

Homework #1

Deadline: Monday, 15 April 2019, 18:00**Question 1** (10 marks)

Recall that the divide-and-conquer cake-cutting protocol guarantees each of the n agents a proportional share of the cake (i.e., a share of at least $\frac{1}{n}$ according to her own valuation), while requiring a total of $O(n \log n)$ cut-queries (and no evaluation-queries). Given that we demand that a cut may only be made in a location returned as the answer to a cut-query, the smallest number of cut-queries any protocol requires is $n - 1$ (otherwise we would end up with fewer than n pieces of cake). The purpose of this exercise is to see that we can achieve a reasonable approximation of the ideal of proportional fairness by means of a protocol that requires only $n - 1$ cut-queries. Answer the following questions:

- (a) Design a protocol for four agents that, using three cut-queries (and any number of evaluation queries), guarantees each agent a share of at least $\frac{1}{6}$ of the cake.
- (b) Design a protocol for $n \geq 2$ agents that, using $n - 1$ cut-queries (and any number of evaluation queries), guarantees each agent a share of at least $\frac{1}{2(n-1)}$ of the cake.

Make sure it is clear from your presentation that your protocols really use no more than the allowed number of cut-queries and that they really provide the required fairness guarantees.

Question 2 (10 marks)

Suppose there are n agents located anywhere on the interval $[0, 1]$. We have to decide where to build an amusement park A , also anywhere on the same interval. The *disutility* of an agent is her distance to A .

- (a) What is the solution selected by the egalitarian CUF?
- (b) What is the solution selected by the elitist (n -rank dictator) CUF?
- (c) For arbitrary $k \leq n$, give a general algorithm to compute a solution that is optimal with respect to the k -rank dictator CUF. What is the complexity of your algorithm?

Question 3 (10 marks)

Let us consider a variant of the model of fair allocation problems with indivisible goods presented in class. In this variant of the model the number of goods is equal to n , the number of agents, and each agent must receive exactly one good. Answer the following questions (and give brief justifications in all cases):

- (a) How many different allocations are there?

- (b) Recall that for the kind of allocation problem discussed in class there exist scenarios where it is impossible to find an allocation under which every agent receives (at least) her maximin share. Show that, in contrast to that result, for the model considered here there always exists such a solution. Can you quantify *how many* allocations will give every agent (at least) her maximin share?
- (c) It is not difficult to verify that for the model of fair allocation problems considered here there exist scenarios in which it is impossible to find an envy-free allocation (just imagine a scenario with two agents and two goods, with both agents agreeing that the first item is more valuable than the second). Suppose we generate allocation scenarios by independently drawing, for every agent $i \in N$ and every good $x \in G$, the value that i assigns to x from the uniform probability distribution over the interval $[0, 1]$. What is the probability of generating a scenario that admits an envy-free allocation? What can you say about this probability for large groups of agents with $n \rightarrow \infty$?
- (d) Is the problem of computing an allocation that maximises utilitarian social welfare tractable or intractable for this model? If you believe it is tractable, prove your claim correct either by providing a polynomial algorithm or by showing how to reduce the problem to a well-known basic algorithmic problem discussed in the literature and provide a reference that establishes the existence of a polynomial algorithm for that problem. If you believe the problem is intractable, provide a polynomial-time reduction from another problem that is known to be intractable (e.g., NP-complete).
- (e) Suppose we start with a random allocation and then ask the agents to contract a sequence of bilateral swap deal (with side payments) that are individually rational. Any such deal involves two agents swapping the goods they hold at the moment. A deal can only go ahead if it is possible for the two agents involved to agree on side payments that leave both of them better off. Show that this process converges: there can be no infinite sequence of such deals. Also show that the final allocation will not always maximise utilitarian social welfare.