Computational Social Choice: Spring 2009
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Plan for Today
Much of social choice theory is about the problems associated with aggregating preferences (linear orders, utility functions, ...). Today we will look into the problem of aggregating judgements: truth assignments to logically interconnected propositions.

- Doctrinal Paradox: a first example demonstrating that JA is difficult (inspired by work in Law and Economics)
- Impossibility Result: a set of reasonable axioms and a theorem showing that there can be no JA procedure satisfying them all
- Conditions under which we can circumvent impossibilities
- Procedures for JA, each satisfying a subset of the axioms

Much of this lecture is based on the tutorial paper by List (2008).


The Doctrinal Paradox
Take a court with three judges. Suppose legal doctrine stipulates that the defendant is liable (C) iff there has been a valid contract (A) and that contract has been breached (B): C ↔ A ∧ B.

\[
\begin{array}{ccc}
A & B & C \\
\text{Judge 1:} & \text{yes} & \text{yes} & \text{yes} \\
\text{Judge 2:} & \text{no} & \text{yes} & \text{no} \\
\text{Judge 3:} & \text{yes} & \text{no} & \text{no} \\
\text{Majority:} & \text{yes} & \text{yes} & \text{no}
\end{array}
\]

Paradox: taking majority decisions issue-by-issue, here A and B, (and deciding on the case C accordingly) gives a different result from taking majority decisions case-by-case (that is, on C directly)


Variants of the Paradox
In the example, individuals were expressing judgements on atomic propositions (A, B, C) and consistency of a judgement set was evaluated wrt. a background theory (C ↔ A ∧ B).

Alternatively, we could allow judgements directly on compound formulas. And we could make the legal doctrine itself a proposition on which individuals can express a judgement.

\[
\begin{array}{ccc}
A & B & A \land B \\
\text{Judge 1:} & \text{yes} & \text{yes} & \text{yes} \\
\text{Judge 2:} & \text{no} & \text{yes} & \text{no} \\
\text{Judge 3:} & \text{yes} & \text{no} & \text{no} \\
\text{Majority:} & \text{yes} & \text{yes} & \text{no}
\end{array}
\]

\[
\begin{array}{cccc}
A & B & C & A \land B \\
\text{Judge 1:} & \text{yes} & \text{yes} & \text{yes} & \text{yes} \\
\text{Judge 2:} & \text{no} & \text{yes} & \text{no} & \text{no} \\
\text{Judge 3:} & \text{yes} & \text{no} & \text{yes} & \text{no} \\
\text{Majority:} & \text{yes} & \text{yes} & \text{yes} & \text{no}
\end{array}
\]

Conclusion: We do not require the notion of a background theory (doctrine) to model the full extent of the problem.
Judgement Aggregation: The Model

- Finite set of variables $PS$, propositional language $L_{PS}$ over $PS$
- An agenda is a (finite) set of formulas $\Phi \subseteq L_{PS}$ that is closed under complementation.
- Judgement set: subset $J \subseteq \Phi$ of formulas in the agenda
  - consistent: if $J \not\models \bot$
  - complete: if for each proposition $\varphi \in \Phi$, $\varphi \in J$ or $\neg \varphi \in J$
- Finite set of (at least two) individuals $I = \{1, \ldots, n\}$, each with a (usually consistent and complete) judgement set
- A judgement aggregation rule is a function mapping each profile of individual judgement sets to a collective judgement set.

Axioms

Possible choices of axioms for judgement aggregation include:

- Universal Domain (UD): the rule should be defined for any profile of consistent and complete judgement sets
- Anonymity (AN): symmetry wrt. individuals
- Neutrality (NE): symmetry wrt. elements of the agenda
- Independence (IN): inclusion of a proposition $\varphi$ (of the agenda) into the collective judgement set should depend solely on (non-)inclusion of $\varphi$ in the individual judgement sets

Independence + neutrality is also known as systematicity.

Preference vs Judgement Aggregation

Naturally, there are close links between PA and JA.
One can (and people do) argue over which is more general . . .

