Computational Social Choice: Autumn 2012

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Plan for Today

Today we will discuss methods for dividing a single *divisible good* between a number of agents. In one line of research this good is referred to as the "*cake*".

Studied seriously since the 1940s (Banach, Knaster, Steinhaus). Simple model, yet still many open problems.

This lecture will be an introduction to *cake-cutting algorithms:*

- Problem definition (formal model, proportionality, envy-freeness)
- Classical procedures (cut-and-choose, Banach-Knaster, ...)
- Some open problems

This material is also covered in my lecture notes.

At the end: review of the course as a whole.

U. Endriss. *Lecture Notes on Fair Division*. ILLC, University of Amsterdam, 2010.

Cake-Cutting: Formal Framework

"Cake-cutting" is the problem of fair division of a single *divisible* (and heterogeneous) good between n *agents* (or *players*).

The *cake* is represented by the unit interval $[0, 1]$:

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|------------------------|
0                        1
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Each agent i has a *utility function* u_i (or *valuation*, *measure*) mapping finite unions of subintervals of $[0, 1]$ to the reals, satisfying:

- Non-negativity: $u_i(B) \geqslant 0$ for all $B \subseteq [0, 1]$
- Normalisation: $u_i(\emptyset) = 0$ and $u_i([0, 1]) = 1$
- Additivity: $u_i(B \cup B') = u_i(B) + u_i(B')$ for disjoint $B, B' \subseteq [0, 1]$
- u_i is continuous: the Intermediate-Value Theorem applies and single points do not have any value.

Cut-and-Choose

The classical approach for dividing a cake between *two agents:*

- *One agent cuts the cake in two pieces (she considers to be of equal value), and the other chooses one of them (the piece she prefers).*

The cut-and-choose procedure satisfies two important properties:

- *Proportionality:* Each agent is guaranteed at least one half (general: $1/n$) according to her own valuation.

  Discussion: In fact, the first agent (if she is risk-averse) will receive exactly $1/2$, while the second will usually get more.

- *Envy-freeness:* No agent will envy (any of) the other(s).

  Discussion: Actually, for two agents, proportionality and envy-freeness amount to the same thing (in this model).

What if there are more than two agents?

The Steinhaus Procedure

This procedure for *three agents* has been proposed by Steinhaus around 1943. Our exposition follows Brams and Taylor (1995).

(1) Agent 1 cuts the cake into three pieces (which she values equally).

(2) Agent 2 "passes" (if she thinks at least two of the pieces are $\geqslant 1/3$) or labels two of them as "bad". — If agent 2 passed, then agents 3, 2, 1 each choose a piece (in that order) and we are done. ✓

(3) If agent 2 did not pass, then agent 3 can also choose between passing and labelling. — If agent 3 passed, then agents 2, 3, 1 each choose a piece (in that order) and we are done. ✓

(4) If neither agent 2 or agent 3 passed, then agent 1 has to take (one of) the piece(s) labelled as "bad" by both 2 and 3. — The rest is reassembled and 2 and 3 play cut-and-choose. ✓

S.J. Brams and A.D. Taylor. An Envy-free Cake Division Protocol. *American Mathematical Monthly*, 102(1):9–18, 1995.

Properties of the Steinhaus Procedure

The Steinhaus procedure —

- Guarantees a *proportional* division of the cake (under the standard assumption that agents are risk-averse, i.e., if they want to maximise their payoff in the worst case).
- Is *not envy-free* (if agents 2 and 3 end up playing cut-and-choose, then agent 1 might envy one of them afterwards);
- Requires *at most 3 cuts* (as opposed to the minimum of 2 cuts). The resulting pieces might *not* be *contiguous* (namely if both 2 and 3 label the middle piece as "bad" and 1 takes it; and if the cut-and-choose cut is different from 1's original cut).

The Dubins-Spanier Procedure

Dubins and Spanier (1961) proposed a *proportional* procedure for *n agents*, producing *contiguous* slices.

(1) A referee moves a knife slowly across the cake, from left to right. Any agent may shout "stop" at any time. Whoever does so receives the piece to the left of the knife.

