Computational Social Choice: Autumn 2012

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Plan for Today

Earlier we have seen the basic judgment aggregation framework and various axioms and rules; a basic impossibility theorem; and several ways around this impossibility.

Today we will cover additional topics in *judgment aggregation:*

- Characterisation of aggregators: *quota rules* and *majority rule*
- *Agenda characterisation* results: types of agendas on which paradoxical outcomes can be avoided. This includes:
  - *Possibility:* existence of acceptable rules on certain agendas
  - *Safety:* guaranteed consistency of outcomes for all relevant rules on certain agendas
- *Complexity results* for safety conditions: *polynomial hierarchy*

Reminder: Formal Framework

Notation: Let $\sim\varphi := \varphi'$ if $\varphi = \neg\varphi'$ and let $\sim\varphi := \neg\varphi$ otherwise.

An *agenda* Φ is a finite nonempty set of propositional formulas (w/o double negation) closed under complementation: $\varphi \in \Phi \;\Rightarrow\; \sim\varphi \in \Phi$.

A *judgment set* J on an agenda Φ is a subset of Φ. We call J:

- *complete* if $\varphi \in J$ or $\sim\varphi \in J$ for all $\varphi \in \Phi$
- *complement-free* if $\varphi \notin J$ or $\sim\varphi \notin J$ for all $\varphi \in \Phi$
- *consistent* if there exists an assignment satisfying all $\varphi \in J$

Let $\mathcal{J}(\Phi)$ be the set of all complete and consistent subsets of Φ.

Now a finite set of *individuals* $\mathcal{N} = \{1, \ldots, n\}$, with $n \geqslant 2$, express judgments on the formulas in Φ, producing a *profile* $\boldsymbol{J} = (J_1, \ldots, J_n)$.

An *aggregation procedure* for agenda Φ and a set $\mathcal{N}$ of individuals is a function mapping a profile of complete and consistent individual judgment sets to a single collective judgment set: $F : \mathcal{J}(\Phi)^{\mathcal{N}} \to 2^{\Phi}$.

Properties of Aggregation Procedures

We extend the concepts of completeness, complement-freeness, and consistency of *judgment sets* to properties of *aggregators* F:

- F is *complete* if $F(\boldsymbol{J})$ is complete for any $\boldsymbol{J} \in \mathcal{J}(\Phi)^{\mathcal{N}}$
- F is *complement-free* if $F(\boldsymbol{J})$ is c.-f. for any $\boldsymbol{J} \in \mathcal{J}(\Phi)^{\mathcal{N}}$
- F is *consistent* if $F(\boldsymbol{J})$ is consistent for any $\boldsymbol{J} \in \mathcal{J}(\Phi)^{\mathcal{N}}$

Only consistency involves logic *proper*. Complement-freeness and completeness are purely syntactic concepts, not involving any model-theoretic ideas (they are also computationally easy to check).

F is called *collectively rational* if it is both complete and consistent (and thus also complement-free).

Axioms

Some natural axioms for JA we have seen before:

- *Unanimity:* if $\varphi \in J_i$ for all i, then $\varphi \in F(\boldsymbol{J})$.
- *Anonymity:* for any profile $\boldsymbol{J}$ and any permutation $\pi : \mathcal{N} \to \mathcal{N}$ we have $F(J_1, \ldots, J_n) = F(J_{\pi(1)}, \ldots, J_{\pi(n)})$.
- *Neutrality:* for any φ, ψ in the agenda Φ and profile $\boldsymbol{J} \in \mathcal{J}(\Phi)^{\mathcal{N}}$, if for all i we have $\varphi \in J_i \Leftrightarrow \psi \in J_i$, then $\varphi \in F(\boldsymbol{J}) \Leftrightarrow \psi \in F(\boldsymbol{J})$.
- *Independence:* for any $\varphi \in \Phi$ and profiles $\boldsymbol{J}$ and $\boldsymbol{J'}$ in $\mathcal{J}(\Phi)^{\mathcal{N}}$, if $\varphi \in J_i \Leftrightarrow \varphi \in J'_i$ for all i, then $\varphi \in F(\boldsymbol{J}) \Leftrightarrow \varphi \in F(\boldsymbol{J'})$.
- *Systematicity* $=$ neutrality $+$ independence