For example, we can model the Condorcet Paradox in JA:

<table>
<thead>
<tr>
<th></th>
<th>$A \succ B$</th>
<th>$A \succ C$</th>
<th>$B \succ C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agent 1:</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>Agent 2:</td>
<td>no</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>Agent 3:</td>
<td>yes</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>Majority:  yes</td>
<td>no</td>
<td>yes</td>
<td>[not a linear order]</td>
</tr>
</tbody>
</table>

And all agents agree on these propositions:

- $\neg [A \succ A], \neg [B \succ B], \neg [C \succ C]$
- $[A \succ B] \lor [B \succ A], [A \succ C] \lor [C \succ A], [B \succ C] \lor [C \succ B]$
- $[A \succ B] \land [B \succ C] \rightarrow [A \succ C]$, etc.

Impossibility Theorem

The original impossibility theorem for judgement aggregation:

**Theorem 1 (List and Pettit, 2002)** If the agenda contains at least $P$, $Q$, and $P \land Q$, then no rule producing consistent and complete judgement sets satisfies (UD), (AN), (NE), and (IN).

**Remark:** The theorem also holds for other sufficiently complex agendas, e.g., any agenda containing at least $P$, $Q$, and $P \rightarrow Q$.

Now for the proof . . .

Judgement Aggregation

Proof

By anonymity and neutrality, collective acceptance of $\varphi$ must be a function of the number of individuals accepting $\varphi$ alone.

Write $\#[\varphi]$ for the number of individuals accepting $\varphi$.

- Suppose the number $n$ of individuals is even:
  
  Due to the universal domain axiom, we must cater for the case where $\#(P \land Q) = \#(\neg (P \land Q))$. As argued above, we need to accept either both or neither. Accepting both contradicts consistency. Accepting neither contradicts completeness. ✓

- Suppose the number $n$ of individuals is odd (and $n > 1$):
  
  Suppose $\frac{n-1}{2}$ accept $P$ and $Q$; 1 each accept exactly one of $P$ and $Q$; and $\frac{n-3}{2}$ accept neither $\Rightarrow \#[P] = \#[Q] = \#(\neg (P \land Q)]$

  Accepting all three formulas contradicts consistency. But if we accept none, completeness forces us to accept their complements, which also contradicts consistency. ✓

Circumventing the Impossibility Theorem

If we are prepared to relax some of the axioms, we may be able to circumvent the impossibility theorems and successfully aggregate judgements. Next, we will explore some such possibilities:

- Relaxing the input conditions: drop the universal domain axiom and design rules for restricted domains
- Relaxing the output conditions: drop the completeness requirement (dropping consistency works but is unattractive)
- Giving up anonymity: dictatorships will surely work, but maybe we can do a little better than that
- Weakening systematicity: maybe neutrality is after all rather inappropriate for logically interconnected propositions (?), and we already know that independence is a very demanding axiom

Agenda Characterisation

Several variants of above impossibility theorem, for different sets of axioms, are discussed in the literature.

There are also so-called agenda characterisation theorems, which give sufficient and necessary conditions for an impossibility to arise.

Some suggestions for further reading are listed below.


Unidimensional Alignment

Call a profile of individual judgement sets unidimensionally aligned iff we can order the individuals such that for each proposition $\varphi$ in the agenda the individuals accepting $\varphi$ are either all to the left or all to the right of those rejecting $\varphi$. Example:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5 (Majority)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>no (no)</td>
</tr>
<tr>
<td>$B$</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>yes</td>
<td>yes (no)</td>
</tr>
<tr>
<td>$A \rightarrow B$</td>
<td>no</td>
<td>no</td>
<td>yes</td>
<td>yes</td>
<td>yes (yes)</td>
</tr>
</tbody>
</table>

Theorem 2 (List, 2003) If profiles are unidimensionally aligned, then the majority rule will produce a consistent outcome.

