Computational Social Choice: Autumn 2011

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Plan for Today

Today's lecture will be devoted to classical *impossibility theorems* in social choice theory. Last week we proved *Arrow's Theorem* using the *"decisive coalition" technique*. Today we'll see two further proofs:

- A proof based on *ultrafilters* (sketch only)
- A proof using the *"pivotal voter" technique*

Then we'll see two further classical impossibility theorems:

- *Sen's Theorem* on the Impossibility of a Paretian Liberal (1970)
- The *Muller-Satterthwaite Theorem* (1977)

The former is easy to prove; for the latter we will again use the "decisive coalition" technique.

Arrow's Theorem

Recall terminology and axioms:

- SWF: $F : \mathcal{L}(\mathcal{X})^{\mathcal{N}} \to \mathcal{L}(\mathcal{X})$
- Pareto: $N^{\boldsymbol{R}}_{x \succ y} = \mathcal{N}$ implies $(x, y) \in F(\boldsymbol{R})$
- IIA: $N^{\boldsymbol{R}}_{x \succ y} = N^{\boldsymbol{R}'}_{x \succ y}$ implies $(x, y) \in F(\boldsymbol{R}) \Leftrightarrow (x, y) \in F(\boldsymbol{R}')$
- Dictatorship: $\exists i \in \mathcal{N}$ s.t. $\forall (R_1, \ldots, R_n)$: $F(R_1, \ldots, R_n) = R_i$

Here is again the theorem:

Theorem 1 (Arrow, 1951) *Any SWF for $\geqslant 3$ alternatives that satisfies the Pareto condition and IIA must be a dictatorship.*

K.J. Arrow. *Social Choice and Individual Values*. John Wiley and Sons, 2nd edition, 1963. First edition published in 1951.

Second Proof: Ultrafilters (Sketch)

Kirman and Sondermann (1972) prove Arrow's Theorem via a reduction to a well-known fact about ultrafilters.

An *ultrafilter* $\mathcal{G}$ for a set $\mathcal{N}$ is a set of subsets of $\mathcal{N}$ such that:

- $\emptyset \notin \mathcal{G}$.
- If $G_1 \in \mathcal{G}$ and $G_2 \in \mathcal{G}$, then $G_1 \cap G_2 \in \mathcal{G}$.
- For all $G \subseteq \mathcal{N}$, either $G \in \mathcal{G}$ or $(\mathcal{N} \setminus G) \in \mathcal{G}$.

$\mathcal{G}$ is called *principal* if there exists a $d \in \mathcal{N}$ s.t. $\mathcal{G} = \{G \subseteq \mathcal{N} \mid d \in G\}$.
By a known fact, every finite ultrafilter must be principal.

Let $\mathcal{N}$ be the set of individuals and $\mathcal{G}$ the set of all decisive coalitions.
Note that $\mathcal{G}$ is principal *iff* there is a dictator (namely the d generating $\mathcal{G}$).

Proving Arrow's Theorem now amounts to showing that $\mathcal{G}$ is an ultrafilter: condition $\emptyset \notin \mathcal{G}$ obviously holds; the rest is similar to last week's proof.

A.P. Kirman and D. Sondermann. Arrow's Theorem, Many Agents, and Invisible Dictators. *Journal of Economic Theory*, 5(3):267–277, 1972.

Third Proof: Pivotal Voters

Our third proof of Arrow's Theorem is due to Geanakoplos (2005). It employs the "pivotal voter" technique, introduced by Barberà (1980).

Approach:

- Let F be a SWF for $\geqslant 3$ alternatives $(x,\ y,\ z,\ \ldots)$ that satisfies the Pareto condition and IIA.
- For any given profile $(R_1, \ldots, R_n)$, let $R := F(R_1, \ldots, R_n)$. Write xRy for $(x, y) \in F(R_1, \ldots, R_n)$: society ranks $x \succ y$.

J. Geanakoplos. Three Brief Proofs of Arrow's Impossibility Theorem. *Economic Theory*, 26(1):211–215, 2005.

S. Barberà (1980). Pivotal Voters: A New Proof of Arrow's Theorem. *Economics Letters*, 6(1):13–16, 1980.

Extremal Lemma

Let y be any alternative.

Claim: For any profile in which every individual ranks y in an extremal position (either top or bottom), society must do the same.

Proof: Suppose otherwise; that is, suppose y is ranked top or bottom by every individual, but not by society.

(1) Then xRy and yRz for distinct alternatives x and z different from y and for the social preference order R.

(2) By IIA, this continues to hold if we move z above x for every individual, as doing so does not affect the extremal y.