A further axiom is monotonicity:

- *Monotonicity:* for any $\varphi \in \Phi$ and $\boldsymbol{J}, \boldsymbol{J'} \in \mathcal{J}(\Phi)^{\mathcal{N}}$, if $\varphi \in J'_{i^\star} \backslash J_{i^\star}$ for some $i^\star$ and $J_i = J'_i$ for all $i \neq i^\star$, then $\varphi \in F(\boldsymbol{J}) \Rightarrow \varphi \in F(\boldsymbol{J'})$.

Quota Rules

Notation: Let $N_\varphi^{\boldsymbol{J}}$ be the set of individuals accepting φ in profile $\boldsymbol{J}$.

A *quota rule* F_q is defined by a function $q : \Phi \rightarrow \{0, 1, \ldots, n{+}1\}$:

$$F_q(\boldsymbol{J}) \quad = \quad \{\varphi \in \Phi \mid \#N_\varphi^{\boldsymbol{J}} \geqslant q(\varphi)\}$$

A quota rule F_q is called *uniform* if q maps any given formula to the same number k. Examples:

- The *unanimous rule* F_n accepts φ *iff* everyone does.
- The *constant rule* F_0 (F_{n+1}) accepts all (no) formulas.
- The *(strict) majority rule* $F_{\textsf{maj}}$ is the quota rule with $q = \lceil \frac{n+1}{2} \rceil$.
- The *weak majority rule* is the quota rule with $q = \lceil \frac{n}{2} \rceil$.

Observe that for *odd* n the majority rule and the weak majority rule coincide. For *even* n the differ (and only the weak one is complete).

Characterisation of Quota Rules

Proposition 1 (Dietrich and List, 2007) *An aggregation procedure is anonymous, independent and monotonic iff it is a quota rule.*

Proof: Clearly, any quota rule has these properties (right-to-left).

For the other direction (proof sketch):

- Independence means that acceptance of φ only depends on the coalition $N_\varphi^{\boldsymbol{J}}$ accepting it.
- Anonymity means that it only depends on the cardinality of $N_\varphi^{\boldsymbol{J}}$.
- Monotonicity means that acceptance of φ cannot turn to rejection as additional individuals accept φ.

Hence, it must be a quota rule. ✓

F. Dietrich and C. List. Judgment Aggregation by Quota Rules: Majority Voting Generalized. *Journal of Theoretical Politics*, 19(4)391–424, 2007.

More Characterisations

A quota rule F_q is uniform *iff* it is neutral. Thus:

Corollary 1 *An aggregation procedure is anonymous, neutral, independent and monotonic (= ANIM) iff it is a uniform quota rule.*

Now consider a uniform quota rule F_q with quota q. Two observations:

- For F_q to be *complete*, we need $q \leqslant \max\limits_{0 \leqslant x \leqslant n}(x, n-x) \Rightarrow q \leqslant \lceil \frac{n}{2} \rceil$.
- For F_q to be *compl.-free*, we need $q > \min\limits_{0 \leqslant x \leqslant n}(x, n-x) \Rightarrow q > \lfloor \frac{n}{2} \rfloor$.

For n *even*, no such q exists. Thus:

Proposition 2 *For n even, no aggregation procedure is ANIM, complete and complement-free.*

For n *odd*, such a q does exist, namely $q = \lceil \frac{n}{2} \rceil = \lceil \frac{n+1}{2} \rceil$. Thus:

Proposition 3 *For n odd, an aggregation procedure is ANIM, complete and complement-free iff it is the (strict) majority rule.*

Agenda Characterisations

Our characterisation results so far only involve *choice-theoretic axioms* (independence, etc.) and *syntactic conditions* on the outcome (completeness and complement-freeness). No logic so far.

