On the Arrow property for symmetric classes of choice functions

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Abstract

We prove a refined version of Shelah's theorem of the Arrow property. Let $\mathfrak{C}_r(A)$ be the set of all choice functions defined on the r-element subsets of a finite set A. Then any non-empty proper symmetric subset $\mathfrak{D}$ of the set $\mathfrak{C}_r(A)$ has the Arrow property whenever $|A| \geq 5$ and $r \geq 3$. We also describe all symmetric sets $\mathfrak{D} \subset \mathfrak{C}_r(A)$ that have the Arrow property if either $r = 2$, or $r = 3$ and $|A| = 4$.

1 Introduction

Arrow's impossibility theorem (see [1], 1950) states that among aggregation rules meeting some natural criteria there exists no rule that preserves the set of all rational choice functions. Several proofs of Arrow's theorem have been proposed, using various techniques: combinatorial ([1], see also [2]), topological (see [3]), model-theoretic (see [4]). Some approaches lead to generalizations of Arrow's theorem, and to some related results. In [5] (2005) a problem of preservation by aggregation rules of arbitrary symmetric sets of choice functions (not necessarily rational ones) is considered, and the impossibility theorem is proved under some additional conditions. In the present paper we remove those conditions. We consider as very promising the clonal approach used in [5]. It can be further developed and used for solving other related problems, and in particular for obtaining "positive" results (i.e. "possibility theorems") in some specific cases.

Let A be a finite set of alternatives. For any natural number r we denote by the symbol $[A]^r$ the set of all r-element subsets of A. The symbol $\mathfrak{C}(A)$ denote the set of all choice functions on the set A and the symbol $\mathfrak{C}_r(A)$ denote the set of restrictions of all functions $\mathfrak{c} \in \mathfrak{C}(A)$ to $[A]^r$. In other terms, $\mathfrak{c} \in \mathfrak{C}_r(A)$ iff

$$\mathfrak{c} \in {}^{[A]^r}A \text{ and } \mathfrak{c}(p) \in p \text{ for all } p \in [A]^r.$$

A function $\mathfrak{c} \in \mathfrak{C}_r(A)$ is called *rational* if there is a linear order $<$ on A such that

$$\mathfrak{c}(q) = \max q,$$

i.e. $(\forall x \in q)\, x = \mathfrak{c}(q) \vee x < \mathfrak{c}(q)$, for all $q \in [A]^r$.

A function $\mathfrak{c} \in \mathfrak{C}_r(A)$ (rational or non-rational) may represent *individual preferences* or *an individual behavior*(see [5], [6]). Shelah gives the following natural examples of non-rational choice functions: $\mathfrak{c}(q)$ is the second largest element in q according to some ordering, or $\mathfrak{c}(q)$ is the median element of q (assume $|q|$ is odd) according to some ordering.

If some partial order $\prec$ is defined on a set of alternatives, then a natural example of a non-rational (in general) choice function is provided by a function that relates every set $q \in [A]^r$ to an element $\mathfrak{c}(q) \in q$, which is not *dominated* by other elements from q, i.e.

$$(\forall x \in q)\, \mathfrak{c}(q) \not\prec x.$$

A set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ is called *symmetric* if for all function $\mathfrak{c} \in \mathfrak{D}$ and permutation $\sigma \in S_A$ the function c_σ defined by

$$(\forall p \in [A]^r)\, \mathfrak{c}_\sigma(p) = \sigma^{-1}\mathfrak{c}(\sigma p)$$

belongs to $\mathfrak{D}$.

Every symmetric set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ can be characterized as a class of all functions $\mathfrak{c} \in \mathfrak{C}_r(A)$ satisfying some condition Θ, where Θ is a closed formula built by applying logical connectives and quantifiers to elementary formulas of the form

$$\mathfrak{c}(\{x_{i_0}, x_{i_1}, \ldots, x_{i_{r-1}}\}) = x_{i_j},$$

where $0 \leq j < r$ and $i_k = i_l \to k = l$ $(0 \leq k, l < r)$. Informally this means that a symmetric set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ represents a set of individual behaviors coordinated by the same "common principle".

