Approval-based Committee Voting

Computational Social Choice 2026

Patrick Lederer
ILLC, University of Amsterdam

Plan for Today

Today, we will consider an alternative voting model: we will study elections where a whole committee needs to be chosen instead of a single winner. To this end, we will

- introduce the framework of approval-based committee elections and numerous voting rules for this setting.
- discuss and formalize the concept of proportional representation.
- see how the axiomatic method can be used in approval-based committee voting.

For a good overview of the topic, see

> M. Lackner and P. Skowron. Multi-winner Voting with Approval Preferences. Springer, 2023.
> https://arxiv.org/abs/2007.01795

From Single-Winner Voting to Committee Voting

Single-Winner Voting

2 voters: $a \succ b \succ c \succ d$

3 voters: $b \succ d \succ c \succ a$ $\xrightarrow{Borda}$ c

4 voters: $c \succ d \succ a \succ b$

- Voters report rankings to indicate their preferences.
- We choose a single winner (or a set of tied winners).
- This models presidential elections, jury decisions, hiring a single candidate.

Approval-based Committee Voting

2 voters: $\{a, b\}$

3 voters: $\{b, d, c\}$ $\xrightarrow{AV,\, k=2}$ $\{b, c\}$

4 voters: $\{c\}$

- Voters report approval sets to indicate their preferences.
- We choose a set of k winners (or a set of tied winning sets).
- This models parliamentary elections, recommender systems, hiring multiple candidates.

The Model

- Finite set of alternatives $A = \{a, b, c, \dots\}$; $|A| = m$.
- Finite set of voters $N = \{1, \dots, n\}$; $|N| = n$.
- Each voter $i \in N$ reports a non-empty approval ballot $A_i \subseteq A$.
  - Voters only distinguish between approved and disapproved alternatives!
  - The set of all approval ballots is $\mathcal{A} = 2^A \setminus \{\emptyset\}$.
- An approval profile $\boldsymbol{A} = (A_1, \dots, A_n) \in \mathcal{A}^n$ specifies the approval ballot of every voter.
- A size-k committee W is a subset of the set of alternatives A with $|W| = k$.
  - The set of all size-k committees is $\mathcal{W}_k = \{W \subseteq A \colon |W| = k\}$.
- An approval-based committee (ABC) voting rule f maps every approval profile $\boldsymbol{A}$ and target committee size k to a non-empty set of size-k committees. Formally:

  $f : \mathcal{A}^n \times \{1, \dots, m\} \to 2^{2^A} \setminus \{\emptyset\}$ such that $f(\boldsymbol{A}, k) \subseteq \mathcal{W}_k$ for all $\boldsymbol{A} \in \mathcal{A}^n$, $k \in \{1, \dots, m\}$.

- Just as for single-winner voting rules, we would like to have a single winning committee, but every natural rule may encounter ties.

Multi-winner approval voting

Multi-winner approval voting (AV): Choose the k alternatives approved by the most voters.

Example: $k = 3$

		Approval scores:			
30 voters:	$\{a, b, c\}$	a:	60	d:	30
15 voters:	$\{a, b, d\}$	b:	45	e:	40
15 voters:	$\{a, c, d\}$	c:	45	f:	40
40 voters:	$\{e, f\}$				

For $k = 3$, AV (uniquely) selects $\{a, b, c\}$.

- Does this committee seem fair?
- AV promotes individual excellence: we choose the k individually best alternatives (in terms of approval scores) at the cost of neglecting fairness towards voters.

Proportional Representation

Idea: In many applications of committee voting, it is important that the selected committee proportionally represents the voters' preferences.

The basic idea of proportional representation is that every group of $\lceil \frac{n}{k} \rceil$ voters should be allowed to determine one seat of a size-k committee.

Problem: $k = 2$

$$1 \text{ voter: } \{a\} \quad 1 \text{ voter: } \{b\} \quad 1 \text{ voter: } \{c\} \quad 1 \text{ voter: } \{d\}$$

Every size-2 committee leaves $2 = \frac{n}{2}$ voters unrepresented!

Solution: A group only deserves representation if it is large enough and agrees on a sufficient number of alternatives.

Justified Representation

Fix a committee size k.

- A group of voters S is called ℓ-cohesive if $|\bigcap_{i \in S} A_i| \geq \ell$ and $|S| \geq \ell \frac{n}{k}$.
- A size-k committee W satisfies justified representation (JR) for an approval profile $\boldsymbol{A}$ if, for each 1-cohesive group S, there exists a voter $i \in S$ such that $|A_i \cap W| \geq 1$.
- A size-k committee W satisfies extended justified representation (EJR) for an approval profile $\boldsymbol{A}$ if, for each $\ell \in \{1, \ldots, k\}$ and ℓ-cohesive group S, there exists a voter $i \in S$ such that $|A_i \cap W| \geq \ell$.