We now turn to a different type of characterisation result:

- We already know that adding *consistency* to our requirements (thus asking for *collective rationality*) is troublesome (doctrinal paradox, original impossibility theorem).
- But if we assume certain *properties of the agenda*, then consistency might be achievable.

Safety of the Agenda under Majority Voting

Previously we saw that the majority rule can produce an inconsistent outcome for *some* (not all) profiles based on agendas $\Phi \supseteq \{p, q, p \wedge q\}$. How can we *characterise* the class of agendas with this problem?

An *agenda* Φ is said to be *safe* for an aggregation procedure F if the *outcome* $F(\boldsymbol{J})$ is *consistent* for *any* admissible profile $\boldsymbol{J} \in \mathcal{J}(\Phi)^{\mathcal{N}}$.

Proposition 4 (Nehring and Puppe, 2007) *An agenda Φ is safe for the (strict) majority rule iff Φ has the median property (for $|\mathcal{N}| \geqslant 3$).*

A set of formulas Φ satisfies the *median property* if every inconsistent subset of Φ does itself have an inconsistent subset of size $\leqslant 2$.

K. Nehring and C. Puppe. The Structure of Strategy-proof Social Choice. Part I: General Characterization and Possibility Results on Median Space. *Journal of Economic Theory*, 135(1):269–305, 2007.

Proof

Claim: Φ is safe [$F_{\text{maj}}(\boldsymbol{J})$ is consistent] $\Leftrightarrow$ Φ has the median property

($\Leftarrow$) Let Φ be an agenda with the median property. Now assume that there exists an admissible profile $\boldsymbol{J}$ such that $F_{\text{maj}}(\boldsymbol{J})$ is *not* consistent.

- $\leadsto$ There exists an inconsistent set $\{\varphi, \psi\} \subseteq F_{\text{maj}}(\boldsymbol{J})$.
- $\leadsto$ Each of φ and ψ must have been accepted by a strict majority.
- $\leadsto$ One individual must have accepted both φ and ψ.
- $\leadsto$ Contradiction (individual judgment sets must be consistent). ✓

($\Rightarrow$) Let Φ be an agenda that violates the median property, i.e., there exists a minimally inconsistent set $\Delta = \{\varphi_1, \ldots, \varphi_k\} \subseteq \Phi$ with $k > 2$.

For simplicity, suppose n (the number of individuals) is divisible by 3. There exists a consistent profile $\boldsymbol{J}$ under which individual i accepts all formulas in Δ except for $\varphi_{1+(i \bmod 3)}$. But then the majority rule will accept all formulas in Δ, i.e., $F_{\text{maj}}(\boldsymbol{J})$ is inconsistent. ✓

Agenda Characterisation for Classes of Rules

Now instead of a single aggregator, suppose we are interested in a *class of aggregators*, possibly determined by a set of *axioms*.

We might ask:

- *Possibility:* Does there exist an aggregator meeting certain axioms that will be consistent for any agenda with a given property?
- *Safety:* Will every aggregator meeting certain axioms be consistent for any agenda with a given property?

Possibility Theorem for Median Spaces

Recall: majority $\Leftrightarrow$ ANIM + completeness + complement-freeness

Now weaken anonymity to *non-dictatoriality* $\Rightarrow$ obtain *class* of rules (includes, e.g., weighted majorities).

We can strengthen the agenda characterisation result for the majority:

Theorem 1 (Nehring and Puppe, 2007) *There exists a neutral, independent, monotonic, nondictatorial, and collectively rational aggregation procedure for an agenda Φ iff Φ has the median property.*

Proof: Omitted (but the main idea is already in Proposition 4).

Various similar results are reviewed by List and Puppe (2009).

K. Nehring and C. Puppe. The Structure of Strategy-proof Social Choice. Part I: General Characterization and Possibility Results on Median Space. *Journal of Economic Theory*, 135(1):269–305, 2007.

C. List and C. Puppe. Judgment Aggregation: A Survey. In: *Handbook of Rational and Social Choice*, Oxford University Press, 2009.

Safety of the Agenda for Systematic Rules

Suppose we know that the group will use *some* aggregation procedure meeting certain requirements, but we do not know which procedure exactly. Can we guarantee that the outcome will be consistent?