(3) By transitivity of R, applied to (1), we get xRz.

(4) But by the Pareto condition, applied to (2), we get zRx. Contradiction. ✓

Existence of an Extremal Pivotal Individual

Fix some alternative y. We call an individual *extremal-pivotal* if there exists a profile at which it can move y from the bottom to the top of the social preference order.

Claim: There exists an extremal-pivotal individual i.

Proof: Start with a profile where every individual places y at the bottom. By the Pareto condition, so does society.

Then let the individuals change their preferences one by one, moving y from the bottom to the top.

By the Extremal Lemma and the Pareto condition, there must be a point when the change in preference of a particular individual causes y to rise from the bottom to the top in the social preference order. ✓

Convention Call the profile just before this switch occurred *Profile I*, and the one just after the switch *Profile II*.

Dictatorship: Case 1

Let i be the extremal-pivotal individual from before (for alternative y).

Claim: Individual i can dictate the social preference order with respect to any alternatives x, z different from y.

Proof: W.l.o.g., suppose i wants to place x above z.

Let *Profile III* be like *Profile II*, except that (1) i makes x its new top choice (that is, xR_iyR_iz), and (2) all the others have rearranged their relative rankings of x and z as they please. Two observations:

- In *Profile III* all relative rankings for x, y are as in *Profile I*. So by IIA, the social rankings must coincide: xRy.
- In *Profile III* all relative rankings for y, z are as in *Profile II*. So by IIA, the social rankings must coincide: yRz.

By transitivity, we get xRz. By IIA, this continues to hold if others change their relative ranking of alternatives other than x, z. ✓

Dictatorship: Case 2

Let y and i be defined as before.

Claim: Individual i can also dictate the social preference order with respect to y and any other alternative x.

Proof: We can use a similar construction as before to show that for a given alternative z, there must be an individual j that can dictate the relative social ranking of x and y (both different from z).

But at least in *Profiles I* and *II*, i can dictate the relative social ranking of x and y. As there can be at most one dictator in any situation, we get $i = j$. ✓

So individual i will be a *dictator* for *any* two alternatives.
Hence, our SWF must be dictatorial, and Arrow's Theorem follows.

Other Proofs

- Nipkow (2009) has encoded Geanakoplos' proof in the language of the higher-order logic *proof assistant* ISABELLE, resulting in an automatic verification of the proof.
- We will discuss further approaches to proving Arrow's Theorem using tools from *automated reasoning* later on in the course.

T. Nipkow. Social Choice Theory in HOL: Arrow and Gibbard-Satterthwaite. *Journal of Automated Reasoning*, 43(3):289–304, 2009.

Social Choice Functions

From now on we consider aggregators that take a profile of preferences and return one or several "winners" (rather than a full social ranking). This is called a *social choice function* (SCF):

$$F : \mathcal{L}(\mathcal{X})^{\mathcal{N}} \to 2^{\mathcal{X}} \setminus \{\emptyset\}$$

A SCF is called *resolute* if $|F(\boldsymbol{R})| = 1$ for any given profile $\boldsymbol{R}$, i.e., if it always selects a unique winner.

Remark: We can think of a SCF as a *voting rule*, particularly if it tends to select "small" sets of winners (we won't make this precise). Voting rules are often required to be resolute ($\leadsto$ *tie-breaking rule*).

Alternative Definition

In the literature you will sometimes find the term SCF being used for functions $F : \mathcal{L}(X)^{\mathcal{N}} \times 2^{\mathcal{X}} \setminus \{\emptyset\} \to 2^{\mathcal{X}} \setminus \{\emptyset\}$. Two readings:

- The input of F is a profile of preferences (as before) + a set of *feasible alternatives*. The output should be a subset of the feasible alternatives (that is "appropriate" given the preference profile).
- The input of F is just a profile of preferences (as before). The output is a *choice function* $C : 2^{\mathcal{X}} \setminus \{\emptyset\} \to 2^{\mathcal{X}} \setminus \{\emptyset\}$ that will select a set of winners from any given set of alternatives.

  Note: $\mathcal{L}(X)^{\mathcal{N}} \times 2^{\mathcal{X}} \setminus \{\emptyset\} \to 2^{\mathcal{X}} \setminus \{\emptyset\} = \mathcal{L}(X)^{\mathcal{N}} \to (2^{\mathcal{X}} \setminus \{\emptyset\} \to 2^{\mathcal{X}} \setminus \{\emptyset\})$

This refinement is not relevant for the results we want to discuss here, so we shall take a SCF to be a function $F : \mathcal{L}(\mathcal{X})^{\mathcal{N}} \to 2^{\mathcal{X}} \setminus \{\emptyset\}$.

