Homework #2

Deadline: Tuesday, 10 November 2026, 10:00

Exercise 1 (10 points)

Recall that under the *antiplurality rule*, also known as the *veto rule*, the voters rank the alternatives, and the alternative(s) ranked last the least often win(s). Thus, this is the positional scoring rule with scoring vector $(0, \ldots, 0, -1)$. The purpose of this exercise is to find a number of different characterisations of this voting rule.

(a) Find a consensus criterion such that the antiplurality rule is characterised by that criterion and the *discrete distance*.

(b) Find a way of measuring distances such that the antiplurality rule is characterised by the *unanimous winner* consensus criterion and that distance.

(c) Find a noise model such that the corresponding maximum likelihood estimator is equivalent to the antiplurality rule.

Exercise 2 (10 points)

Recall that we proved Arrow's Theorem for resolute social choice functions $F : \mathcal{L}(A)^n \to A$ in class. For this exercise we will slightly change the underlying model of social choice and explore how this change affects the impossibility result. Suppose we want to allow for the possibility that the outcome of an election is *undecided* (we'll use the symbol $\perp$ for this). So now we are interested in aggregation rules of the form $F : \mathcal{L}(A)^n \to A \cup \{\perp\}$.

Both the Pareto Principle and Independence of Irrelevant Alternatives, exactly as formulated in class, still represent meaningful requirements in this new model of social choice. The notion of dictatorship also still is well-defined. But it turns out that Arrow's Theorem does not hold any longer. The purpose of this exercise is to understand why this is the case.

(a) Pinpoint where the proof of Arrow's Theorem presented in class would break down if we were to attempt to adapt it to our new model of social choice.

(b) Of course, finding a gap in the proof does not mean that the theorem is false. To demonstrate it really is, we need a counterexample. A single example of a nondictatorial aggregation rule that meets all of our requirements would be enough. But to make this more interesting, present an entire *family* of nondictatorial aggregation rules for this model that are all Paretian and independent. Define your family of rules for arbitrary n (number of voters) and arbitrary $m \geqslant 3$ (number of alternatives). For $n = 100$ and $m = 5$ your family should include at least 1,000 aggregation rules.

Briefly comment on the significance of your family of counterexamples. Are these attractive aggregation rules that offer a true way out of Arrow's impossibility?

Exercise 3 (10 points)

Consider a scenario in which $n \geqslant 3$ voters are situated on a social network and each voter can observe at most k of the other voters. We want to run an election to choose an alternative from a set $A = \{a, b, c\}$ of three alternatives using a voting rule F. If necessary, we will break ties lexicographically, meaning that any tie involving alternative a will be broken in favour of a, and any tie involving b but not a will be broken in favour of b. We are concerned that one of the voters may have gained knowledge of the voting intentions of the other voters she can observe and is considering to manipulate the election. But we also know that she will report an untruthful preference only in case (i) she considers it *possible* that doing so will yield a better election outcome for herself than voting truthfully and (ii) she is *certain* that doing so will not yield a worse election outcome for herself than voting truthfully.

Intuitively speaking, the lower the number k, the less we have to worry about manipulation. This exercise is about trying to better understand this link between the voting rule F and the value of k (relative to n). Given n, let us call F *safe* for k in case we can be certain that no manipulation will occur when F is used for the election and no voter can observe more than k of the other $n - 1$ voters. Answer the following questions:

(a) Suppose $n = 4$. Show that the Borda rule is not safe for $k = n - 2 = 2$.

(b) For any given n, characterise the range of values of k (in relation to n) for which the plurality rule is safe. An approximate characterisation (abstracting away from issues such as whether n is odd or even) is perfectly acceptable.

(c) Intuition suggests that every voting rule should be safe for $k = 0$. Either prove that this intuition is correct or provide a counterexample.

(d) By the Gibbard-Satterthwaite Theorem, no voting rule that—when combined with our lexicographic tie-breaking rule—is both surjective and nondictatorial can possibly be safe for $k = n - 1$. This raises the question of whether we can do better for $k = n - 2$.

Either present a voting rule that (in combination with our tie-breaking rule) not only is surjective and nondictatorial but also safe for $k = n - 2$ or show that no such rule exists. (In the former case, your rule should be well-defined for any $n \geqslant 3$. In the latter case, a counterexample for one specific value of n is sufficient.)

Can you comment on the significance of your finding?