Example: $k = 3$

30 voters: $\{a, b, c\}$

30 voters: $\{a, b, d, e\}$

30 voters: $\{d, e, f, g\}$

- The voters in rows 1 and 2 are a 2-cohesive group.
- The voters in rows 2 and 3 are a 2-cohesive group.
- The voters in each individual row are a 1-cohesive group.
- The committee $\{a, b, c\}$ fails JR.
- The committee $\{a, f, g\}$ satisfies JR but fails EJR.
- The committees $\{a, b, d\}$ and $\{a, d, e\}$ satisfy EJR.

Thiele Rules

Question: How can we compute committees satisfying JR or EJR?

Idea: AV chooses the committees $W \in \mathcal{W}_k$ that maximize $\sum_{i \in N} |W \cap A_i|$. This assumes that the satisfaction of each voter grows linearly in the number of the selected alternatives they approve. Maybe another satisfaction function works?

An ABC voting rule f is a Thiele rule if there is a non-decreasing score function $s : \mathbb{N}_0 \to \mathbb{R}$ such that $f(\boldsymbol{A}, k) = \arg\max_{W \in \mathcal{W}_k} \sum_{i \in N} s(|A_i \cap W|)$.

Thiele rules can be seen as analogues of PSRs for ABC elections: each voter gives a score to each committee and we choose the committees with maximal total score. The class of Thiele rules is even characterized by essentially the same axioms as the class of PSRs.

C. Dong and P. Lederer. Refined Characterizations of Approval-Based Committee Scoring Rules. AAAI, 2024. https://arxiv.org/abs/2312.08799

Thiele Rules

Important Thiele rules include:

- AV given by $s_{\text{AV}}(x) = x$ for all $x \in \mathbb{N}_0$.
- Chamberlin-Courant approval voting (CCAV) given by $s_{\text{CCAV}}(0) = 0$ and $s_{\text{CCAV}}(x) = 1$ for all $x \in \mathbb{N}$.
- Proportional approval voting (PAV) given by $s_{\text{PAV}}(0) = 0$ and $s_{\text{PAV}}(x) = \sum_{j=1}^{x} \frac{1}{j}$ for all $x \in \mathbb{N}$.

Example: $k = 3$

1 voter:	$\{a\}$
1 voter:	$\{a, b\}$
6 voters:	$\{a, b, c\}$
4 voters:	$\{d\}$
2 voters:	$\{e\}$

- AV chooses $\{a, b, c\}$ (with a score of $1+2+6\cdot 3+4\cdot 0+2\cdot 0 = 21$).
- CCAV chooses $\{a, d, e\}$ (with a score of $1+1+6\cdot 1+4\cdot 1+2\cdot 1 = 14$).
- PAV chooses $\{a, b, d\}$ (with a score of $1+\frac{3}{2}+6\cdot\frac{3}{2}+4\cdot 1+2\cdot 0 = \frac{31}{2}$).

Thiele Rules

Question: Do some Thiele rules guarantee to select committees satisfying JR or EJR?

Theorem

Every committee selected by CCAV satisfies JR.

Proof:

- Assume for contradiction that there is a profile $\boldsymbol{A}$ and a committee size k such that a committee W selected by CCAV fails JR.
- There is a group of voters S such that $|S| \geq \frac{n}{k}$, $\bigcap_{i \in S} A_i \neq \emptyset$, and $W \cap A_i = \emptyset$ for all $i \in S$. Let $y \in \bigcap_{i \in S} A_i$.
- For each alternative $x \in W$, let $\Delta_x = \sum_{i \in N} \big(s_{\text{CCAV}}(|W \cap A_i|) - s_{\text{CCAV}}(|(W \setminus \{x\}) \cap A_i|)\big)$. Note that $\sum_{x \in W} \Delta_x \leq n - |S| \leq \frac{(k-1)n}{k}$.
- Thus, there is an alternative $x \in W$ with $\Delta_x \leq \frac{(k-1)n}{k^2} < \frac{n}{k}$.
- Let $W' = (W \setminus \{x\}) \cup \{y\}$. It holds that $\sum_{i \in N} \big(s_{\text{CCAV}}(|W' \cap A_i|) - s_{\text{CCAV}}(|W \cap A_i|)\big) \geq |S| - \Delta_x > 0$. Contradiction.

