Homework #4

Deadline: Wednesday, 2 November 2011, 11:00

Question 1 (10 marks)

In the context of the formal framework for *ranking sets of objects*, we have seen the axioms making up the preference extension principles proposed by Kelly and Gärdenfors, as well as the independence axiom due to Kannai and Peleg. Consider the following two additional axioms. The first is a weak form of the independence axiom:

$$A \cup \{c\} \mathrel{\hat{\succeq}} B \cup \{c\} \text{ if } A \mathrel{\hat{\succ}} B \text{ and either } c \succ \max(A \cup B) \text{ or } \min(A \cup B) \succ c$$

That is, adding object c to both of two sets A and B will only guarantee not to invert the preference relation between those sets if c is either strictly better or strictly worse than any of the elements of those sets (the original independence axiom only requires c to be different from any of the elements of those sets). The second additional axiom is the following:

$$\{a, c\} \mathrel{\hat{\succ}} \{b\} \text{ if } a \succ b \text{ and } b \succ c$$

This is appropriate for a decision maker who is willing to take risks. Recall that the axioms presented in class did not allow us to rank $\{a, c\}$ and $\{b\}$.

Does the impossibility flagged by the Kannai-Peleg Theorem persist if we weaken the independence axiom as indicated above and additionally assume that our decision maker is willing to take risks in the above sense? Give a proof or a counterexample. (Addendum: There is a mistake in the statement of this question. Inspection of the proof of the Kannai-Peleg Theorem shows that the impossibility persists when we weaken independence as indicated, even without imposing the additional risk-related axiom. The intended task was to check what happens for smaller domain sizes, particularly size 4.)

Question 2 (10 marks)

A social welfare function $F : \mathcal{L}(\mathcal{X})^{\mathcal{N}} \to \mathcal{L}(\mathcal{X})$ is said to be *weakly monotonic* if $(x^\star, x) \in F(\boldsymbol{R})$ implies $(x^\star, x) \in F(\boldsymbol{R'})$ for any two alternatives $x^\star$ and x and any distinct profiles $\boldsymbol{R}$ and $\boldsymbol{R'}$ with $N^{\boldsymbol{R}}_{x^\star \succ y} \subseteq N^{\boldsymbol{R'}}_{x^\star \succ y}$ and $N^{\boldsymbol{R}}_{y \succ z} = N^{\boldsymbol{R'}}_{y \succ z}$ for all alternatives $y, z \in \mathcal{X} \setminus \{x^\star\}$. That is, if some individuals raise $x^\star$ in their ranking and there are no other changes, then $x^\star$ should still beat any opponent that it was able to beat before being raised. Now recall the two main approaches to embedding the Arrovian framework of social welfare functions into a logical system discussed in class:

(a) using the modal logic proposed by Ågotnes, van der Hoek, and Wooldridge; and

(b) using classical first-order logic, as proposed by Grandi and Endriss.

For each one of these approaches, either show how to model weak monotonicity (and explain your solution), or argue why weak monotonicity cannot be modelled adequately using the logic in question. Use the notation from the lecture slides wherever possible.