# *Computational Social Choice: Spring 2017*

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# *What is Judgment Aggregation?*

Voting deals with aggregating preferences provided by agents into a collective decision that reflects the views of the group.

| | *Lectures* | *HW* | *COMSOC Course* |
|---|---|---|---|
| Student 1 | ✓ | ✓ | ✓ |
| Student 2 | × | ✓ | × |
| Student 3 | ✓ | × | × |

# *What is Judgment Aggregation?*

JA deals with aggregating Yes/No opinions provided by agents into a collective decision that reflects the views of the group.

| | *Lectures* | *HW* | *COMSOC Course* |
|---|---|---|---|
| Student 1 | ✓ | ✓ | ✓ |
| Student 2 | × | ✓ | × |
| Student 3 | ✓ | × | × |

# *Doctrinal Paradox*

Aggregating judges' opinions in a legal case.

$p :=$ 'document is a binding contract'
$q :=$ 'the promise in the document was breached'
$r :=$ 'the defendant is liable'

All agents accept that $p \wedge q \leftrightarrow r$.

| | $p$ | $q$ | $p \wedge q$ |
|---|---|---|---|
| Judge 1 | ✓ | ✓ | ✓ |
| Judge 2 | × | ✓ | × |
| Judge 3 | ✓ | × | × |
| Majority | ✓ | ✓ | × |

Kornhauser, L.A. and Sager, L.G. The One and the Many: Adjudication in Collegial Courts. *California Law Review*, 81(1), 1-59, 1993
C. List and P. Pettit. Aggregating Sets of Judgments: An Impossibility Result. *Economics and Philosophy*, 18(1), 89-110, 2002.

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All agents accept that $p \wedge q \leftrightarrow r$.

| | $p$ | $q$ | $(p \wedge q) \leftrightarrow r$ | $r$ |
|---|---|---|---|---|
| Judge 1 | ✓ | ✓ | ✓ | ✓ |
| Judge 2 | × | ✓ | ✓ | × |
| Judge 3 | ✓ | × | ✓ | × |
| Majority | ✓ | ✓ | ✓ | × |

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C. List and P. Pettit. Aggregating Sets of Judgments: An Impossibility Result. *Economics and Philosophy*, 18(1), 89-110, 2002.

# *Plan for Today*

We'll see the (very) general framework of judgment aggregation, and get a feeling for how exactly it is more general than aggregating preferences.

- Axioms which have counterparts in voting theory
- A simple impossibility theorem related to the doctrinal paradox
- Results which depend on the structure of the agenda
- Strategic manipulation in JA

As always, consult the Handbook for more details on everything!

# Formal Framework

Notation: Let $\sim\varphi := \varphi'$ if $\varphi = \neg\varphi'$ and $\sim\varphi := \neg\varphi$ otherwise.

- An agenda $\Phi$ is a finite, nonempty set of propositional formulas closed under complementation ($\varphi \in \Phi \Rightarrow \sim\varphi \in \Phi$).
- A judgment set $J$ is a subset of $\Phi$. $J$ is:
  - complete if $\varphi \in J$ or $\sim\varphi \in J$ for all $\varphi \in \Phi$
  - complement-free if $\varphi \notin J$ or $\sim\varphi \notin J$ for all $\varphi \in \Phi$
  - consistent if there is an assignment making all $\varphi \in J$ true

  $\mathcal{J}(\Phi)$ is the set of all complete and consistent subsets of $\Phi$.

A set of agents $\mathcal{N} = \{1, \ldots, n\}$ report their judgment sets, giving us a profile $\boldsymbol{J} = (J_1, \ldots J_n)$.

A (resolute) aggregation rule $F$ for an agenda $\Phi$ and a set of agents $\mathcal{N}$ is a function mapping a profile of complete and consistent judgment sets to a single (collective) judgment set:

$$F : \mathcal{J}(\Phi)^n \to 2^{\Phi}$$

.