Thiele Rules

Question: Do some Thiele rules guarantee to select committees satisfying JR or EJR?

However, CCAV is easily seen to fail EJR. (Why?)

Theorem

Every committee selected by PAV satisfies EJR.

Proof: Similar swapping argument as for CCAV. Details in

> H. Aziz, M. Brill, V. Conitzer, E. Elkind, R. Freeman, and T. Walsh. Justified Representation in Approval-Based Committee Voting. Social Choice and Welfare, 2017.
> https://arxiv.org/abs/1407.8269

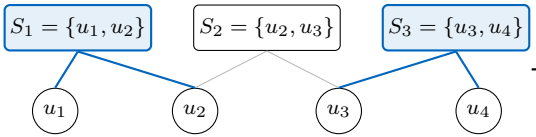
Thiele Rules

Problem: Can we efficiently compute Thiele rules?

Theorem

Given a profile $\boldsymbol{A}$, an integer k, a threshold t, and a non-linear concave score function s, it is NP-hard to decide whether there is a size-k committee W with $\sum_{i \in N} s(|A_i \cap W|) \geq t$.

Proof: For CCAV, this follows from a simple reduction from set cover.



$k = 2$: cover $\{S_1, S_3\}$

Voter	Approval set
u_1	$\{S_1\}$
u_2	$\{S_1, S_2\}$
u_3	$\{S_2, S_3\}$
u_4	$\{S_3\}$

$k = 2$: committee $\{S_1, S_3\}$; CCAV score $= 4$

P. Skowron, P. Faliszewski, and J. Lang. Finding a Collective Set of Items: From Proportional Multirepresentation to Group Recommendation. Artificial Intelligence, 2016.
https://arxiv.org/abs/1402.3044

The Method of Equal Shares

The Method of Equal Shares (MES) repeatedly adds alternatives to the winning committee and makes voters pay for the added alternatives. In every round j, it does the following for an approval profile $\boldsymbol{A}$ and a target committee size k:

- Let W_{j-1} denote the set of previously selected alternatives; initially $W_0 = \emptyset$.
- For each voter i, let $b_j(i)$ denote her remaining budget; initially $b_1(i) = \frac{k}{n}$ for all $i \in N$.
- Alternatives have a cost of 1. Let $X_j = \{x \in A \setminus W_{j-1} \colon \sum_{i \in N \colon x \in A_i} b_j(i) \geq 1\}$ denote the set of affordable alternatives in the current round. Stop if $X_j = \emptyset$.
- For each $x \in X_j$, let $\rho_x = \min \left\{\rho \geq 0 \colon \sum_{i \in N : x \in A_i} \min(b_j(i), \rho) = 1\right\}$.
- We add the alternative $x^* \in X_j$ minimizing ρ_x to the committee: $W_j = W_{j-1} \cup \{x^*\}$.
- We charge each voter i with $x^* \in A_i$ a price of $\min(b_j(i), \rho_{x^*})$:
  - $b_{j+1}(i) = b_j(i) - \min(b_j(i), \rho_{x^*})$ for all $i \in N$ with $x^* \in A_i$.
  - $b_{j+1}(i) = b_j(i)$ for all $i \in N$ with $x^* \notin A_i$.

The Method of Equal Shares

Example: $k = 3$

First round: $W_0 = \emptyset$

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
A_i	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{b,d\}$	$\{b,d\}$	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\{c,d\}$	$\{c,d\}$
$b_1(i)$	$\frac{3}{12}$	$\frac{3}{12}$	$\frac{3}{12}$	$\frac{3}{12}$	$\frac{3}{12}$	$\frac{3}{12}$	$\frac{3}{12}$	$\frac{3}{12}$	$\frac{3}{12}$	$\frac{3}{12}$	$\frac{3}{12}$	$\frac{3}{12}$

- a is affordable for $\rho_a = \frac{1}{8}$.
- b is affordable for $\rho_b = \frac{1}{7}$.
- c is affordable for $\rho_c = \frac{1}{5}$.
- d is affordable for $\rho_d = \frac{1}{4}$.
- a will be added to W next and we reduce the budget of all voters approving a by $\frac{1}{8}$.

