

The Art and Science of Cake Cutting

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Why cake cutting?

- *Useful*: how to best allocate *resources* to several *agents* is relevant to Economics, Policy Making, Computer Science, ...
- *Interesting*: easy to understand the *questions*, but some *answers* still outstanding after 70+ years of research.

Plan for this lecture:

- Definition of the problem: how can you cut a cake fairly?
- Presentation of several procedures for cutting a cake
- Complexity analysis: how many cuts do you need?

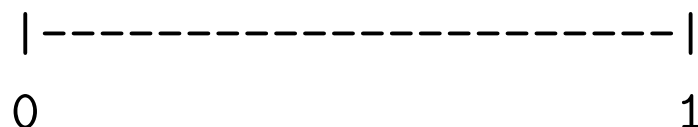






Formal Model

The *cake* is the interval $[0, 1]$ of the real numbers from 0 to 1:



We need to divide the cake between n *agents* (with $n = 2, 3, 4, 5, \dots$). A *piece* of cake is a subinterval (or a collection of subintervals).

Each agent i has a *valuation function* v_i to measure how much she likes any given piece of cake. Assumptions:

- Normalisation: $v_i(\text{full_cake}) = 1$ and $v_i(\text{nothing}) = 0$
- Additivity: $v_i(A \cup B) = v_i(A) + v_i(B)$ if A and B don't overlap
- Continuity: small increases in cake \Rightarrow small increases in value

Fairness

We want to design procedures that are “fair”. What does that mean?

One possible definition:

*A cake-cutting procedure is **proportionally fair**, if every agent can ensure she gets a piece that she values **at least at $\frac{1}{n}$** .*

Reminder:

- 1 = value of the full cake (for everyone)
- n = number of agents

Cut-and-Choose Procedure

For the case of *2 agents*, you all know how to do this:

- ▶ *One agent cuts the cake in two pieces (of equal value to her), and the other chooses one of them (the piece she prefers).*

This clearly is *proportionally fair!*

What about 3 agents? Or more?

Moving-Knife Procedure

Dubins and Spanier (1961) proposed this procedure (for any n):

- (1) A referee moves a *knife* slowly across the cake, from left to right. Any agent may shout “*stop*” at any time. Whoever does so receives the piece to the left of the knife.
- (2) When a piece has been cut off, we *continue* with the remaining agents, until just one agent is left (who takes the rest).

This is *proportionally fair!* (Of course: right-to-left works as well.)

Exercise 1: You *love* strawberries. There is a single large strawberry on the right end of the cake. Do you prefer left-to-right or right-to-left?

Exercise 2: How would you program a computer to play for you?

L.E. Dubins and E.H. Spanier. How to Cut a Cake Fairly. *American Mathematical Monthly*, 68(1):1–17, 1961.

Problem

Each agent has to continuously monitor the knife as it moves over all the real numbers from 0 to 1. For each number, the agent has to evaluate the piece to the left of the knife. This is impossible.

Simulating the Moving-Knife Procedure

We can “discretise” the moving-knife procedure to solve our problem:

(1) Ask each agent to *mark* the cake where she *would* shout “stop”.

Then *cut* the cake at the *leftmost mark* and give the resulting piece to the agent who made that mark.

(2) When a piece has been cut off, we *continue* with the remaining agents, until just one agent is left (who takes the rest).

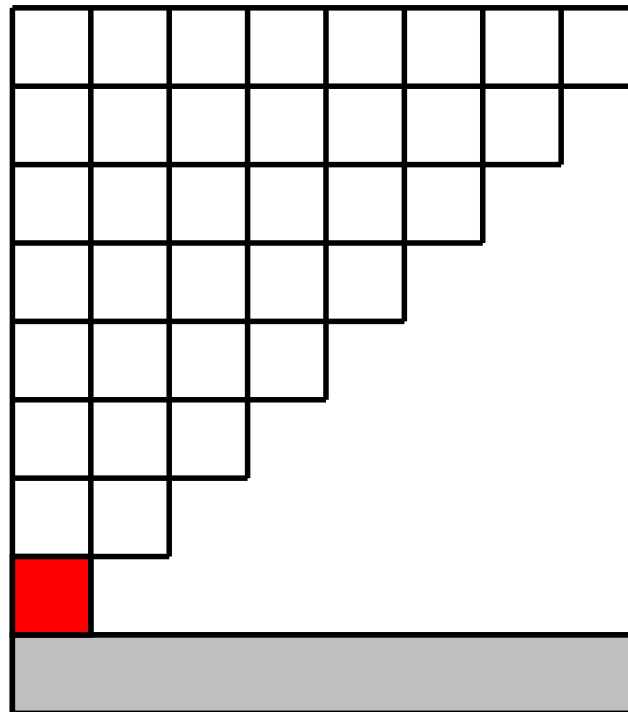
Next: How *complex* is this? That is: *how many marks* do we need?