Note that the other axioms are all satisfied by the majority rule also in the general case (completeness only if $n$ is odd).

For simplicity, suppose the number \( n \) of individuals is odd.

Here is again our example, for illustration:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>(Majority)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>(no)</td>
</tr>
<tr>
<td>B</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>yes</td>
<td>(no)</td>
</tr>
<tr>
<td>A → B</td>
<td>no</td>
<td>no</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>(yes)</td>
</tr>
</tbody>
</table>

Call the \( \lceil \frac{n}{2} \rceil \)th individual according to our left-to-right ordering establishing unidimensional alignment the \textit{median individual}.

1. By definition, for each \( \varphi \) in the agenda, at least \( \lceil \frac{n}{2} \rceil \) individuals (a majority) accept \( \varphi \) iff the median individual does.

2. As the judgement set of the median individual is consistent, so is the collective judgement set under the majority rule. \( \checkmark \)

\begin{itemize}
  \item \textbf{Value Restriction}
  
  For simplicity, assume the agenda \( \Phi \) doesn’t contain contradictions.
  
  A set \( X \subseteq \Phi \) is called \textit{minimally inconsistent} if it is inconsistent and every proper subset \( Y \subset X \) is consistent.
  
  Call a profile of individual judgement sets \textit{value-restricted} iff every minimally inconsistent \( X \subseteq \Phi \) has a two-element subset \( Y \subseteq X \) that is not a subset of any of the judgement sets.
  
  \textbf{Theorem 3 (Dietrich and List, 2007)} If profiles are value-restricted, then the majority rule will produce a consistent outcome.

  \textbf{Remark:} Unidimensional alignment entails value-restriction, so the former is more powerful a criterion (Dietrich and List, 2007).
\end{itemize}

\textbf{Interlude: Single-Peaked Preferences}

Unidimensional alignment roughly corresponds to the case of single-peaked preferences in preference aggregation.

A profile of individual preferences over a set of alternatives \( A \) is called \textit{single-peaked} iff there exists a “left-to-right” ordering \( < \) on \( A \) such that for each individual’s most preferred candidate \( x \) we have that \( y \) is preferred over \( z \) whenever \( x < y < z \) or \( z < y < x \).

On single-peaked domains, social choice works very well: the \textit{Condorcet Paradox}, \textit{Arrow’s Theorem}, and the \textit{Gibbard-Satterthwaite Theorem} all go away.

\footnote{D. Black. \textit{The Theory of Committees and Elections}. Cambridge, 1958.}

\textbf{Proof}

Assume the profile \( \langle J_1, \ldots, J_n \rangle \) is value-restricted.

Now (for the sake of contradiction) suppose \( J \) is inconsistent.

Then there exists a set \( X \subseteq J \) that is minimally inconsistent.

By value restriction, there exists a set \( Y = \{ p, q \} \subseteq X \) such that \( Y \not\subseteq J_i \) for all \( i \in \{1, \ldots, n\} \).

On the other hand, due to \( Y \subseteq J \), there must have been a (strict) majority for both \( p \) and \( q \). Hence, there must exist at least one \( i \in \{1, \ldots, n\} \) such that \( Y \subseteq J_i \Rightarrow \) contradiction. \( \checkmark \)
Supermajority Rules

Or we could drop completeness from our list of requirements. If the collective judgement set need not be complete, we can get judgement aggregation rules satisfying the remaining axioms:

- **Unanimous rule**: include \( \varphi \) in the collective judgement set iff \( \varphi \) is in every individual judgement set. Always works.
- **Consider this variant of the original doctrinal paradox:**

| Judges 1–10: | yes | yes | yes |
| Judges 11–20: | no | yes | yes |
| Judges 21–30: | yes | no | yes |

Here the 4/5-supermajority rule, accepting \( \varphi \) iff \( \geq 25 \) judges do, produces a consistent (but not necessarily complete) outcome.

- For general results of this sort, see Dietrich and List (2007).