# *The Majority Rule*

Notation: $N_\varphi^{\boldsymbol{J}}$ is the set of agents who accept $\varphi$ in profile $\boldsymbol{J}$

The (strict) majority rule $F_{Maj}$ takes a (complete and consistent) profile and returns the set of formulas that are accepted by more than half the agents.

$$F_{Maj} : \boldsymbol{J} \mapsto \{\varphi \mid |N_\varphi^{\boldsymbol{J}}| > \frac{n}{2}\}$$

# *Other Rules*

Premise based rules: divide the agenda into premises and conclusions, aggregate opinion on premises, then accept a conclusion $C$ if accepted premises imply $C$.

Kemeny Rule:

$$F(\boldsymbol{J}) = \underset{J \in \mathcal{J}(\Phi)}{\text{argmin}} \sum_{i \in \mathcal{N}} H(J, J_i)$$

Where $H(J, J') = |J \setminus J'|$ is the Hamming distance.

- Similar to Kemeny in voting (which minimises the sum of pairwise disagreements with agents' ballots).
- Guarantees consistency

# *Embedding Voting in JA*

We can use the JA framework to simulate the standard framework of preference aggregation.

Take the following preference profile:

$$a \succ b \succ c$$
$$c \succ a \succ b$$
$$b \succ c \succ a$$

- for each $a$ and $b$: add proposition $p_{a \succ b}$ – 'a is preferable to b'
- We build the preference agenda $\Phi$:
  - $p_{a \succ b}, p_{a \succ c}, p_{b \succ c}, p_{b \succ a}, p_{c \succ a}, p_{c \succ a} \in \Phi$.
  - $(p_{a \succ b} \leftrightarrow \neg p_{b \succ a}) \in \Phi$ for all pairs $a, b$
  - $(p_{a \succ b} \wedge p_{b \succ c} \rightarrow p_{a \succ c}) \in \Phi$ for all $a, b, c$

These encode the properties of linear orders.

# *Condorcet Paradox in JA*

Voting

$a \succ b \succ c$

$c \succ a \succ b$

$b \succ c \succ a$

Judgment Aggregation

| | $p_{a \succ b}$ | $p_{a \succ c}$ | $p_{b \succ c}$ |
|---|---|---|---|
| Judge 1: | ✓ | ✓ | ✓ |
| Judge 2: | ✓ | × | × |
| Judge 3: | × | × | ✓ |
| Majority: | ✓ | × | ✓ |

- Translating back to preferences: $a \succ b \succ c \succ a$.
- In JA: the majority judgment is inconsistent: the majority accepts $p_{a \succ b}$, $p_{b \succ c}$, $p_{a \succ b} \wedge p_{b \succ c} \rightarrow p_{a \succ c}$, but not $p_{a \succ c}$.

# *Collective Rationality Requirements*

Recall that we can require judgment sets to be complete, complement-free and consistent. We can also require that an aggregation rule "lifts" these requirements.

$F$ is:

- complete if $F(\boldsymbol{J})$ is complete for all profiles $\boldsymbol{J}$
- complement-free if $F(\boldsymbol{J})$ is complement-free for all profiles $\boldsymbol{J}$
- consistent if $F(\boldsymbol{J})$ is consistent for all profiles $\boldsymbol{J}$

We already saw the majority rule is not always consistent. We will see now that this problem occurs more generally.

# *Axioms*

The following three axioms have obvious counterparts in voting.

- Anonymity: for any profile $\boldsymbol{J}$ and any permutation $\pi : \mathcal{N} \to \mathcal{N}$, we have that $F(J_1, \ldots J_n) = F(J_{\pi(1)}, \ldots J_{\pi(n)})$.
- Neutrality: for any $\varphi, \psi \in \Phi$ and any profile $\boldsymbol{J}$, if $\varphi \in J_i \Leftrightarrow \psi \in J_i$ for all $i \in \mathcal{N}$, then $\varphi \in F(\boldsymbol{J}) \Leftrightarrow \psi \in F(\boldsymbol{J})$.
- Independence: for any $\varphi \in \Phi$ and any two profiles $\boldsymbol{J}$ and $\boldsymbol{J'}$, if $\varphi \in J_i \Leftrightarrow \varphi \in J'_i$ for all $i \in \mathcal{N}$, then $\varphi \in F(\boldsymbol{J}) \Leftrightarrow \varphi \in F(\boldsymbol{J'})$.