Complexity Analysis: Number of Marks

In each round, each participating agent makes one mark. The number of participating agents goes down from n to 2. Thus:

$$n + (n - 1) + (n - 2) + \dots + 3 + 2 = \frac{n \cdot (n + 1)}{2} - 1 \approx \frac{1}{2} \cdot n^2$$

Proof:



Can we do better?

Divide-and-Conquer Procedure

Even and Paz (1984) introduced the divide-and-conquer procedure:

(1) Ask each agent to put a *mark* on the cake.

(2) *Cut* the cake at the $\lfloor \frac{n}{2} \rfloor$ *th mark* (counting from the left).

Associate the agents who made the *leftmost* $\lfloor \frac{n}{2} \rfloor$ *marks* with the *lefthand part*, and the *remaining agents* with the *righthand part*.

(3) *Repeat* for each group, until only one agent is left.

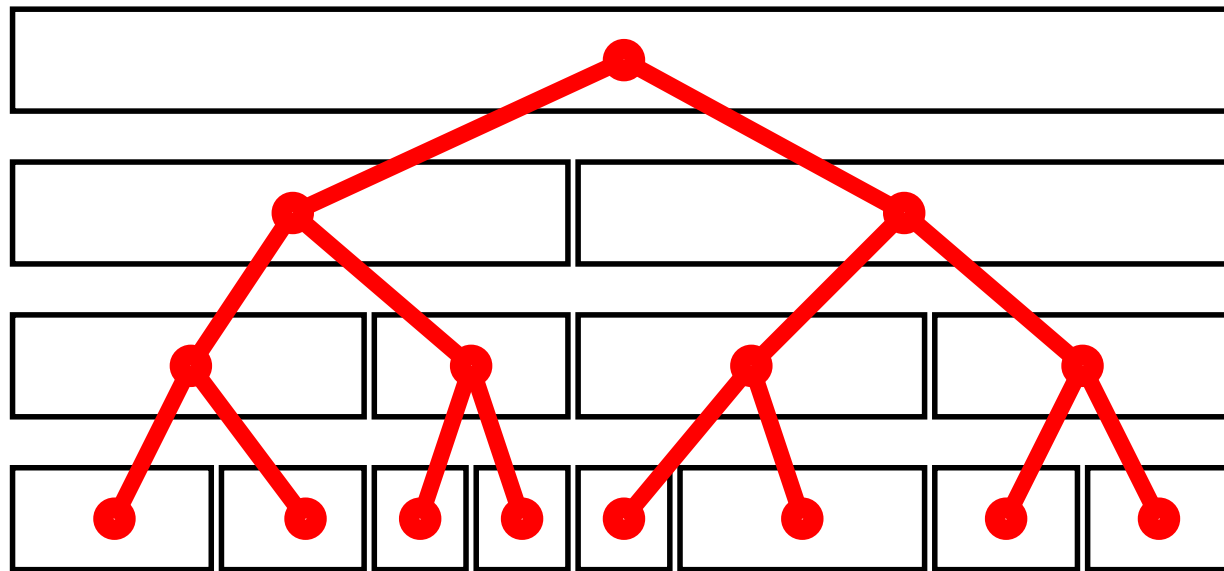
This also is *proportionally fair!* But how *complex* is it?

S. Even and A. Paz. A Note on Cake Cutting. *Discrete Applied Mathematics*, 7(3):285–296, 1984.

Complexity Analysis: Number of Marks

In each round, every agent makes one mark. So: *n marks per round*

But how many rounds?