Examples

The *plurality rule* and the *Borda rule* (defined last week) are both examples for voting rules (i.e., for SCFs). A few more examples:

- *Positional scoring rules:* Fix a (decreasing) *scoring vector* $\langle s_1, \ldots, s_m \rangle$. An alternative gets s_k points for every voter placing her at position k. Special cases: *Borda*: $\langle m{-}1, m{-}2, \ldots, 0\rangle$; *Plurality*: $\langle 1, 0, \ldots, 0\rangle$
- *Plurality with runoff:* Each voter initially votes for one alternative. The winner is elected in a second round by using the plurality rule with the two top alternatives from the first round.
- *Condorcet:* An alternative that beats every other alternative in pairwise majority contests is called a *Condorcet winner*. Rule: elect the Condorcet winner if it exists; otherwise elect all alternatives.
- *Copeland:* Run a majority contest for every pair of alternatives. Award $+1$ point to an alternative for every contest won, and -1 for any contest lost. The alternative with the most points wins.

Note that none of these voting rules is resolute.

The Pareto Condition for Social Choice Functions

A SCF F satisfies the *Pareto condition* if, whenever all individuals rank x above y, then y cannot win:

$$N^{\boldsymbol{R}}_{x \succ y} = \mathcal{N} \text{ implies } y \notin F(\boldsymbol{R})$$

Liberalism

Think of $\mathcal{X}$ as the set of all possible social states. Certain aspects of such a state will be some individual's private business. Example:

> If x and y are identical states, except that in x I paint my bedroom white, while in y I paint it pink, then I should be able to dictate the relative social ranking of x and y.

Sen (1970) proposed the following axiom:

A SCF F satisfies the axiom of *liberalism* if, for every individual $i \in \mathcal{N}$, there exist two distinct alternatives $x, y \in \mathcal{X}$ such that i is *two-way decisive* on x and y:

$$i \in N^{\boldsymbol{R}}_{x \succ y} \text{ implies } y \notin F(\boldsymbol{R}) \text{ and } i \in N^{\boldsymbol{R}}_{y \succ x} \text{ implies } x \notin F(\boldsymbol{R})$$

A.K. Sen. The Impossibility of a Paretian Liberal. *Journal of Political Economics*, 78(1):152–157, 1970.

The Impossibility of a Paretian Liberal

Sen (1970) showed that liberalism and the Pareto condition are incompatible (recall that we required $|\mathcal{N}| \geqslant 2$, which matters here):

Theorem 2 (Sen, 1970) *No SCF satisfies both liberalism and the Pareto condition.*

As we shall see, the theorem holds even when liberalism is enforced for only two individuals. The number of alternatives does not matter.

Again, a surprising result (but easier to prove than Arrow's Theorem).

A.K. Sen. The Impossibility of a Paretian Liberal. *Journal of Political Economics*, 78(1):152–157, 1970.

Proof

Let F be a SCF satisfying Pareto and liberalism. Get a contradiction:

Take two distinguished individuals i_1 and i_2, with:

- i_1 is two-way decisive on x_1 and y_1
- i_2 is two-way decisive on x_2 and y_2

Assume x_1, y_1, x_2, y_2 are pairwise distinct (other cases: easy).

Consider a profile with these properties:

(1) Individual i_1 ranks $x_1 \succ y_1$.
(2) Individual i_2 ranks $x_2 \succ y_2$.
(3) All individuals rank $y_1 \succ x_2$ and $y_2 \succ x_1$.
(4) All individuals rank x_1, x_2, y_1, y_2 above all other alternatives.

From liberalism: (1) rules out y_1 and (2) rules out y_2 as winner.
From Pareto: (3) rules out x_1 and x_2 and (4) rules out all others.

Thus, there are no winners. Contradiction. ✓

Monotonicity

Next we want to formalise the idea that when a winner receives increased support, she should not become a loser.

We focus on *resolute* SCFs. Write $x^\star = F(\boldsymbol{R})$ for $\{x^\star\} = F(\boldsymbol{R})$.

