## Natural versus Mechanical Reasoning

- Natural deduction proofs are good in the sense that – once done – they very clearly structure the sequence of logical arguments that have led to a proof. They are quite easy to understand, they are almost *natural*.
- However, when *searching* for a proof it can sometimes be rather tricky to decide what rule to use next.
- This also makes the *implementation* of ND a difficult business.
- For the purpose of automatisation it would be desirable to have a somewhat more *mechanical* deduction system.





## Goal-directed Deduction

- Over the years a number of deduction systems have been devised. Each one has certain advantages and disadvantages. Amongst the more "mechanical" ones the best known examples are *resolution* and *analytic tableaux*.
- In this course we will introduce the *goal-directed* system. The name stems from the fact that each proof step depends directly on the form of the current *goal* (the formula we are trying to prove).

## Arguments

With the goal-directed system we try to prove the validity of arguments of the form

$$\mathcal{P} \vdash A,$$

where

- $\mathcal{P}$ is a set of given formulas (called *data*, *premises*, or *assumptions*) and
- $A$ (the *goal* or *conclusion*) is the formula for which we are trying to show that it can be inferred from $\mathcal{P}$.

Example:

$$P,\ (P \wedge Q) \Rightarrow R,\ (R \Rightarrow Q) \Rightarrow \bot,\ Q \quad \vdash \quad R$$





## Proofs

Giving a goal-directed proofs means iteratively replacing the original argument with a slightly modified one according to a number of rules until we either reach

- an argument which obviously holds (*success*) or
- a state where no more rules apply (*failure*).

Example: The following argument obviously holds. (Why?)

$$(P \wedge R) \Rightarrow \bot,\ P,\ Q \Rightarrow R \quad \vdash \quad P$$

## Introducing the Rules: Conjunction

We can show that $A \wedge B$ follows from a set of formulas $\mathcal{P}$ by showing that both $A$ and $B$ can be shown from $\mathcal{P}$, i.e. by giving *two* separate proofs.

**Rule for conjunction:**

From $\mathcal{P} \vdash A \wedge B$ step to $\mathcal{P} \vdash A$ and $\mathcal{P} \vdash B$.

**Remark.** In natural deduction this would correspond to an application of ($\wedge$I) at the end of a proof.





## Implication

In order to show an implication $A \Rightarrow B$ we may add $A$ to the assumptions and show $B$ instead.

**Rule for implication:**

From $\mathcal{P} \vdash A \Rightarrow B$ step to $\mathcal{P}, A \vdash B$.

**Remark.** This corresponds to the "subcomputation-rule" ($\Rightarrow$I) in natural deduction.

## Atoms

If we want to show $B$ (a propositional atom) and the implication $A \Rightarrow B$ can be found in the data the we may show $A$ instead. An application of modus ponens would then yield the desired $B$.

**Rule for atoms:**

From $\mathcal{P}, A \Rightarrow B \vdash B$ step to $\mathcal{P}, A \Rightarrow B \vdash A$.





## Atoms (continued)

Suppose the atom $B$ is our current goal, but no fitting implication with $B$ in the consequent can be found in the data. If, however, we can find $A \Rightarrow \bot$ (for some formula $A$), it is enough to show $A$.

This is so, because from $A$ we then could get $\bot$ (modus ponens) and from $\bot$ we could infer anything, including our original goal $B$.

**Rule for atoms (with $\bot$):**

From $\mathcal{P}, A \Rightarrow \bot \vdash B$ step to $\mathcal{P}, A \Rightarrow \bot \vdash A$.

## Success

If $A$ is our goal and $A$ is also in the data, then this argument certainly holds, i.e. the proof is a success.

We can also stop if $\bot$ is in the data, because from $\bot$ (from an inconsistent set of data) one may infer anything, including the current goal.





## Example

Show that $(A \Rightarrow B) \wedge A \Rightarrow B$ is a tautology.

$$
\begin{array}{rcl}
 & \vdash & (A \Rightarrow B) \wedge A \Rightarrow B \\
 & & \text{rule for implication} \\
A \Rightarrow B,\ A & \vdash & B \\
 & & \text{rule for atoms with } A \Rightarrow B \\
A \Rightarrow B,\ A & \vdash & A \\
 & & \text{success}
\end{array}
$$

Note how *conjunctions in the data* are rewritten as the set of their conjuncts (after the first proof step).

## Negation and Disjunction

We have introduced most of the goal-directed system so far and can already prove a number of arguments. But a few things are still missing. In particular:

- How do we treat negation?
- How do we treat disjunction?

Fortunately, both these operations can be expressed in terms of $\wedge$, $\Rightarrow$, and $\bot$:

- $\neg A \;\equiv\; A \Rightarrow \bot$
- $A \vee B \;\equiv\; \neg A \Rightarrow B \;\equiv\; (A \Rightarrow \bot) \Rightarrow B$





## Normal Form

For a goal-directed proof all formulas have to be in a certain normal form. Otherwise the rules wouldn't be applicable.

**Definition.** A formula is said to be *ready for computation*

(i) if it is atomic (that is either a propositional atom or $\bot$) *or*

(ii) if it is an implication whose consequent is atomic and whose antecedent is a conjunction of formulas in ready for computation form.

Data formulas have to be in ready for computation form. Goal formulas may be conjunctions of formulas in ready for computation form.

## Translation

Any propositional formula can be rewritten as a conjunction of formulas in ready for computation form by applying the following *rewrite rules*:

$$
\begin{array}{lcl}
A \Rightarrow (B \wedge C) & \rightsquigarrow & (A \Rightarrow B) \wedge (A \Rightarrow C) \\
A \Rightarrow (B \Rightarrow C) & \rightsquigarrow & (A \wedge B) \Rightarrow C \\
\neg A & \rightsquigarrow & A \Rightarrow \bot \\
A \vee B & \rightsquigarrow & (A \Rightarrow \bot) \Rightarrow B
\end{array}
$$





## Summary of Rules so far

1. If the current goal can be found in the data, **succeed**.
2. If $\bot$ can be found in the data, **succeed**.
3. **Implication:** to show $\mathcal{P} \vdash A \Rightarrow B$ show $\mathcal{P}, A \vdash B$.
4. **Conjunction:** to show $\mathcal{P} \vdash A \wedge B$ show $\mathcal{P} \vdash A$ and $\mathcal{P} \vdash B$.
5. **Atoms:** to show $\mathcal{P}, A \Rightarrow B \vdash B$ ($B$ atomic) show $\mathcal{P}, A \Rightarrow B \vdash A$.
6. **Atoms (with $\bot$):** to show $\mathcal{P}, A \Rightarrow \bot \vdash B$ ($B$ atomic) show $\mathcal{P}, A \Rightarrow \bot \vdash A$.

**Remarks.** All formulas need to be ready for computation. One rule is still missing; the current system is *not complete*.