Axiomatic Analysis of Approval-Based Scoring Rules

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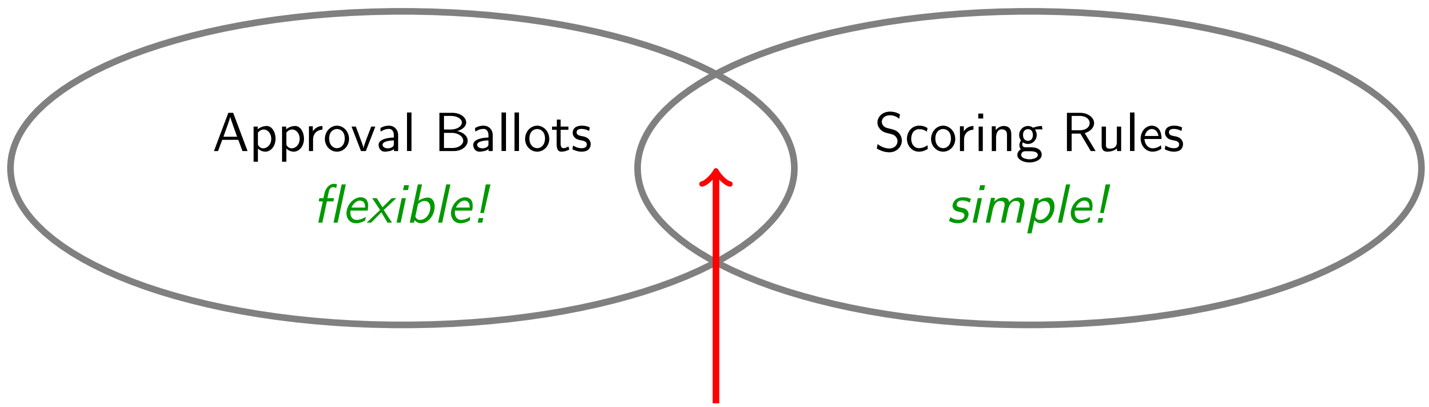
[joint work with Tuva Bardal (Warwick)]

Voting



What are normatively appealing voting rules in this space?

T. Bardal and U. Endriss. Axiomatic Analysis of Approval-Based Scoring Rules. Available at SSRN (ssrn.com/abstract=5082951).

The Model

Fix: Set C of $m = |C|$ *candidates* and universe $\mathbb{N}$ of *potential voters*.
A *profile* A maps each voter $i \in \mathbb{N}$ to her *approval ballot* $A(i) \subseteq C$.

At any given time, voters from a finite *electorate* $N \subset \mathbb{N}$ actually vote, resulting in *response profile* A_N (profile A restricted to electorate N).

A *voting rule* f maps response profiles to nonempty sets of candidates.

A *simple scoring rule* $f_{\boldsymbol{w}}$ is induced by *weights* $\boldsymbol{w} = (w_1, \ldots, w_m)$:

$$f_{\boldsymbol{w}}(A_N) \quad = \quad \operatorname*{argmax}_{c \in C} \sum_{i \in N} \mathbb{1}_{c \in A(i)} \cdot w_{|A(i)|}$$

Thus: A voter who approves of k candidates gets a weight of w_k.

