Computational Social Choice: Spring 2008

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Plan for Today

Much work on multiagent resource allocation, in particular in the AI and MAS communities, is about the allocation of several *indivisible goods* (and we have seen examples in previous lectures).

However, the classical problem in *fair division* is that of dividing a *cake* (a single divisible good) amongst several agents (or "players", as they are usually called in this kind of literature).

This lecture will be an introduction to such *cake-cutting procedures*.

Cakes

We will discuss the division of a single divisible good, commonly referred to as a *cake* (amongst *n players*). It's a cake where you can cut off slices with a single cut (so not a round *tart*).

More abstractly, you may think of a cake as the unit interval $[0, 1]$:

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|----------------------|
0                      1
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Each player i has a valuation function v_i mapping finite unions of subintervals (slices) to the reals, satisfying the following conditions:

- Non-negativity: $v_i(X) \geq 0$ for all $X \subseteq [0, 1]$
- Additivity: $v_i(X \cup Y) = v_i(X) + v_i(Y)$ for disjoint $X, Y \subseteq [0, 1]$
- v_i is continuous (the Intermediate-Value Theorem applies) and single points do not have any value.
- $v_i([0, 1]) = 1$ (*i.e.* it's like a probability measure)

Cut-and-Choose

The classical approach for dividing a cake between *two players:*

> *One player cuts the cake in two pieces (which she considers to be of equal value), and the other one chooses one of the pieces (the piece she prefers).*

The cut-and-choose procedure satisfies two important properties:

- *Proportionality:* Each player is guaranteed at least one half (general: $1/n$) according to her own valuation.

  Discussion: In fact, the first player (if she is risk-averse) will receive exactly $1/2$, while the second will usually get more.

- *Envy-freeness:* No player will envy (any of) the other(s).

  Discussion: Actually, for two players, proportionality and envy-freeness amount to the same thing.

Further Properties

We may also be interested in the following properties:

- *Equitability:* Under an equitable division, each player assigns the same value to the slice they receive.

  Discussion: Cut-and-choose clearly violates equitability. Furthermore, for $n > 2$, equitability is often in conflict with envy-freeness, and we shall not discuss it any further today.

- *Efficiency:* Under an efficient (Pareto optimal) division, no other division will make somebody better and nobody worse off.

  Discussion*:* Generally speaking, cut-and-choose violates efficiency: suppose player 1 really likes the middle of the cake and player 2 really like the two outer parts (then *no* one-cut procedure will be efficient). But amongst all divisions into two contiguous slices, the cut-and-choose division will be efficient.

Operational Properties

The properties discussed so far all relate to the fairness (or efficiency) of the resulting division of the cake. Beyond that we may also be interested in the "operational" properties of the procedures themselves:

- Does the procedure guarantee that each player receives a single *contiguous* slice (rather than the union of several subintervals)?
- Is the *number of cuts* minimal? If not, is it at least bounded?
- Does the procedure require an active *referee*, or can all actions be performed by the players themselves?
- Is the procedure a proper algorithm (a.k.a. a *protocol*), requiring a finite number of specific actions from the participants (no need for a "continuously moving knife"—to be discussed)?

Cut-and-choose is ideal and as simple as can be with respect to all of these properties. For $n > 2$, it won't be quite that easy though ...

Proportionality and Envy-Freeness

For $n \geq 3$, proportionality and envy-freeness are not the same properties anymore (unlike for $n = 2$):

Fact *Any envy-free division is also proportional, but there are proportional divisions that are not envy-free.*

Over the next few slides, we are going to focus on cake-cutting procedures that achieve proportional divisions.

► Any ideas how to find a proportional division for three players?

The Steinhaus Procedure

This procedure for *three players* has been proposed by Steinhaus around 1943. Our exposition follows Brams and Taylor (1995).

(1) Player 1 cuts the cake into three pieces (which she values equally).

(2) Player 2 "passes" (if she thinks at least two of the pieces are $\geq 1/3$) or labels two of them as "bad". — If player 2 passed, then players 3, 2, 1 each choose a piece (in that order) and we are done. ✓

(3) If player 2 did not pass, then player 3 can also choose between passing and labelling. — If player 3 passed, then players 2, 3, 1 each choose a piece (in that order) and we are done. ✓

(4) If neither player 2 or player 3 passed, then player 1 has to take (one of) the piece(s) labelled as "bad" by both 2 and 3. — The rest is reassembled and 2 and 3 play cut-and-choose. ✓

S.J. Brams and A.D. Taylor. An Envy-free Cake Division Protocol. *American Mathematical Monthly*, 102(1):9–18, 1995.

Properties

The Steinhaus procedure —

- Guarantees a *proportional* division of the cake (under the standard assumption that players are risk-averse: they want to maximise their payoff in the worst case).
- Is *not envy-free*. However, observe that players 2 and 3 will not envy anyone. Only player 1 may envy one of the others in case the situation where 2 and 3 play cut-and-choose occurs.
- Is a discrete procedure that does not require a referee.
- Requires *at most 3 cuts* (as opposed to the minimum of 2 cuts). The resulting pieces do not have to be contiguous (namely if both 2 and 3 label the middle piece as "bad" and 1 takes it; and if the cut-and-choose cut is different from 1's original cut).