Oligarchic Rules

As we have seen, supermajority rules (with suitable quota) can circumvent impossibility if we are prepared to give up completeness.

Instead, we may try replacing completeness by deductive closure:

\[ \varphi \in \Phi \text{ and } J \models \varphi \Rightarrow \varphi \in J \text{ for the (collective) judgement set } J. \]

The **oligarchic rule** for the set of individuals \( X \subseteq I \) is the rule that accepts \( \varphi \) iff everyone in \( X \) does. Special cases:

- **dictatorial rule**: \( |X| = 1 \)
- **unanimous rule**: \( |X| = n \)

It is easy to check that any oligarchic rule satisfies:

- **consistency** and **deductive closure** (if individuals do);
- **universal domain**, **neutrality**, and **independence**;
- but not anonymity (except if \( |X| = n \)).

Gärdenfors (2006) gives a more precise axiomatic characterisation.


Premise-Based Procedure

For the original doctrinal paradox, the premise-based procedure consists in using the majority rule for \( A \) and \( B \) (“premises”), and then inferring the collective judgement on \( A \land B \) (“conclusion”).

<table>
<thead>
<tr>
<th>( A )</th>
<th>( B )</th>
<th>( A \land B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judge 1:</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>Judge 2:</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>Judge 3:</td>
<td>yes</td>
<td>no</td>
</tr>
</tbody>
</table>

Collective: yes yes __

The premise-base procedure (for this agenda) satisfies consistency and completeness, but violates neutrality and independence.

General Premise-Based Procedures

How can we distinguish “premises” from “conclusions” in the general case? \( \Rightarrow \) We can’t. But we can do this:

1. Label any logically independent subset \( \Delta \) of the propositions in the agenda as “premises”

   A set of formulas \( \Delta \) is **logically independent** iff, for any \( \Gamma \subseteq \Delta \), the set \( \Gamma \cup \{ \neg \varphi \mid \varphi \in \Delta \setminus \Gamma \} \) is consistent.

2. Make collective judgements on each of these premises using the majority rule.

3. Add any further propositions from the agenda that are logical consequences of these decisions to the collective judgement set.

This procedure satisfies **consistency** and **deductive closure**. If \( \Delta \) is maximally logically independent, then it also satisfies **completeness**.
Logically Independent Agendas

A (very) special case is when some $\Delta \subseteq \Phi$ with $|\Delta| = \frac{1}{2} \cdot |\Phi|$ is logically independent (i.e., pick one from each pair of complements). Then the majority rule will always produce a consistent outcome.

This roughly corresponds to the case of separable preferences discussed during the lecture on voting in combinatorial domains.

Distance-Based Procedures

Idea: enforce consistency by choosing collective judgement set “closest” to some ideal (possibly inconsistent) aggregated set.

Assumption: For simplicity, assume the agenda $\Phi$ is such that any consistent and complete judgement set forces a unique model (e.g., assume $\Phi$ includes all atomic propositions).

Define a distance-based procedure in two steps:
- Fix a distance metric between models (and judgement sets), e.g., the Hamming distance.
- Fix an objective function to optimise, e.g., (minimise) the sum of the individual distances to the collective choice.

This procedure (Hamming/$\Sigma$) behaves like the majority rule in case that is consistent, and makes a “reasonable” choice otherwise.

References

A good starting point for learning about judgement aggregation is List’s introductory paper (also the main reference for this lecture):

Some additional material is covered in this survey paper:
What next?

Next we will move on to problems related to distributive justice, fair division, and multiagent resource allocation.

- Rather than choosing one alternative for all individuals, now we need to divide a common resource and individuals have preferences over their lot (still a social choice problem!).
- Preferences will typically be modelled as utility functions, rather than as linear orders.

By restricting attention to more specific problems and allowing for richer preference structures, we will encounter fewer impossibilities.

Plan for the next few weeks:
- axiomatic treatment of different criteria for judging solutions
- procedures for different types of domains: cake-cutting, distributed resource allocation, combinatorial auctions