(2) When a piece has been cut off, we continue with the remaining $n-1$ agents, until just one agent is left (who takes the rest). ✓

Observe that this is also *not envy-free*.

Note that this is not a discrete "protocol" (i.e., *not an algorithm*) in the narrow sense of the word (you cannot actually implement this!).

L.E. Dubins and E.H. Spanier. How to Cut a Cake Fairly. *American Mathematical Monthly*, 68(1):1–17, 1961.

The Robertson-Webb Model

Robertson and Webb have proposed a model for cake-cutting algorithms allowing only two types of queries to agents:

- *Evaluate:* What is your value for the piece $[x, y]$?
- *Cut:* Indicate y such that your value for piece $[x, y]$ is α!

All algorithms we have seen (and will see) that do not involve a moving knife fit into this model.

Having such a model allows proving negative results ("there exists no algorithm such that ...").

J. Robertson and W. Webb. *Cake-Cutting Algorithms: Be Fair if You Can*. A.K. Peters, 1998.

A Discrete Variant of the Dubins-Spanier Procedure

We can "discretise" the Dubins-Spanier procedure as follows:

- Ask each agent to make a mark at their $1/n$ point. Cut the cake at the leftmost mark (or anywhere between the two leftmost marks) and give that piece to the respective agent.
- Continue with $n - 1$ agents, until only one is left. ✓

This removes the need for an (active) referee, can be implemented in the Robertson-Webb model, and results in contiguous pieces.

We need $n + (n-1) \cdots + 2 = \frac{n \cdot (n-1)}{2} \in O(n^2)$ *"cut" queries*.

▶ Can we do better?

The Banach-Knaster Last-Diminisher Procedure

In the first ever paper on fair division, Steinhaus (1948) reports on a *proportional* procedure for n *agents* due to Banach and Knaster.

(1) Agent 1 cuts off a piece (that she considers to represent $1/n$).

(2) That piece is passed around the agents. Each agent either lets it pass (if she considers it too small) or trims it down further (to what she considers $1/n$).

(3) After the piece has made the full round, the last agent to cut something off (the "last diminisher") is obliged to take it.

(4) The rest (including the trimmings) is then divided amongst the remaining $n-1$ agents. Play cut-and-choose once $n = 2$. ✓

Also requires $O(n^2)$ cuts (though arguably fewer cuts "on average").

May not be *contiguous* (unless you always trim "from the right").

H. Steinhaus. The Problem of Fair Division. *Econometrica*, 16:101–104, 1948.

The Even-Paz Divide-and-Conquer Procedure

Even and Paz (1984) introduced the *divide-and-conquer* procedure:

(1) Ask each agent to cut the cake at her $\lfloor \frac{n}{2} \rfloor : \lceil \frac{n}{2} \rceil$ mark.

(2) Associate the union of the leftmost $\lfloor \frac{n}{2} \rfloor$ pieces with the agents who made the leftmost $\lfloor \frac{n}{2} \rfloor$ cuts, and the rest with the others.

(3) Recursively apply the same procedure to each of the two groups, until only a single agent is left. ✓

Each agent is guaranteed a *proportional* piece. Takes $O(n \log n)$ cuts.

Woeginger and Sgall (2007) later showed that we cannot do much better: $\Omega(n \log n)$ is a lower bound on the number of queries for any proportional procedure producing contiguous pieces.

S. Even and A. Paz. A Note on Cake Cutting. *Discrete Applied Mathematics*, 7(3):285–296, 1984.

G.J. Woeginger and J. Sgall. On the Complexity of Cake Cutting. *Discrete Optimization*, 4(2):213–220, 2007.

Proportionality and Envy-Freeness

Except for cut-and-choose, we have not yet seen any procedure that would guarantee envy-freeness.