- *Weak monotonicity:* F is weakly monotonic if $x^\star = F(\boldsymbol{R})$ implies $x^\star = F(\boldsymbol{R}')$ for any alternative $x^\star$ and distinct profiles $\boldsymbol{R}$ and $\boldsymbol{R}'$ with $N^{\boldsymbol{R}}_{x^\star \succ y} \subseteq N^{\boldsymbol{R}'}_{x^\star \succ y}$ and $N^{\boldsymbol{R}}_{y \succ z} = N^{\boldsymbol{R}'}_{y \succ z}$ for all $y, z \in \mathcal{X} \setminus \{x^\star\}$.
- *Strong monotonicity:* F is strongly monotonic if $x^\star = F(\boldsymbol{R})$ implies $x^\star = F(\boldsymbol{R}')$ for any alternative $x^\star$ and distinct profiles $\boldsymbol{R}$ and $\boldsymbol{R}'$ with $N^{\boldsymbol{R}}_{x^\star \succ y} \subseteq N^{\boldsymbol{R}'}_{x^\star \succ y}$ for all $y \in \mathcal{X} \setminus \{x^\star\}$.

The latter property is also known as *Maskin monotonicity* or *strong positive association*.

Example

Even *weak monotonicity* is not satisfied by some common voting rules. Consider *plurality with runoff* (with any tie-breaking rule).

27 voters:	$A \succ B \succ C$
42 voters:	$C \succ A \succ B$
24 voters:	$B \succ C \succ A$

B is eliminated in the first round and C beats A 66:27 in the runoff. But if 4 of the voters in the first group *raise C to the top* (i.e., join the second group), then B will win.

But other procedures (e.g., *plurality*) do satisfy weak monotonicity. How about *strong monotonicity*?

The Muller-Satterthwaite Theorem

Strong monotonicity turns out to be (desirable but) too demanding:

Theorem 3 (Muller and Satterthwaite, 1977) *Any resolute SCF for $\geqslant 3$ alternatives that is surjective and strongly monotonic must be a dictatorship.*

Here, a resolute SCF F is called *surjective* if for every alternative $x \in \mathcal{X}$ there exists a profile $\boldsymbol{R}$ such that $F(\boldsymbol{R}) = x$.

And: a SCF F is a *dictatorship* if there exists an $i \in \mathcal{N}$ such that $F(R_1, \ldots, R_n) = \mathrm{top}(R_i)$ for every profile $(R_1, \ldots, R_n)$.

Remark: Above theorem, which is what is nowadays usually referred to as the Muller-Satterthwaite Theorem, is in fact a corollary of their main theorem and the Gibbard-Satterthwaite Theorem.

E. Muller and M.A. Satterthwaite. The Equivalence of Strong Positive Association and Strategy-Proofness. *Journal of Economic Theory*, 14(2):412–418, 1977.

Proof

We use again the "decisive coalition" technique. Full details are available in the review paper cited below.

Claim: *Any resolute SCF for $\geqslant 3$ alternatives that is surjective and strongly monotonic must be a dictatorship.*

Let F be a SCF for $\geqslant 3$ alt. that is surjective and strongly monotonic.

Proof Plan:

- Show that F must be *independent* (to be defined).
- Show that F must be *Pareto* efficient.
- Prove a version of Arrow's Theorem for SCFs.

U. Endriss. Logic and Social Choice Theory. In J. van Benthem and A. Gupta (eds.), *Logic and Philosophy Today*, College Publications. In press (2011).

Independence

Call a SCF F *independent* if it is the case that $x \neq y$, $F(\boldsymbol{R}) = x$, and $N^{\boldsymbol{R}}_{x \succ y} = N^{\boldsymbol{R}'}_{x \succ y}$ together imply $F(\boldsymbol{R}') \neq y$.

That is, if y lost to x under profile $\boldsymbol{R}$, and the relative rankings of x vs. y do not change, then y will still lose (possibly to a different winner).

Claim: F is independent.

Proof: Suppose $x \neq y$, $F(\boldsymbol{R}) = x$, and $N^{\boldsymbol{R}}_{x \succ y} = N^{\boldsymbol{R}'}_{x \succ y}$.

Construct a third profile $\boldsymbol{R}''$:

- All individuals rank x and y in the top-two positions.
- The relative rankings of x vs. y are as in $\boldsymbol{R}$, i.e., $N^{\boldsymbol{R}''}_{x \succ y} = N^{\boldsymbol{R}}_{x \succ y}$.
- Rest: whatever

By strong monotonicity, $F(\boldsymbol{R}) = x$ implies $F(\boldsymbol{R}'') = x$.
By strong monotonicity, $F(\boldsymbol{R}') = y$ *would* imply $F(\boldsymbol{R}'') = y$.
Thus, we must have $F(\boldsymbol{R}') \neq y$. ✓

Pareto Condition

Recall the Pareto condition: if everyone ranks $x \succ y$, then y won't win.

Claim: F satisfies the Pareto condition.