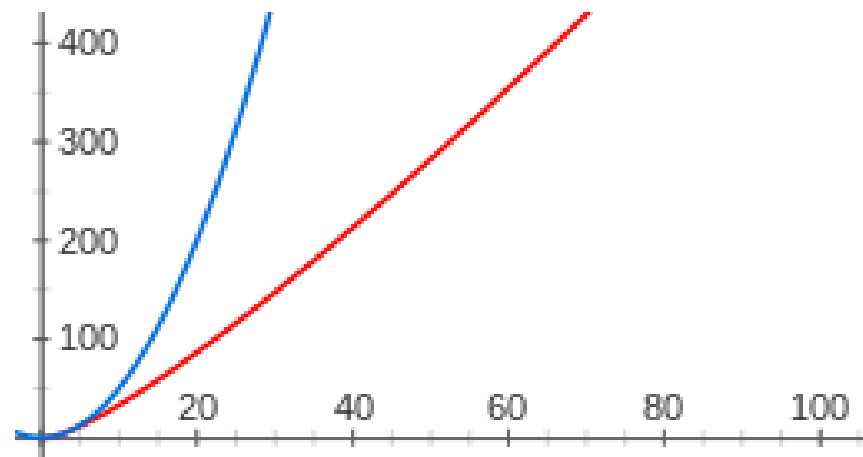


rounds = number of times you can divide n by 2 before hitting ≤ 1
 $\approx \log_2 n$ (example: $\log_2 8 = 3$)

Thus: number of *marks* required $\approx n \cdot \log_2 n$

Comparison

Recall: *simulated moving-knife* requires around $\frac{1}{2} \cdot n^2$ marks and *divide-and-conquer* requires around $n \cdot \log_2 n$ marks.



So: divide-and-conquer is much better (for large n , complexity-wise).

Remark: Woeginger and Sgall (2007) proved that you can't do better.

G.J. Woeginger and J. Sgall. On the Complexity of Cake Cutting. *Discrete Optimization*, 4(2):213–220, 2007.

Envy

Proportional fairness is but one formalisation of “fairness”:

*A cake-cutting procedure is called **envy-free**, if every agent can ensure that she will receive a subjectively largest piece.*

Connections between these two notions of fairness:

- Observe that for $n = 2$ agents, we have:

$$\text{envy-freeness} \iff \text{proportional fairness}$$

- But for $n \geq 3$ agents, we only have:

$$\text{envy-freeness} \implies \text{proportional fairness}$$

Indeed, of our procedures only **cut-and-choose** guarantees envy-freeness.

No fully satisfactory solution for envy-free cake cutting is known!

We will now see a moving-knife procedure for the special case of $n = 3$.

Four Simultaneously Moving Knives

Stromquist (1980) found this *envy-free* procedure for *3 agents*:

- A referee slowly moves a knife across the cake, from left to right (supposed to eventually cut somewhere around the $\frac{1}{3}$ mark).
- At the same time, each agent is moving her own knife so that it would cut the righthand piece in half (w.r.t. her own valuation).
- The first agent to call “stop” receives the piece to the left of the referee’s knife. The righthand part is cut by the middle one of the three agent knives. If neither of the other two agents hold the middle knife, they each obtain the piece at which their knife is pointing. If one of them does hold the middle knife, then the other one gets the piece at which her knife is pointing.

W. Stromquist. How to Cut a Cake Fairly. *American Mathematical Monthly*, 87(8):640–644, 1980.

Last Slide

This has been an introduction to cake cutting. We've seen:

- usable procedures for guaranteeing *proportional fairness*
- how to analyse the *complexity* of a cake-cutting procedure
- less usable procedure for guaranteeing *envy-freeness* ($n = 3$)

Take-home messages:

- simple mathematical models useful to understand real problems
- intuitive concepts (“fairness”) permit mathematical definitions
- different criteria to assess quality of a solution (complexity, ...)
- simple models can give rise to challenging questions

For further reading, see my lecture notes (available from my website).

U. Endriss. *Lecture Notes on Fair Division*. Institute for Logic, Language and Computation, University of Amsterdam, 2009/2010.