A typical result (for the majority rule axioms, minus monotonicity):

Theorem 2 (Endriss et al., 2010) *An agenda Φ is safe for any anonymous, neutral, independent, complete and complement-free aggregation procedure iff Φ has the simplified median property .*

An agenda Φ has the *simplified median property* if every inconsistent subset of Φ has itself an inconsistent subset $\{\varphi, \psi\}$ with $\models \varphi \leftrightarrow \neg\psi$.

Note: This is more restrictive than the median property: $\{\neg p, p \wedge q\}$.

U. Endriss, U. Grandi and D. Porello. Complexity of Judgment Aggregation: Safety of the Agenda. Proc. AAMAS-2010.

Proof

<u>Claim:</u> Φ is safe for any ANI/complete/comp-free rule $F \Leftrightarrow \Phi$ has SMP

($\Leftarrow$) Suppose Φ has the SMP. For the sake of contradiction, assume $F(\boldsymbol{J})$ is inconsistent. Then $\{\varphi, \psi\} \subseteq F(\boldsymbol{J})$ with $\models \varphi \leftrightarrow \neg\psi$. Now:

- $\leadsto$ $\varphi \in J_i \Leftrightarrow {\sim}\psi \in J_i$ for each individual i (from $\models \varphi \leftrightarrow \neg\psi$ together with consistency and completeness of individual judgment sets)
- $\leadsto$ $\varphi \in F(\boldsymbol{J}) \Leftrightarrow {\sim}\psi \in F(\boldsymbol{J})$ (from neutrality)
- $\leadsto$ both ψ and ${\sim}\psi$ in $F(\boldsymbol{J}) \leadsto$ contradiction (with complement-freeness) ✓

($\Rightarrow$) Suppose Φ violates the SMP. Take any minimally inconsistent $\Delta \subseteq \Phi$. If $|\Delta| > 2$, then also the MP is violated and we already know that the majority rule is not consistent. ✓ So can assume $\Delta = \{\varphi, \psi\}$.

W.l.o.g., must have $\varphi \models \neg\psi$ but $\neg\psi \not\models \varphi$ (otherwise SMP holds).

But now we can find a rule F that is not safe: accept a formula if at most one individual does and take a profile with $J_1 = \{{\sim}\varphi, {\sim}\psi, \ldots\}$, $J_2 = \{{\sim}\varphi, \psi, \ldots\}$, and $J_3 = \{\varphi, {\sim}\psi, \ldots\}$. Then $F(\boldsymbol{J}) = \{\varphi, \psi, \ldots\}$. ✓

Comparing Possibility and Safety Results

Possibility theorems and safety theorems are closely related:

- Possibility: *some* aggregator in the class determined by the given axioms will produce consistent outcomes *iff* the agenda has a given property
- Safety: *all* aggregators in the class determined by the given axioms will produce consistent outcomes *iff* the agenda has a given property

In what situations do we need these results?

- Possibility: a mechanism designer wants to know whether she can design an aggregation rule meeting a given list of requirements
- Safety: a system might know certain properties of the aggregator users will employ (but not all properties) and we want to be sure there won't be any problem (we might want to check this again and again)

For safety problems in particular we might want to develop *algorithms*, i.e., *complexity* plays a role.

Complexity Theory: The Polynomial Hierarchy

The *polynomial hierarchy* is an infinite sequence of complexity classes: $\Sigma_1^p := \mathrm{NP}$ and Σ_i^p (for $i > 1$) is the class of problems solvable in polynomial time by a nondeterministic machine that has access to an *oracle* that decides Σ_{i-1}^p-complete problems in constant time.

Also define: $\Pi_i^p := \mathrm{co}\Sigma_i^p$ (complements).

SAT for *quantified boolean formulas* with $< i$ quantifier alternations is a *complete problem* for Σ_i^p (Π_i^p) if the first quantifier is $\exists$ ($\forall$).

