Computational Social Choice: Autumn 2010

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Plan for Today

We will introduce a few (more) *voting procedures* and study some of their *properties*, including these:

- a few intuitive advantages and disadvantages
- some social choice-theoretic properties, most importantly the *Condorcet principle*
- the *computational complexity* of the problem of determining the *winner* of an election

This discussion will give some initial guidelines for choosing a suitable voting procedure for a specific situation at hand (a very difficult problem that we won't fully resolve).

Voting Procedures

We'll discuss procedures for n *voters* (or *individuals*, *agents*, *players*) to collectively choose from a set of m *alternatives* (or *candidates*):

- Each voter votes by submitting a *ballot*, e.g., the name of a single alternative, a ranking of all alternatives, or something else.
- The procedure defines what are *valid ballots*, and how to *aggregate* the ballot information to obtain a winner.

Remark 1: There could be *ties*. So our voting procedures will actually produce *sets of winners*. Tie-breaking is a separate issue.

Remark 2: Formally, *voting rules* (or *resolute* voting procedures) return single winners; *voting correspondences* return sets of winners.

Many Voting Procedures

There are vast variety of voting procedures around. Many, not all, of them are defined in the survey paper by Brams and Fishburn (2002).

> Positional Scoring Rules, including Plurality, Borda, Antiplurality/Veto, and k-approval; Plurality with Runoff; Single Transferable Vote (STV)/Hare; Approval Voting; Condorcet-consistent methods based on the simple majority graph (e.g., Cup Rule/Voting Trees, Copeland, Banks, Slater, Schwartz, and the basic Condorcet rule itself), based on the weighted majority graph (e.g., Maximin/Simpson, Kemeny, and Ranked Pairs/Tideman), or requiring full ballot information (e.g., Bucklin, Dodgson, and Young); Majoritarian Judgment; Cumulative Voting; Range Voting.

S.J. Brams and P.C. Fishburn. Voting Procedures. In K.J. Arrow et al. (eds.), *Handbook of Social Choice and Welfare*, Elsevier, 2002.

Plurality Rule

Under the *plurality rule* each voter submits a ballot showing the name of one alternative. The alternative(s) receiving the most votes win(s).

Remarks:

- Also known as the *simple majority* rule ($\neq$ *absolute majority* rule).
- This is the most widely used voting procedure in practice.
- If there are only two alternatives, then it is a very good procedure.
- The information on voter preferences other than who their favourite candidate is gets ignored.
- Dispersion of votes across ideologically similar candidates.
- Encourages voters not to vote for their true favourite, if that candidate is perceived to have little chance of winning.

Plurality with Run-Off

Under the *plurality rule with run-off*, each voter initially votes for one alternative. The winner is elected in a second round by using the plurality rule with the two top alternatives from the first round.

Remarks:

- Used to elect the president in France.
- Addresses some of the noted problems: elicits more information from voters; realistic "second best" candidate gets another chance.
- Still: heavily criticised after Le Pen entered the run-off in 2002.

The No-Show Paradox

Under plurality with run-off, it may be better to abstain than to vote for your favourite candidate! Example:

25 voters: $A \succ B \succ C$
46 voters: $C \succ A \succ B$
24 voters: $B \succ C \succ A$

Given these voter preferences, B gets eliminated in the first round, and C beats A 70:25 in the run-off.

Now suppose two voters from the first group abstain:

23 voters: $A \succ B \succ C$
46 voters: $C \succ A \succ B$
24 voters: $B \succ C \succ A$

A gets eliminated, and B beats C 47:46 in the run-off.

P.C. Fishburn and S.J Brams. Paradoxes of Preferential Voting. *Mathematics Magazine*, 56(4):207-214, 1983.

Single Transferable Vote (STV)

Also known as the *Hare system*. Voters rank the candidates. Then:

- If one of the candidates is the 1st choice for over 50% of the voters (*quota*), she wins.
- Otherwise, the candidate who is ranked 1st by the fewest voters gets *eliminated* from the race.
- Votes for eliminated candidates get *transferred:* delete removed candidates from ballots and "shift" rankings (i.e., if your 1st choice got eliminated, then your 2nd choice becomes 1st).

In practice, voters need not be required to rank all candidates (non-ranked candidates are assumed to be ranked lowest).

STV is used in several countries (e.g., Australia, New Zealand, . . .).

For three candidates, STV and Plurality with Run-off coincide.