The Method of Equal Shares

Example: $k = 3$

Second round: $W_1 = \{a\}$

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
A_i	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{b,d\}$	$\{b,d\}$	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\{c,d\}$	$\{c,d\}$
$b_2(i)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{12}$	$\frac{3}{12}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{12}$	$\frac{3}{12}$

- b is affordable for $\rho_b = \frac{3}{16}$: $v_1, \ldots, v_5$ pay $\frac{1}{8}$; v_6 and v_7 pay $\frac{3}{16}$.
- c is not affordable anymore: voters approving c only have a total budget of $\frac{7}{8}$.
- d is affordable for $\rho_d = \frac{1}{4}$.
- b will be added to W next. We reduce the budget of the voters $v_1, \ldots, v_5$ by $\frac{1}{8}$ and the budget of voters v_6 and v_7 by $\frac{3}{16}$.

The Method of Equal Shares

Example: $k = 3$

Third round: $W_2 = \{a, b\}$

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
A_i	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{b,d\}$	$\{b,d\}$	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\{c,d\}$	$\{c,d\}$
$b_3(i)$	0	0	0	0	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{12}$	$\frac{3}{12}$

- c is not affordable anymore: voters approving c only have a total budget of $\frac{7}{8}$.
- d is not affordable anymore: voters approving d only have a total budget of $\frac{5}{8}$.

The Method of Equal Shares

Problem: MES may select fewer than k alternatives.

Solution: There are different completion methods:

- Fill up the committee with the unselected alternatives with the maximal approval score (AV-completion).
- Continuously increase the budget of each voter at a constant rate. Add alternatives once their approvers have a total budget of 1, then reset these voters' budgets to 0 (Phragmén-completion).
- Start MES with initial budgets larger than $\frac{k}{n}$ (AddOne-completion).

From an axiomatic perspective, the choice of the completion method has little influence.

The Method of Equal Shares

Theorem

The committee returned by the Method of Equal Shares (with any completion method) satisfies EJR.

Proof:

- Assume for contradiction that there is a profile $\boldsymbol{A}$ and a committee size k such that a committee W chosen by MES fails EJR.
- There is an integer ℓ and a group of voters S such that $|\bigcap_{i \in S} A_i| \geq \ell$, $|S| \geq \ell \cdot \frac{n}{k}$, and $|A_i \cap W| < \ell$ for all $i \in S$.
- The total budget of S after the execution of MES must be less than 1; otherwise, there is an alternative $x \in \bigcap_{i \in S} A_i$ that is affordable and could be added.
- There must be a round j where a voter $i \in S$ pays more than $\frac{k}{n \cdot \ell}$. Let j^* be the first such round.
- It holds for the alternative x selected in this round that $\rho_x > \frac{k}{n \cdot \ell}$.

The Method of Equal Shares

Theorem

The committee returned by the Method of Equal Shares (with any completion that retains all selected alternatives) satisfies EJR.

Proof:

- Before round j^*, each voter in S has paid at most $\frac{k}{n \cdot \ell}$ per alternative. So each voter in S has spent at most $\frac{k \cdot (\ell - 1)}{n \cdot \ell}$ and has a budget of at least $\frac{k}{n \cdot \ell}$ remaining.
- The total budget of the voters in S is at least $|S| \cdot \frac{k}{n \cdot \ell} \geq 1$.
- There is an alternative $y \in \bigcap_{i \in S} A_i$ that is not yet selected, and the voters in S can afford this alternative for a price of $\rho_y \leq \frac{k}{n \cdot \ell}$.
- This contradicts the definition of MES which always chooses the alternative that minimizes the price ρ_x.

D. Peters and P. Skowron. Proportionality and the Limits of Welfarism. ACM-EC, 2020.
https://arxiv.org/abs/1911.11747

Core-stability

Question: Can we guarantee stronger proportionality notions?

A size-k committee W is core-stable for an approval profile $\boldsymbol{A}$ if there is no group of voters S and set of alternatives T such that $S \neq \emptyset$, $\frac{|S|}{n} \cdot k \geq |T|$, and $|T \cap A_i| > |W \cap A_i|$ for all $i \in S$.

Proposition

If a committee is core-stable for an approval profile, it also satisfies EJR.

Proof: Any violation of EJR translates to a violation of core-stability.

Theorem

For every approval profile $\boldsymbol{A}$ and committee size k, there is a core-stable committee.

Proof: Via a complicated rule called the Harmonic Entropy rule.

P. Becker, M. Greger, and D. Peters. Existence of the Core in Approval-Based Committee Elections. Working paper, 2026. https://arxiv.org/pdf/2609.11912

Summary

We learned about approval-based committee voting rules today.

- Multi-winner Approval Voting is a simple rule that fails proportional representation.
- Proportional Approval Voting always returns committees satisfying EJR but cannot be computed efficiently.
- The Method of Equal Shares is a computationally efficient rule that always returns committees satisfying EJR.

Next up: Discussion on project progress, then automated reasoning in COMSOC.