We will work with Π_2^p (sometimes written $\mathrm{coNP}^{\mathrm{NP}}$). The satisfiability problem for formulas of the following type is complete for this class:

$$\forall x_1 \cdots x_r \exists y_1 \cdots y_s . \varphi(x_1, \ldots, x_r, y_1, \ldots, y_s)$$

S. Arora and B. Barak. *Computational Complexity: A Modern Approach*. Cambridge University Press, 2009.

Complexity of the Median Property

How hard is it to decide whether a given agenda will be safe for a given (class of) aggregation procedure(s)?

Recall that we have seen that Φ is safe for the majority rule iff Φ satisfies the median property. Let MP be the problem of deciding whether a given set of formulas has the median property.

Lemma 1 (Endriss et al., 2010) *Deciding* MP *is* Π_2^p*-complete.*

Next we give a proof of Π_2^p-membership and some basic intuitions regarding Π_2^p-hardness. The full proof is in the paper cited below.

There are similar results for similar agenda properties. Hence, checking safety of the agenda is typically intractable.

U. Endriss, U. Grandi and D. Porello. Complexity of Judgment Aggregation: Safety of the Agenda. Proc. AAMAS-2010.

Proof of Π_2^p-Membership

Claim: Deciding whether a set Φ has the median property is in Π_2^p.

That is: We need to show that a machine equipped with a *SAT-oracle* can, in *polynomial time*, verify the correctness of a *certificate* claiming to establish a *violation* of the median property.

Use as certificate a set $\Delta \subseteq \Phi$ with $|\Delta| > 2$ that is inconsistent but has no subset of size $\leqslant 2$ that is inconsistent.

We can verify the correctness of such a certificate using a polynomial number of queries to the SAT-oracle:

- one query to check that Δ is inconsistent
- $|\Delta|$ queries to check that each subset of size 1 is consistent
- $O(|\Delta|^2)$ queries to check that each subset of size 2 is consistent

Done. ✓

Π_2^p-Hardness

We won't give a proof, only some intuition about what SAT for QBF's of the form $\forall\exists\varphi$ has to do with properties like the median property.

- Consider this QBF:

$$\forall x_1 \cdots x_r \exists y_1 \cdots y_s . \varphi(x_1, \ldots, x_r, y_1, \ldots, y_s)$$

Now construct this agenda:

$$\Phi \quad := \quad \{x_1, \neg x_1, x_2, \neg x_2, \ldots, x_r, \neg x_r, \varphi, \neg\varphi\}$$

The QBF is unsatisfiable (i.e., false) *iff* there is a subset of Φ (incl. φ) that is inconsistent but does not include complementary formulas.

- Another way of seeing a connection between the two problems:

  The MP asks: for *all* subsets of the agenda that are inconsistent, does there *exist* a subset with a certain property?

Summary

We have seen several types of results in judgment aggregation:

- Characterising aggregation rules via axioms (cf. voting theory):
  - quota rules
  - majority rule
- Characterising agendas permitting consistent aggregation:
  - possibility theorems
  - safety theorems
  - complexity of deciding safety of the agenda

Most of these results are negative: consistent judgment aggregation tends to be possible only on structurally simplistic agendas and deciding whether a given agenda is simple enough is intractable.

Further Reading

For general background reading on judgment aggregation:

- C. List. The Theory of Judgment Aggregation: An Introductory Review. *Synthese*, 187(1):179–207, 2012.
- C. List and C. Puppe. Judgment Aggregation: A Survey. In P. Anand, P. Pattanaik and C. Puppe (eds.), *Handbook of Rational and Social Choice*, Oxford University Press, 2009.
- D. Grossi and G. Pigozzi. *Introduction to Judgment Aggregation*. Lecture Notes, 23rd European Summer School in Logic, Language and Information (ESSLLI-2011), Ljubljana, 2011.
- U. Endriss. Logic and Social Choice Theory. In A. Gupta and J. van Benthem (eds.), *Logic and Philosophy Today*, College Publications, 2011. Section 5.

What next?

The final topic of the course will be *fair division*.

We will see axiomatic results, discuss concrete division procedures, and discuss algorithmic considerations.