Proof: Take any two alternatives x and y.

From surjectivity: x will win for *some* profile $\boldsymbol{R}$.

Starting in $\boldsymbol{R}$, have everyone move x above y (if not above already).
From strong monotonicity: x still wins.

From independence: y does not win for *any* profile where all individuals continue to rank $x \succ y$. ✓

Plan for the Rest of the Proof

We now know that F must be a SCF for $\geqslant 3$ alternatives that is *independent* and *Pareto* efficient. We want to infer that F must be a *dictatorship*.

Call a coalition $G \subseteq \mathcal{N}$ *decisive* on (x, y) iff $G \subseteq N^{\boldsymbol{R}}_{x \succ y} \Rightarrow y \neq F(\boldsymbol{R})$.

Proof plan:

- Pareto condition $=$ $\mathcal{N}$ is decisive for all pairs of alternatives
- Lemma: G with $|G| \geqslant 2$ *decisive* for all pairs $\Rightarrow$ some $G' \subset G$ as well
- Thus (by induction), there's a decisive coalition of size 1 (a *dictator*).

About Decisiveness

Recall: $G \subseteq \mathcal{N}$ *decisive* on (x,y) iff $G \subseteq N^{\boldsymbol{R}}_{x \succ y} \;\Rightarrow\; y \neq F(\boldsymbol{R})$

Call $G \subseteq \mathcal{N}$ *weakly decisive* on (x,y) iff $G = N^{\boldsymbol{R}}_{x \succ y} \;\Rightarrow\; y \neq F(\boldsymbol{R})$.

Claim: G weakly decisive on $(x,y) \;\Rightarrow\; G$ decisive on *any* pair (x',y')

Proof: Suppose x, y, x', y' are all distinct (other cases: similar).

Consider a profile where individuals express these preferences:

- Members of G: $x' \succ x \succ y \succ y'$
- Others: $x' \succ x$, $y \succ y'$, and $y \succ x$ (note that x'-vs.-y' is not specified)
- All rank x, y, x', y' above all other alternatives.

From G being weakly decisive for $(x,y) \Rightarrow y$ must lose
From Pareto $\Rightarrow x$ must lose (to x') and y' must lose (to y)

Thus, x' must win (and y' must lose). By independence, y' will still lose when everyone changes their non-x'-vs.-y' rankings.

Thus, for *any* profile $\boldsymbol{R}$ with $G \subseteq N^{\boldsymbol{R}}_{x' \succ y'}$ we get $y' \neq F(\boldsymbol{R})$. ✓

Contraction Lemma

Claim: If $G \subseteq \mathcal{N}$ with $|G| \geqslant 2$ is a coalition that is decisive on all pairs of alternatives, then so is some nonempty coalition $G' \subset G$.

Proof: Take any nonempty G_1, G_2 with $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \emptyset$.

Recall that there are $\geqslant 3$ alternatives. Consider this profile:

- Members of G_1: $x \succ y \succ z \succ$ *rest*
- Members of G_2: $y \succ z \succ x \succ$ *rest*
- Others: $z \succ x \succ y \succ$ *rest*

As $G = G_1 \cup G_2$ is decisive, z cannot win (it loses to y). Two cases:

(1) The winner is x: Exactly G_1 ranks $x \succ z \Rightarrow$ By independence, in any profile where exactly G_1 ranks $x \succ z$, z will lose (to x) $\Rightarrow G_1$ is weakly decisive on (x,z). Hence (previous slide): G_1 is decisive on all pairs.

(2) The winner is y, i.e., x loses (to y). Exactly G_2 ranks $y \succ x \Rightarrow \cdots \Rightarrow$ G_2 is decisive on all pairs.

Hence, one of G_1 and G_2 will always be decisive. ✓

Summary

We have by now see three important impossibility theorems, establishing the incompatibility of certain desirable properties:

- *Arrow:* Pareto, IIA, nondictatoriality
- *Sen:* Pareto, liberalism
- *Muller-Satterthwaite:* surjectivity, strong monotonicity, nondictat.

We have discussed these results in two formal frameworks (none of the results heavily depend on the choice of framework):

- social welfare functions (*SWF*)
- (resolute) social choice functions (*SCF*)

This has also been an introduction to the *axiomatic method:*

- formulate desirable properties of aggregators as axioms
- explore the consequences of imposing several such axioms

What next?

As discussed, the impossibility theorems we have seen can also be interpreted as axiomatic characterisations of the class of dictatorships.

Next week we will see *characterisations* of more attractive (classes of) voting rules:

- using (again) the axiomatic method; and
- using different methods.