Fact 1 *Under additive preferences, a division for two agents is proportional if and only if it is envy-free.*

For $n \geqslant 3$, proportionality and envy-freeness are not the same properties anymore (unlike for $n = 2$):

Fact 2 *Any envy-free division is also proportional, but there are proportional divisions that are not envy-free.*

Envy-Free Procedures

Achieving *envy-freeness* is much harder than achieving proportionality:

- For $n = 2$ the problem is easy: cut-and-choose does the job.
- For $n = 3$ we will see two solutions. They are already quite complicated: either the number of cuts is *not minimal* (but > 2), or *several simultaneously moving knives* are required.
- For $n = 4$, to date, no procedure producing *contiguous pieces* is known. Barbanel and Brams (2004), for example, give a moving-knife procedure requiring up to 5 cuts.
- For $n \geqslant 6$, to date, only procedures requiring an *unbounded* number of cuts are known (see, e.g., Brams and Taylor, 1995).

J.B. Barbanel and S.J. Brams. Cake Division with Minimal Cuts. *Mathematical Social Sciences*, 48(3):251–269, 2004.

S.J. Brams and A.D. Taylor. An Envy-free Cake Division Protocol. *American Mathematical Monthly*, 102(1):9–18, 1995.

The Selfridge-Conway Procedure

The first discrete protocol achieving *envy-freeness* for $n = 3$ agents has been discovered independently by Selfridge and Conway (around 1960). It doesn't ensure contiguous pieces. Our exposition follows Brams and Taylor (1995).

(1) Agent 1 cuts the cake in three pieces (she considers equal).

(2) Agent 2 either "passes" (if she thinks at least two pieces are tied for largest) or trims one piece (to get two tied for largest pieces). — If she passed, then let agents 3, 2, 1 pick (in that order). ✓

(3) If agent 2 did trim, then let 3, 2, 1 pick (in that order), but require 2 to take the trimmed piece (unless 3 did). Keep the trimmings unallocated for now (note: the partial allocation is envy-free).

(4) Now divide the trimmings. Whoever of 2 and 3 received the *un*trimmed piece does the cutting. Let agents choose in this order: non-cutter, agent 1, cutter. ✓

S.J. Brams and A.D. Taylor. An Envy-free Cake Division Protocol. *American Mathematical Monthly*, 102(1):9–18, 1995.

The Stromquist Procedure

Stromquist (1980) found an *envy-free* procedure for $n = 3$ producing *contiguous* pieces, though requiring four simultaneously *moving knifes:*

- A referee slowly moves a knife across the cake, from left to right (supposed to cut somewhere around the $1/3$ mark).
- At the same time, each agent is moving her own knife so that it would cut the righthand piece in half (wrt. her own valuation).
- The first agent to call "stop" receives the piece to the left of the referee's knife. The righthand part is cut by the middle one of the three agent knifes. If neither of the other two agents hold the middle knife, they each obtain the piece at which their knife is pointing. If one of them does hold the middle knife, then the other one gets the piece at which her knife is pointing. ✓

W. Stromquist. How to Cut a Cake Fairly. *American Mathematical Monthly*, 87(8):640–644, 1980.

Summary

We have discussed various procedures for fairly dividing a cake (a metaphor for a single divisible good) amongst several agents.

- Fairness criteria: *proportionality* and *envy-freeness* (but other notions, such as equitability, Pareto efficiency, strategy-proofness . . . are also of interest)
- Distinguish *discrete* procedures (cf. Robertson-Webb model) from hypothetical *moving-knife* procedures.
- The problem becomes non-trivial for more than two agents, and there are many open problems relating to finding procedures with “good” properties for larger numbers.

S.J. Brams and A.D. Taylor. *Fair Division: From Cake-Cutting to Dispute Resolution*. Cambridge University Press, 1996.

J. Robertson and W. Webb. *Cake-Cutting Algorithms: Be Fair if You Can*. A.K. Peters, 1998.

End of Course

Models of collective decision making:

- preference aggregation and voting, fair division, judgment aggregation
- ordinal preferences, preferences as utilities, no preferences at all

Central themes regarding methodology:

- axiomatic method: fairness, characterisations, impossibilities
- complexity theory: winners, manipulation, possible winners, safety
- communication and information: compilation of election outcomes, convergence in negotiation, number of cuts
- representation: languages for modelling preferences / utility functions
- variety of applications: politics, multiagent systems, ...

Final words:

- the specific results are less important than the general approach (but do try to remember some of the classics!)
- COMSOC is a young research field with many opportunities