For any natural number $n \geq 1$ a function $f\colon (\mathfrak{C}_r(A))^n \to \mathfrak{C}_r(A)$ is called an (n-ary) *aggregation rule*. The set of all aggregation rules is denoted by the symbol $\mathcal{O}(A, r)$.

Definition 1.1. An aggregation rule $f\colon (\mathfrak{C}_r(A))^n \to \mathfrak{C}_r(A)$ is *normal* if for any $p \in [A]^r$ there is a function $f_p\colon p^n \to p$ such that

1. $f(\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1})(p) = f_p(\mathfrak{c}_0(p), \mathfrak{c}_0(p), \ldots, \mathfrak{c}_{n-1}(p))$ for all $\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1} \in \mathfrak{C}_r(A)$,
2. $\bigvee\limits_{0 \leq i < n} f_p(a_0, a_1, \ldots, a_{n-1}) = a_i$ for all $a_0, a_1, \ldots, a_{n-1} \in p$.

Item 1 of Definition 1.1 means that the aggregation rule f has the *IIA property* (Independence of Irrelevant Alternatives), and item 2 is slightly stronger than the Pareto efficiency of the aggregation rule f (if $r = 2$ then item 2 can be replaced by the condition $f_p(a, a) = a$ for all $a \in p$; this is equivalent to the Pareto efficiency of f).

We denote the set of all normal aggregation rules $f\colon (\mathfrak{C}_r(A))^n \to \mathfrak{C}_r(A)$ by the symbol $\mathcal{N}(A, r)$.

Definition 1.2. An aggregation rule $f\colon (\mathfrak{C}_r(A))^n \to \mathfrak{C}_r(A)$ is called

– *simple* if f is normal and f_p does not depend on p, i.e.

$$(\forall p, q \in [A]^r)(\forall \mathbf{a} \in p^n \cap q^n) f_p(\mathbf{a}) = f_q(\mathbf{a});$$

– *dictatorial* (or *monarchical*) if f is a projection, i.e.

$$(\exists j \in \{0, 1, \ldots n-1\})(\forall \mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1} \in \mathfrak{C}_r(A)) f(\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1}) = \mathfrak{c}_j.$$

Any normal aggregation rule $f \in \mathcal{N}(A, 2)$ is simple. Also any dictatorial aggregation rule is normal and simple. We denote the set of all dictatorial aggregation rules $f\colon (\mathfrak{C}_r(A))^n \to \mathfrak{C}_r(A)$ by the symbol $\mathcal{M}(A, r)$.

Remark 1.3. We call an aggregation rule $f\colon (\mathfrak{C}_r(A))^n \to \mathfrak{C}_r(A)$ is *almost normal* (or *IIA-aggregation rule*) if for any $p \in [A]^r$ there is a function $f_p\colon p^n \to p$ such that only condition 1 of Definition 1.1 holds. Let f is almost normal and *simple* in the sense of Definition 1.2 (with "normal" replaced by "almost normal"). Then if $r < |A|$, the condition 2 of Definition 1.1 holds, i.e. f is normal.

Let $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ and $f \in \mathcal{O}(A, r)$. We say that f *preserves* $\mathfrak{D}$ and $\mathfrak{D}$ is *preserved* under f if

$$f(\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n) \in \mathfrak{D} \text{ for all } \mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n \in \mathfrak{D}.$$

The set of all $f \in \mathcal{O}(A, r)$ that preserves $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ is denoted by the symbol $\operatorname{Pol} \mathfrak{D}$ and the set of all $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ that is preserved under $f \in \mathcal{O}(A, r)$ is denoted by the symbol $\operatorname{Inv} f$.

