Goal-directed Deduction for First-order Logic

Goal-directed deduction for FOL is similar to the propositional case, but we need to consider the following questions:

1. How do we treat quantifiers ($\forall$ and $\exists$) on either side of an argument? Does it make a difference whether they appear in the data or the goal?
2. How do we treat free variables?
3. What about a normal form (ready for computation form)?

Rewriting with Free Variables

Rewriting data formulas. If we encounter a formula of the form $\forall x.A(x)$ in the *data* we can *add* $A(u)$ to the data where u is a (new) free variable. We may do this several times.

Rewriting the goal. If the *goal* has the form $\exists x.A(x)$ we step to $A(u)$ as the new goal, where u is a (new) free variable. Using the restart rule, we may do this several times.

We usually use the letters u, v, $\ldots$ for free variables.

Skolemization

Skolemizing data formulas. If we encounter a formula of the form $\exists x.A(x)$ with free variables $y_1, \ldots, y_n$ in the *data*, we *replace* it with $A(f(y_1, \ldots, y_n))$ for a new Skolem function symbol f.

Skolemizing the goal. If the *goal* has the form $\forall x.A(x)$ with free variables $y_1, \ldots, y_n$ we step to $A(f(y_1, \ldots, y_n))$ for a new Skolem function symbol f.

A function with 0 arguments is a constant. That is, if there are no free variables in A we Skolemize using a *constant symbol* k rather than a function symbol f.

We usually use k, l, ... for constant symbols and f, g, ... for function symbols.

Unification and Substitution

Substitution. A *substitution* is a set of pairs (bindings) of variables and terms. Notation: $\{v_1 = t_1, \ldots, v_n = t_n\}$ or $\{v_1/t_1, \ldots, v_n/t_n\}$.

Applying a substitution to a formula means replacing all free occurrences of v_i with the associated term t_i. Example:

$$[A(x) \wedge \forall x.B(x, f(y))]\,\{x/k,\, y/g(u)\} \;=\; A(k) \wedge \forall x.B(x, f(g(u)))$$

Unification. Two formulas A and B are said to *unify* if there exists a suitable substitution σ such that $A\sigma = B\sigma$.

Example: $A(x, f(k))$ and $A(g(y), y)$ unify. Why?

Normal Forms

Prenex normal form. A formula of first order logic with all its quantifiers on the outside is said to be in *prenex normal form*.

Example: $\forall x.\forall y.(A(x,y) \Rightarrow B(x))$ is in prenex normal form, but $\forall x.((\exists y.A(x,y)) \Rightarrow B(x))$ isn't.

Any formula can be transformed into an equivalent formula in prenex normal form (using the equivalences from p. 164 in the textbook).

Ready for computation form. A first-order formula is in ready for computation form if it is in prenex normal form and the quantifier-free part is ready for computation according to the definition for the propositional case.

Translation

One way of performing a goal-directed proof in FOL would be to first translate all formulas into ready for computation form and then apply the rules.

A more convenient way, however, will be to interleave applications of proof rules and translation steps. That is, we only translate (part of) a formula when it is necessary for the next rule application.

Additional translation rules. Strictly speaking we don't need them, but the following two rewrite rules can often simplify a proof:

$$\begin{aligned} \neg\forall x.A(x) &\leadsto \exists x.\neg A(x) \\ \neg\exists x.A(x) &\leadsto \forall x.\neg A(x) \end{aligned}$$

Summary

- Formulas with *universal variables* ($\forall$ in data or $\exists$ in goal) are rewritten with *free variables.*
- Formulas with *existential variables* ($\exists$ in data or $\forall$ in goal) are Skolemized.
- Use *substitution* to be able to unify formulas, which may be necessary for success rule or rule for atoms.
- Use new translation rules to get rid of $\neg$ in front of a quantifier.

Remark. Reasoning in FOL is much more complex than reasoning in propositional logic. In fact, *FOL is not decidable* (Church's Theorem). In practice this means that we cannot tell how many times we have to apply the restart rule or rewrite a formula with a new universal variable before a proof succeeds.