The Banach-Knaster Last-Diminisher Procedure

In the first ever paper on fair division, Steinhaus (1948) reports on his own solution for $n = 3$ and a generalisation to *arbitrary n* proposed by Banach and Knaster.

(1) Player 1 cuts off a piece (that she considers to represent $1/n$).

(2) That piece is passed around the players. Each player either lets it pass (if she considers it too small) or trims it down further (to what she considers $1/n$).

(3) After the piece has made the full round, the last player to cut something off (the "last diminisher") is obliged to take it.

(4) The rest (including the trimmings) is then divided amongst the remaining $n{-}1$ players. Play cut-and-choose once $n = 2$. ✓

The procedure's properties are similar to that of the Steinhaus procedure (proportional; not envy-free; not contiguous; bounded number of cuts).

H. Steinhaus. The Problem of Fair Division. *Econometrica*, 16:101–104, 1948.

The Dubins-Spanier Procedure

Dubins and Spanier (1961) proposed an alternative *proportional* procedure for *arbitrary n*. It produces *contiguous* slices (and hence uses a minimal number of cuts), but it is *not discrete* anymore and it requires the active help of a *referee*.

(1) A referee moves a knife slowly across the cake, from left to right. Any player may shout "stop" at any time. Whoever does so receives the piece to the left of the knife.

(2) When a piece has been cut off, we continue with the remaining $n{-}1$ players, until just one player is left (who takes the rest). ✓

Observe that this is also *not envy-free*. The last chooser is best off (she is the only one who can get more than $1/n$).

L.E. Dubins and E.H. Spanier. How to Cut a Cake Fairly. *American Mathematical Monthly*, 68(1):1–17, 1961.

Discretising the Dubins-Spanier Procedure

We may "discretise" the Dubins-Spanier procedure as follows:

- Ask each player to make a *mark* at their $1/n$ point. Cut the cake at the leftmost mark (or anywhere between the two leftmost marks) and give that piece to the respective player.
- Continue with $n{-}1$ players, until only one is left. ✓

This also removes the need for an (active) referee.

This is a *discrete* procedure guaranteeing a proportional *contiguous* division (in this sense it is superior to both Dubins-Spanier and Banach-Knaster). The number of *actual* cuts is *minimal* (although purists will object to this: the marks are like virtual cuts).

The Even-Paz Divide-and-Conquer Procedure

Even and Paz (1984) investigated *upper bounds* for the number of cuts required to produce a proportional division for *n players*, without allowing either a moving knife or "virtual cuts" (marks).

They conjectured the following *divide-and-conquer* protocol to be optimal in this sense (at least for $n > 4$):

(1) Ask each player to cut the cake at her $\lfloor \frac{n}{2} \rfloor$ / $\lceil \frac{n}{2} \rceil$ mark.

(2) Associate the union of the leftmost $\lfloor \frac{n}{2} \rfloor$ pieces with the players who made the leftmost $\lfloor \frac{n}{2} \rfloor$ cuts (group 1), and the rest with the others (group 2).

(3) Recursively apply the same procedure to each of the two groups, until only a single player is left. ✓

S. Even and A. Paz. A Note on Cake Cutting. *Discrete Applied Mathematics*, 7:285–296, 1984.

Recap: Big-O Notation

To specify upper bounds on the runtime of an algorithm in view of the size n of the input, we use the familiar Big-O notation:

$$O(g(n)) \; = \; \{f : \mathbb{N} \to \mathbb{N} \mid \exists c \in \mathbb{R}^+ \; \exists n_0 \in \mathbb{N} \; \forall n \geq n_0 : f(n) \; \leq \; c \cdot g(n)\}$$

For instance, if we have an algorithm that will compute the function $f(n)$ in (at most) quadratic time, then we write $f(n) \in O(n^2)$.

The following notation is useful for specifying lower bounds:

$$\Omega(g(n)) \; = \; \{f : \mathbb{N} \to \mathbb{N} \mid \exists c \in \mathbb{R}^+ \; \exists n_0 \in \mathbb{N} \; \forall n \geq n_0 : f(n) \geq c \cdot g(n)\}$$

A different way of putting it:

$$f(n) \in \Omega(g(n)) \text{ iff } g(n) \in O(f(n))$$

Complexity of Divide-and-Conquer

Fact *The Even-Paz procedure requires $O(n \log n)$ cuts.*

Proof: The procedure may be understood a taking place along a binary tree. Branching corresponds to dividing the remaining set of players into two groups. At each node, the number of cuts is equal to the number of players in the respective group. At each level of the tree, the number of cuts adds up to n. The overall depth of the tree is $\lceil \log_2 n \rceil$: the number of times we can divide n by 2 before we get down to a single player. ✓

So $O(n \log n)$ is certainly an *upper bound*. Sgall and Woeginger (2003) give a matching *lower bound* of $\Omega(n \log n)$ — under some technical restrictions (you need to be more precise about what is and what is not allowed if you want to prove a lower bound ...).