# *Axioms*

The following three axioms have obvious counterparts in voting.

- Anonymity: Treating all agents symmetrically.
- Neutrality: Treating all formulas the same.
- Independence: Outcome on $\varphi$ depends only on agents' judgment on $\varphi$.

Note that the majority rule satisfies all three axioms.

# *An Impossibility Result*

**Theorem 1 (List and Pettit, 2002)** No judgment aggregation rule for an agenda $\Phi$ with $\{p, q, p \wedge q\} \subseteq \Phi$ satisfies anonymity, neutrality, independence, completeness and consistency.

C. List and P. Pettit. Aggregating Sets of Judgments: An Impossibility Result. *Economics and Philosophy*, 18(1), 89-110, 2002.

*Proof. . .*

<u>Notation:</u> $N_\varphi^{\boldsymbol{J}}$ is the set of agents who accept $\varphi$ in profile $\boldsymbol{J}$

Let $F$ be some anonymous, neutral and independent aggregation rule.

- $F$ is independent: whether $\varphi \in F(\boldsymbol{J})$ depends only on $N_\varphi^{\boldsymbol{J}}$.
- $F$ is anonymous: we only need to look at $|N_\varphi^{\boldsymbol{J}}|$.
- $F$ is neutral: the way in which the status of $\varphi \in F(\boldsymbol{J})$ depends on $|N_\varphi^{\boldsymbol{J}}|$, cannot depend on $\varphi$.

Then, if $\varphi$ and $\psi$ are accepted by the same number of individuals, $F$ must either accept both or reject both.

## . . . *Proof.*

Let $\{p, q, p \wedge q\} \subseteq \Phi$.

For odd $n$, consider a profile $\boldsymbol{J}$ where $\frac{n-1}{2}$ accept both $p$ and $q$, 1 agents accepts $p$ but not $q$, one agent accepts $q$ but not $p$, and the remaining $\frac{n-3}{2}$ agents accept neither $p$ nor $q$.

Then $|N_p^{\boldsymbol{J}}| = |N_q^{\boldsymbol{J}}| = |N_{\neg(p \wedge q)}^{\boldsymbol{J}}|$, and by the previous slide, we have to accept either all or none of them.

- Accept all: Not consistent. 👍
- Accept none: Not complete. 👍

For even $n$, take any profile $\boldsymbol{J}$ where $|N_p^{\boldsymbol{J}}| = |N_{\neg p}^{\boldsymbol{J}}|$.

- Accept both: Not consistent. 👍
- Accept neither: Not complete. 👍

# *Quota Rules*

We define a quota rule by a function $q : \Phi \to \{0, \ldots, n+1\}$.

$$F_q(\boldsymbol{J}) = \{\varphi \mid |N_\varphi^{\boldsymbol{J}}| \geqslant q(\varphi)\}$$

A quota rule is uniform if $q(\varphi)$ is the same for all $\varphi \in \Phi$.

The (strict) majority rule is uniform quota rule with q $= \lfloor \frac{n}{2} \rfloor + 1$.

# *Another Axiom*

<u>Notation:</u> $\boldsymbol{J} =_{-i} \boldsymbol{J'}$ means for all agents $j \neq i$, $J_j = J'_j$.

- Anonymity: for any profile $\boldsymbol{J}$ and any permutation $\pi : \mathcal{N} \to \mathcal{N}$, we have that $F(J_1, \dots J_n) = F(J_{\pi(1)}, \dots J_{\pi(n)})$.
- Neutrality: for any $\varphi, \psi \in \Phi$ and any profile $\boldsymbol{J}$, if $\varphi \in J_i \Leftrightarrow \psi \in J_i$ for all $i \in \mathcal{N}$, then $\Phi \in F(\boldsymbol{J}) \Leftrightarrow \in F(\boldsymbol{J})$.
- Independence: for any $\varphi \in \Phi$ and any two profiles $\boldsymbol{J}$ and $\boldsymbol{J'}$, if $\varphi \in J_i \Leftrightarrow \varphi \in J'_i$ for all $i \in \mathcal{N}$, then $\varphi \in F(\boldsymbol{J}) \Leftrightarrow F(\boldsymbol{J'})$.
- Monotonicity: for any $\varphi \in \Phi$ and profiles $\boldsymbol{J}$ and $\boldsymbol{J'}$, $\boldsymbol{J} =_{-i} \boldsymbol{J'}$, and $\varphi \in J'_i \setminus J_i$ for some agent $i \in \mathcal{N}$, then $\varphi \in F(\boldsymbol{J}) \Rightarrow \varphi \in F(\boldsymbol{J'})$.