Exercise: *What would be reasonable choices for the weight vector?*

Size-Approval Rules

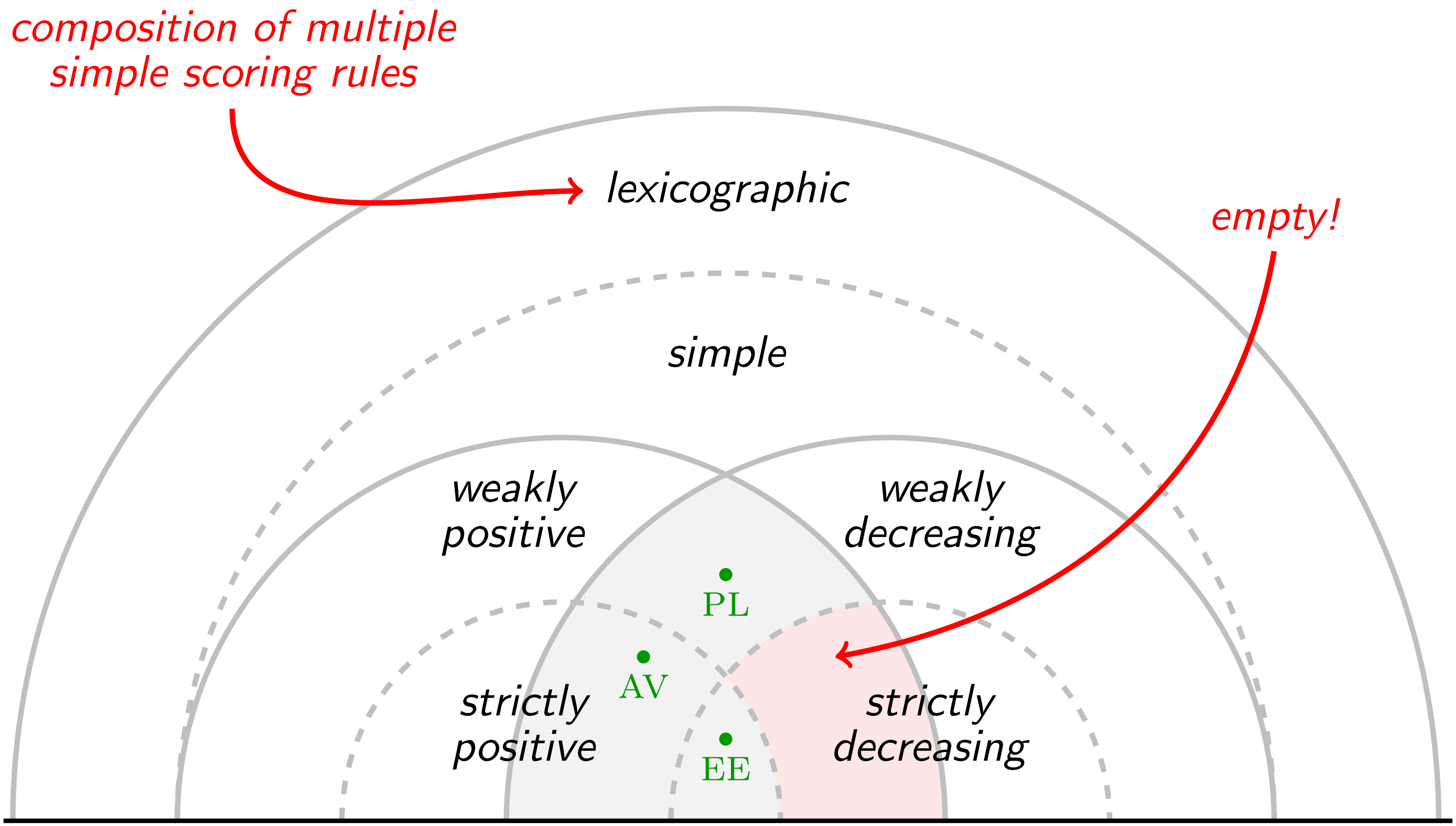
A simple scoring rule is a *size-approval rule* if it can be represented by a weight vector that is *weakly positive* and *weakly decreasing*.

Exercise: $(4, 3, 2, 1)$ *and* $(8, 6, 4, -17)$ *induce the same rule. Why?*

Archetypal representatives of the class of size-approval rules:

- Approval Voting (AV): $(1, \dots, 1)$
- Even-and-Equal (EE): $(1, 1/2, \dots, 1/m)$
- Plurality Rule (PL): $(1, 0, \dots, 0)$

Classes of Scoring Rules



Existing Characterisation

Alcalde-Unzu and Vorsatz characterised the size-approval rules:

Theorem 1 (Alcalde-Unzu and Vorsatz, 2009) *An approval-based voting rule satisfies Anonymity, Neutrality, Reinforcement, Continuity, Congruity and Contraction iff it is a size-approval rule.*

Important result, but not offering much insight into role of individual axioms or help with characterising related classes of voting rules.

Remark: Their proof takes up 11 pages of dense mathematical text.

J. Alcalde-Unzu and M. Vorsatz. Size Approval Voting. *Journal of Economic Theory*, 144(3):1187–1210, 2009.

Lexicographic Scoring Rules

Three classical axioms characterise our largest class:

Theorem 2 (Fishburn, 1979) *An approval-based voting rule satisfies Anonymity, Neutrality and Reinforcement iff it is a lexico. scoring rule.*

Axioms involved:

- *Anonymity:* treat all voters the same!
- *Neutrality:* treat all candidates the same!
- *Reinforcement:* handle subelectorates in a consistent manner!
  $f(A_N) \cap f(A_M) \neq \emptyset$ implies $f(A_N) \cap f(A_M) = f(A_N + A_M)$

P.C. Fishburn. Symmetric and Consistent Aggregation with Dichotomous Voting. In J.J. Laffont (ed.), *Aggregation and Revelation of Preferences*, 1979.

Simple Scoring Rules

Adding one more axiom yields a more natural class of rules:

Theorem 3 (Fishburn, 1979) *A lexicographic scoring rule satisfies the axiom of Continuity iff it is a simple scoring rule.*

Continuity requires that sufficiently many coalitions that would all elect the same candidates cannot be ignored entirely.

Details differ in work of Fishburn, Myerson, Alcalde-Unzu & Vorsatz.

Theorem 4 *You can freely switch between Continuity axioms!*

P.C. Fishburn. Symmetric and Consistent Aggregation with Dichotomous Voting. In J.J. Laffont (ed.), *Aggregation and Revelation of Preferences*, 1979.