Borda Rule

Under the voting procedure proposed by Jean-Charles de Borda, each voter submits a complete ranking of all m candidates.

For each voter that places a candidate first, that candidate receives $m-1$ points, for each voter that places her 2nd she receives $m-2$ points, and so forth. The *Borda count* is the sum of all the points.

The candidates with the highest Borda count win.

Remarks:

- Takes care of some of the problems identified for plurality voting, e.g., this form of balloting is more informative.
- Disadvantage (of any system where voters submit full rankings): higher elicitation and communication costs

J.-C. de Borda. *Mémoire sur les élections au scrutin*. Histoire de l'Académie Royale des Sciences, Paris, 1781.

Example

Consider (again) this example:

49%:	Bush $\succ$ Gore $\succ$ Nader
20%:	Gore $\succ$ Nader $\succ$ Bush
20%:	Gore $\succ$ Bush $\succ$ Nader
11%:	Nader $\succ$ Gore $\succ$ Bush

Our voting procedures give different winners:

- Plurality: Bush wins
- Plurality with run-off: Gore wins (Nader eliminated in round 1)
- Borda: Gore wins ($49 + 40 + 40 + 11 > 98 + 20 > 20 + 22$)
- Gore is also the *Condorcet winner* (wins any pairwise contest).

Positional Scoring Rules

We can generalise the idea underlying the Borda rule as follows:

A *positional scoring rule* is given by a *scoring vector* $s = \langle s_1, \ldots, s_m \rangle$ with $s_1 \geq s_2 \geq \cdots \geq s_m$ and $s_1 > s_m$.

Each voter submits a ranking of the m alternatives. Each alternative receives s_i points for every voter putting it at the ith position.

The alternatives with the highest score (sum of points) win.

Remarks:

- The *Borda rule* is is the positional scoring rule with the scoring vector $\langle m-1, m-2, \ldots, 0 \rangle$.
- The *plurality rule* is the positional scoring rule with the scoring vector $\langle 1, 0, \ldots, 0 \rangle$.
- The *antiplurality* or *veto rule* is the positional scoring rule with the scoring vector $\langle 1, \ldots, 1, 0 \rangle$.

The Condorcet Principle

An alternative that beats every other alternative in pairwise majority contests is called a *Condorcet winner*.

There may be no Condorcet winner; witness the *Condorcet paradox:*

Ann:	$A \succ B \succ C$
Bob:	$B \succ C \succ A$
Cesar:	$C \succ A \succ B$

Whenever a Condorcet winner exists, then it must be *unique*.

A voting procedure satisfies the *Condorcet principle* if it elects (only) the Condorcet winner whenever one exists.

M. le Marquis de Condorcet. *Essai sur l'application de l'analyse à la probabilté des décisions rendues a la pluralité des voix*. Paris, 1785.

Positional Scoring Rules violate Condorcet

Consider the following example:

$$
\begin{array}{ll}
\text{3 voters:} & A \succ B \succ C \\
\text{2 voters:} & B \succ C \succ A \\
\text{1 voter:} & B \succ A \succ C \\
\text{1 voter:} & C \succ A \succ B
\end{array}
$$

A is the *Condorcet winner*; she beats both B and C $4:3$. But any *positional scoring rule* assigning strictly more points to a candidate placed 2nd than to a candidate placed 3rd ($s_2 > s_3$) makes B win:

$$
\begin{array}{ll}
A\text{:} & 3 \cdot s_1 + 2 \cdot s_2 + 2 \cdot s_3 \\
B\text{:} & 3 \cdot s_1 + 3 \cdot s_2 + 1 \cdot s_3 \\
C\text{:} & 1 \cdot s_1 + 2 \cdot s_2 + 4 \cdot s_3
\end{array}
$$

Thus, *no positional scoring rule* for three (or more) alternatives will satisfy the *Condorcet principle*.

The Banks Rule

Let $\mathcal{X}$ be the set of alternatives. Define the *majority graph* $(\mathcal{X}, \succ_M)$:

$$x \succ_M y \text{ \textit{iff} a strict majority of voters rank } x \text{ above } y$$

If $(\mathcal{X}, \succ_M)$ is complete, then it is called a *tournament*. That is, if the number n of voters is odd, then $(\mathcal{X}, \succ_M)$ is a tournament.

Under the *Banks rule*, a candidate x is a winner if it is a top element in a maximal acyclic subgraph of the majority graph.

Fact: The Banks rule respects the Condorcet principle.