# *Characterisation of Quota Rules*

**Theorem 2 (Dietrich and List, 2007).** An aggregation rule $F$ is anonymous, independent and monotonic iff it is a quota rule.

*Proof.*

- ⇐ Clear from the definition of a quota rule. 👍
- ⇒ By independence, we decide formula by formula. By anonymity, only the size of the coalitions matter. By monotonicity if a set of agents can get $\varphi$ accepted, then a superset of those can also get $\varphi$ accepted. This means that for every formula $\varphi$, there is some number $k$ such that $\varphi$ is accepted if and only if at least $k$ agents accept $\varphi$. I.e. $k = q(\varphi)$. 👍

A quota rule is neutral if and only if it is a uniform quota rule.

**Corollary 1.** $F$ is ANIM iff it is a uniform quota rule .

Dietrich, F. and List, C. Judgment Aggregation by Quota Rules: Majority Voting Generalized. *Journal of Theoretical Politics*, 19(4), 391-424, 2007.

# *Characterisation of Majority Rule*

**Corollary 2.** for odd $n$: $F$ is ANIM, complete and complement-free if and only if $F$ is the (strict) majority rule.

*Intuition:*

- Majority is a uniform quota rule, so we get ANIM for free.
- If $q$ is high we get complement-freeness. If $q$ is low, we get completeness. The majority rule hits the sweet spot.

Note: For even $n$, no rule satisfies ANIM + C & C.

# *Agenda Characterisation Results*

In JA, the logical structure of the agenda plays a big role! We already saw the Impossibility result by List & Pettit depended on the agenda.

Restricting the agenda can also give us a way out of some impossibilities, though if we restrict too much, one could argue these possibilities don't carry much weight.

- Existential results: there exists a rule satisfying some axioms which is consistent for every agenda with a given property.
- Universal results: all rules satisfying some axioms are consistent for agendas with a given property.

Dokow, E. and Holzman, R. Aggregation of Binary Evaluations. *Journal of Economic Theory*, 145(2), 495-511, 2010.
Endriss, U., Grandi, U., and Porello, D. Complexity of Judgment Aggregation. *Journal of Artificial Intelligence Research (JAIR)*, 45, 481-514, 2012.

# *Consistent Majority Outcome*

We say an agenda $\Phi$ satisfies the median property (MP) if every inconsistent subset of $\Phi$ has an inconsistent subset of size $\leqslant 2$.

Ex. $\{p, q, (p \wedge q), \neg p, \neg q, \neg(p \wedge q)\}$ does not satisfy MP.

**Theorem 3 (Nehring and Puppe, 2007).** The (strict) majority rule is consistent for a given agenda $\Phi$ iff $\Phi$ has the MP.

Nehring, K. and Puppe, C. The Structure of Strategy-Proof Social Choice. Part I: General Characterization and Possibility Results on Median Spaces. *Journal of Economic Theory*, 135(1), 269-305, 2007.

# *Consistent Majority Outcome*

We say an agenda $\Phi$ satisfies the median property (MP) if every minimally inconsistent (MI) subset of $\Phi$ is of size $\leqslant 2$.

Ex. $\{p, q, (p \wedge q), \neg p, \neg q, \neg(p \wedge q)\}$ does not satisfy MP.

**Theorem 3 (Nehring and Puppe, 2007).** The (strict) majority rule is consistent for a given agenda $\Phi$ iff $\Phi$ has the MP.

