

(3.1) $\neg A$ assumption

(3.2) $\neg\neg A$ from (1)

(4) A (¬E) with (2) and (3)