J.S. Banks. Sophisticated Voting Outcomes and Agenda Control. *Social Choice and Welfare*, 1(4)295–306, 1985.

Complexity of Winner Determination: Banks Rule

A desirable property of any voting procedure would be that it should be easy (computationally tractable) to compute the winner(s).

For the Banks rule, we formulate the problem wrt. the majority graph (which we can compute in polynomial time given the ballot profile):

> BANKS-WINNER
> **Instance:** majority graph $G = (\mathcal{X}, \succ_M)$ and alternative $x^\star \in \mathcal{X}$
> **Question:** Is $x^\star$ a Banks winner for G?

Unfortunately, recognising Banks winners is intractable:

Theorem 1 (Woeginger, 2003) BANKS-WINNER *is NP-complete.*

Proof: NP-membership: certificate = maximal acyclic subgraph
NP-hardness: reduction from GRAPH 3-COLOURING (see paper). ✓

G.J. Woeginger. Banks Winners in Tournaments are Difficult to Recognize. *Social Choice and Welfare*, 20(3)523–528, 2003.

Easiness of Computing Some Winner

We have seen that checking whether x is a Banks winner is NP-hard. So computing *all* Banks winners is also NP-hard.

But computing just *some* Banks winner is easy! Algorithm:

(1) Let $S := \{x_1\}$ and $i := 1$. (candidates $\mathcal{X} = \{x_1, \ldots, x_m\}$)

(2) While $i < m$, repeat:

- Let $i := i + 1$.
- If the majority graph restricted to $S \cup \{x_i\}$ is acyclic, then let $S := S \cup \{x_i\}$.

(3) Return the top element in S (it is a Banks winner).

This algorithm has complexity $O(m^2)$ if given the majority graph, which in turn can be constructed in time $O(n \cdot m^2)$.

O. Hudry. A Note on "Banks Winners in Tournaments are Difficult to Recognize" by G.J. Woeginger. *Social Choice and Welfare*, 23(1):113–114, 2004.

Complexity of Winner Determination

Bartholdi et al. (1989) were the first to study the complexity of computing election winners. They showed that checking whether a candidate's Dodgson score exceeds K is NP-complete. Other results include:

- Checking whether a candidate is a Dodgson winner it is *complete for parallel access to NP*. There are similar results for the Kemeny rule. Other hard voting procedures include those of Young and Slater. Consult the COMSOC Survey (Chevaleyre et al., 2007) for references.
- More recent work has also analysed the *parametrised complexity* of computing election winners (e.g., Betzler et al., 2010).

J.J. Bartholdi III, C.A. Tovey, and M.A. Trick. Voting schemes for which it can be difficult to tell who won the election. *Soc. Choice Welf.*, 6(2):157–165, 1989.

Y. Chevaleyre, U. Endriss, J. Lang, and N. Maudet. *A Short Introduction to Computational Social Choice*. Proc. SOFSEM-2007.

N. Betzler, J. Guo and R. Niedermeier. Parameterized Computational Complexity of Dodgson and Young Elections. *Inf. Comput.*, 208(2):165–177, 2010.

Summary

We have seen several examples for voting procedures:

- the *positional scoring rules*, including Borda, Plurality and Veto
- two *staged procedures:* STV and Plurality with Run-off
- a *Condorcet-consistent* rule based on the *majority graph:* Banks

Helpful references for these and other voting procedures are the works of Brams and Fishburn (2002) and Nurmi (1987).

We have also discussed three important properties:

- *Participation:* a voting procedure should not give incentives not to vote (i.e., it should not suffer from the *no-show paradox*)
- *Condorcet principle:* elect the Condorcet winner whenever it exists
- *Complexity of winner determination:* computing the winner(s) of an election should be computationally tractable

S.J. Brams and P.C. Fishburn. Voting Procedures. In K.J. Arrow et al. (eds.), *Handbook of Social Choice and Welfare*, Elsevier, 2002.

H. Nurmi. *Comparing Voting Systems*. Kluwer Academic Publishers, 1987.

What next?

We have already seen some indications that it might be difficult to satisfy all of the desirable properties we can think of.

Indeed, some of the most important results in social choice theory establish the impossibility of certain combinations of desiderata.

Next week will be devoted to:

- an introduction to the so-called *axiomatic method*, which has been developed to give a precise account of what is and what is not possible in social choice; and
- a discussion of some fundamental *impossibility theorems*, including in particular *Arrow's Theorem*.