J. Sgall and G.J. Woeginger. *A Lower Bound for Cake Cutting.* ESA-2003.

Envy-Free Procedures

Next we discuss procedures for achieving *envy-free* divisions.

- For $n = 2$ the problem is easy: cut-and-choose does the job.
- For $n = 3$ we will see two solutions. They are already quite complicated: either the number of cuts is *not minimal* (but > 2), or *several simultaneously moving knives* are required.
- For $n = 4$, to date, no procedure producing *contiguous pieces* is known. Barbanel and Brams (2004), for example, give a moving-knife procedure requiring up to 5 cuts.
- For $n \geq 5$, to date, only procedures requiring an *unbounded* number of cuts are known. (see e.g. Brams and Taylor, 1995)

J.B. Barbanel and S.J. Brams. Cake Division with Minimal Cuts. *Mathematical Social Sciences*, 48(3):251–269, 2004.

S.J. Brams and A.D. Taylor. An Envy-free Cake Division Protocol. *American Mathematical Monthly*, 102(1):9–18, 1995.

The Selfridge-Conway Procedure

The first discrete proportional protocol for $n = 3$ has been discovered independently by Selfridge and Conway (around 1960). Our exposition follows Brams and Taylor (1995).

(1) Player 1 cuts the cake in three pieces (she considers equal).

(2) Player 2 either "passes" (if she thinks at least two pieces are tied for largest) or trims one piece (to get two tied for largest pieces). — If she passed, then let players 3, 2, 1 pick (in that order). ✓

(3) If player 2 did trim, then let 3, 2, 1 pick (in that order), but require 2 to take the trimmed piece (unless 3 did). Keep the trimmings unallocated for now (note: the partial allocation is envy-free).

(4) Now divide the trimmings. Whoever of 2 and 3 received the *un*trimmed piece does the cutting. Let players choose in this order: non-cutter, player 1, cutter. ✓

S.J. Brams and A.D. Taylor. An Envy-free Cake Division Protocol. *American Mathematical Monthly*, 102(1):9–18, 1995.

The Stromquist Procedure

Stromquist (1980) has come up with a proportional procedure for $n = 3$ producing contiguous pieces, albeit requiring the use of four simultaneously moving knifes:

- A referee slowly moves a knife across the cake, from left to right (supposed to cut somewhere around the 1/3 mark).
- At the same time, each player is moving her own knife so that it would cut the righthand piece in half (wrt. her own valuation).
- The first player to call "stop" receives the piece to the left of the referee's knife. The righthand part is cut by the middle one of the three player knifes, and the other two pieces are allocated in the obvious manner (ensuring proportionality). ✓

W. Stromquist. How to Cut a Cake Fairly. *American Mathematical Monthly*, 87(8):640–644, 1980.

Summary

We have discussed various procedures for fairly dividing a cake (a metaphor for a single divisible good) amongst several players.

- Fairness properties: *proportionality* and *envy-freeness* (but other notions, such as equitability, efficiency, strategy-proofness . . . are also of interest)
- Distinguish discrete procedures (*protocols*) and continuous (*moving-knife*) procedures.
- The problem becomes non-trivial for more than two players, and there are many open problems relating to finding procedures with "good" properties for larger numbers.

Overview of Procedures

Procedure	**Players**	**Type**	**Division**	**Pieces**	**Cuts**
Cut-and-choose	$n = 2$	protocol	envy-free (∗)	contiguous	minimal
Steinhaus	$n = 3$	protocol	proportional	not contig.	min.+1
Banach-Knaster (last-diminisher)	any n	protocol	proportional	not contig. (but could be)	bounded
Dubins-Spanier	any n	1 knife	proportional	contiguous	minimal
Discrete D-S	any n	protocol	proportional	contiguous	min.(∗∗)
Even-Paz (divide-and-conquer)	any n	protocol	proportional	contiguous	$O(n \log n)$
Selfridge-Conway	$n = 3$	protocol	envy-free (∗)	not contig.	≤ 5
Stromquist	$n = 3$	4 knives	envy-free (∗)	contiguous	minimal

(∗) Recall that envy-freeness entails proportionality.

(∗∗) Count does not include *marks* (virtual cuts).

References

Books on fair division and cake-cutting include the following:

- S.J. Brams and A.D. Taylor. *Fair Division: From Cake-Cutting to Dispute Resolution*. Cambridge University Press, 1996.
- J. Robertson and W. Webb. *Cake-Cutting Algorithms: Be Fair if You Can*. A.K. Peters, 1998.

The following paper by Brams and Taylor not only introduces their original procedure for envy-free division for more than three players, but is also particularly nice in presenting several of the older procedures in a very systematic and accessible manner:

- S.J. Brams and A.D. Taylor. An Envy-free Cake Division Protocol. *American Mathematical Monthly*, 102(1):9–18, 1995.