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*Proof.*

- Let $\Phi$ be an agenda with the MP and assume there is some (consistent) profile $\boldsymbol{J}$ such that $F_{Maj}(\boldsymbol{J})$ is not consistent.
  - Then there is some inconsistent set $\{\varphi, \psi\} \subseteq F_{Maj}(\boldsymbol{J})$...
  - meaning each of $\varphi$, $\psi$ must be accepted by $> \frac{n}{2}$ agents...
  - but then, there is an agent who must have accepted both...
  - which contradicts our assumption that $\boldsymbol{J}$ is consistent. 👍
- Suppose $\Phi$ violates the MP. Then there is a MI set $X = \{\varphi_1, \ldots, \varphi_k\} \subseteq \Phi$ where $k > 2$. Consider a profile $\boldsymbol{J}$ where (roughly) a third of agents accept all formulas in $X$ except $\varphi_1$, another (distinct) third accept all but $\varphi_2$, and another (distinct) third accept all but $\varphi_3$. Then there is a majority for all formulas in $X$, so the majority will be inconsistent. 👍

# *Strategic Manipulation*

As in voting, agents can manipulate (submit untruthful judgment sets) to get a more favourable outcome.

Example: Suppose we are using the premise-based rule, where $p, q$ are the premises.

| | $p$ | $q$ | $p \vee q$ |
|---|---|---|---|
| Agent 1 | $\times$ | $\times$ | $\times$ |
| Agent 2 | $\checkmark$ | $\times$ | $\checkmark$ |
| Agent 3 | $\times$ | $\checkmark$ | $\checkmark$ |
| Majority | $\times$ | $\times$ | $\Rightarrow \times$ |

If agent 3 only cares about the outcome wrt. the conclusion $p \vee q$, she has incentive to manipulate.

# *Strategic Manipulation*

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| | $p$ | $q$ | $p \vee q$ |
|---|---|---|---|
| Agent 1 | $\times$ | $\times$ | $\times$ |
| Agent 2 | ✓ | $\times$ | ✓ |
| Agent 3 | ✓ | ✓ | ✓ |
| Majority | ✓ | $\times$ | $\Rightarrow$ ✓ |

If agent 3 only cares about the outcome wrt. the conclusion $p \vee q$, she has incentive to manipulate.

# *Strategy-proofness*

Agent $i$'s preferences are modelled as a weak order $\succeq_i$ over judgment sets. We will only look at closeness-respecting preferences.

- $\succeq_i$ is closeness-respecting iff $(J' \cap J_i) \subset (J \cap J_i) \Rightarrow J \succeq_i J'$.

Example: If $J_i = \{p, q, r\}$, $J' = \{p, \neg q\}$, $J = \{p, \neg q, r\}$, then $J \succeq_i J'$ because $J' \cap J_i = \{p\} \subset \{p, q\} = J \cap J_i$.

- Agent $i$ manipulates if she reports a judgment set $J \neq J_i$.
- She has incentive to do so if $F(\boldsymbol{J}_{-i}, J_i') \succ_i F(\boldsymbol{J})$ for some $J_i' \in \mathcal{J}(\Phi)$.

An aggregation rule $F$ is strategy-proof for a given class of preferences if no agent (with such preferences) has incentive to manipulate.

# *Strategy-proof Rules*

**Theorem 7 (Dietrich and List, 2007)** $F$ is independent and monotonic iff $F$ is strategy-proof for all closeness-respecting preferences.

*Proof.*

- ➡ Independence means we (and the manipulator) can consider one formula at a time. Monotonicity means it is always in a manipulator's interest to increase support for formulas in her (truthful) judgment set and reduce support for formulas not in her judgment set. Thus, it is always in her best interest to report her truthful judgment set. 👍
- ⬅ Omitted.

Dietrich, F. and List, C. Strategy-Proof Judgment Aggregation. *Economics and Philosophy*, 23(3), 269-300, 2007.

# *Summary*

Things I talked about:

- The Doctrinal Paradox & Failure of Collective Rationality
- Embedding PA in JA
- Axiomatic Characterisation of Quota Rules
- Agenda Characterisations
- Strategic Manipulation

Things I didn't talk about:

- Domain restrictions (similar to single-peakedness)
- Complexity results: there are many
- Binary Aggregation: an alternate framework
- Opinion Diffusion on Networks using JA framework