R. B. Myerson. Axiomatic Derivation of Scoring Rules without the Ordering Assumption. *Social Choice and Welfare*, 12(1):59–74, 1995.

J. Alcalde-Unzu and M. Vorsatz. Size Approval Voting. *Journal of Economic Theory*, 144(3):1187–1210, 2009.

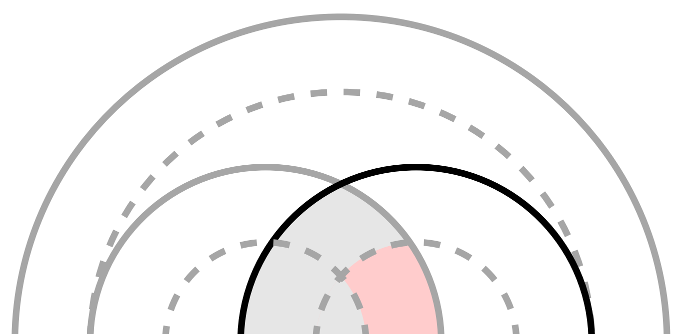
Weakly Decreasing Scoring Rules

The axiom of *Contraction* asks that one voter reducing her approval set (without dropping all winners) will get reflected at the outcome level:

$$A'(i) \subset A(i) \text{ such that } f(A_N) \cap A'(i) \neq \emptyset \text{ implies}$$
$$f(A_N) \cap A'(i) \;\subseteq\; f(A'_N) \;\subseteq\; f(A_N)$$

Call the lefthand inclusion *Weak Contraction*.

Theorem 5 *A simple scoring rule satisfies the axiom of Contraction <u>or</u> the axiom of Weak Contraction <u>iff</u> it is weakly decreasing.*



Weakly Positive Scoring Rules

Theorem 6 *A simple scoring rule satisfies the axiom of Congruity <u>or</u> the axiom of Weak Faithfulness <u>iff</u> it is weakly positive.*

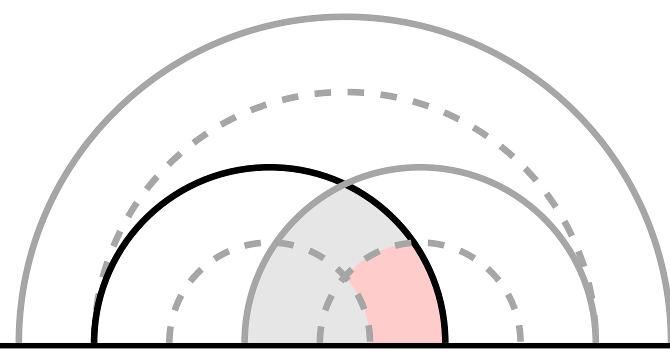
Axioms involved:

- *Congruity:* not approving losers should not make them win!

  $c \notin f(A_N)$ and $c \notin A(i)$ for all $i \in M$ imply $c \notin f(A_N + A_M)$
- *Weak Faithfulness:* lone voters can nominate!

  $f(A_{\{i\}}) \supseteq A(i)$

<u>Remark:</u> Combining results for weakly positive and weakly decreasing scoring rules, we obtain characterisations of the class of *size-approval rules*.



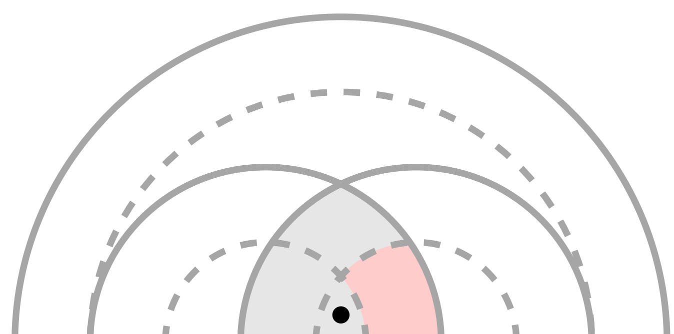
Even-and-Equal Cumulative Voting

For *plurality* and *approval voting*, multiple characterisations exist, but not so for our third example of an archetypal size-approval rule. Now:

Theorem 7 *The rule of even-and-equal cumulative voting is the unique simple scoring rule satisfying Faithfulness and Splitting.*

Axioms involved:

- *Faithfulness:* lone voters can dictate! $[f(A_{\{i\}}) = A(i)]$
- *Splitting:* outcome shouldn't change when k voters each voting for a different singleton instead all vote for all of those k candidates!



Last Slide

Should use *approval ballots* due to flexibility and *scoring rules* due to simplicity. The *size-approval rules* stand out for being most natural.

Combining our results, we obtain $3 \times 2 \times 2 = 12$ *characterisations* of the size-approval rules, including that of Alcalde-Unzu and Vorsatz.

- clear understanding of impact of individual axioms
- logically stronger results due to use of weaker axioms
- significantly simpler proof (not shown here)

Also characterised "most typical" such rule: *even-and-equal voting*.

Paper: ssrn.com/abstract=5082951