For any $\mathcal{F} \subseteq \mathcal{O}(A,r)$ and any $\mathbb{D} \subseteq P(\mathfrak{C}_r(A))$ the symbol $\operatorname{Inv}\mathcal{F}$ denotes the set $\bigcap_{f\in\mathcal{F}} \operatorname{Inv} f$, and the symbol $\operatorname{Pol}\mathbb{D}$ denotes the set $\bigcap_{\mathfrak{D}\in\mathbb{D}} \operatorname{Pol}\mathfrak{D}$. Obviously,

$$\operatorname{Pol}\mathfrak{C}_r(A) = \operatorname{Pol}\varnothing = \operatorname{Pol}\{\varnothing, \mathfrak{C}_r(A)\} = \mathcal{O}(A,r) \text{ and } \operatorname{Inv}\mathcal{M}(A,r) = P(\mathfrak{C}_r(A)).$$

Let $\mathfrak{R}_2(A)$ be the set of all rational function $\mathfrak{c} \in \mathfrak{C}_2(A)$. Arrow's impossibility theorem (see [1]) asserts that if $|A| \geq 3$ then any normal aggregation rule with preserves $\mathfrak{R}_2(A)$ is dictatorial, i.e. $\operatorname{Pol}\mathfrak{R}_2(A) \cap \mathcal{N}(A,2) = \mathcal{M}(A,2)$.

Definition 1.4. A set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ *has the Arrow property* if $\operatorname{Pol}\mathfrak{D} \cap \mathcal{N}(A,r) = \mathcal{M}(A,r)$.

In paper [5] S. Shelah proved that Arrow's theorem can be extended to the case when the individual choices are not rational in a very general setting:

Theorem 1.5. *There are natural numbers r_1, r_2 (e.g. $r_1 = r_2 = 7$) such that for any natural number r, $r_1 \leq r \leq |A| - r_2$, any non-empty proper symmetric subset $\mathfrak{D}$ of the set $\mathfrak{C}_r(A)$ has the Arrow property.*

We proved that if $|A| \geq 5$ this theorem is true if $r_1 = 3$ and $r_2 = 0$. Conversely, if either $r = 2$, or $r = 3$ and $|A| = 4$, then there are non-empty proper symmetric subsets $\mathfrak{D}$ of the set $\mathfrak{C}_r(A)$ which do not have the Arrow property. We describe all such sets $\mathfrak{D}$. Therefore we give a complete description of all symmetric $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ which have the Arrow property (if either $|A| \leq 2$ or $r \leq 1$, Theorem 1.5 holds trivially).

Remark 1.6. In a more general situation, individual preferences can be represented by choice functions $\mathfrak{c}$ defined on the entire set $P(A) \setminus \{\varnothing\}$. In the context of our considerations this general case can be easily reduced to a "particular" one (that is, when $\mathfrak{c} \in \mathfrak{C}_r(A)$) by considering restrictions $\mathfrak{c} \restriction [A]^r$, $1 \leq r \leq |A|$, without creating any new effect.

2 Main theorem

Let $|A| = 4$ and let K be the *Klein four-group*. For any sets $p, q \in [A]^3$ there is only one permutation $\sigma_{p,q} \in K$ for which $q = \sigma_{p,q}(p)$. We denote by the symbol $\mathfrak{C}_3^K(A)$ the set of all function $\mathfrak{c} \in \mathfrak{C}_3(A)$ such that

$$\mathfrak{c}(q) = \sigma_{p,q}\mathfrak{c}(p) \text{ for all } p, q \in [A]^3.$$

It is easy to check that $\mathfrak{C}_3^K(A)$ is the non-empty proper symmetric subset of the set $\mathfrak{C}_r(A)$. The set $\mathfrak{C}_3^K(A)$ is preserved under any simple binary function $f \in \mathcal{N}(A,3)$ satisfying the condition

$$\sigma f_q(\mathbf{a}) = f_{\sigma q}(\sigma\mathbf{a}) \text{ for all } q \in [A]^3,\ \mathbf{a} \in q^2 \text{ and } \sigma \in K.$$

(The set of these functions is not equal to $\mathcal{M}(A,r)$).

Table 1: $A = \{a,b,c,d\}$, $\mathfrak{C}_3^K(A) = \{\mathfrak{c}_0, \mathfrak{c}_1, \mathfrak{c}_2\}$, $f \in \operatorname{Pol}\mathfrak{C}_3^K(A) \cap \mathcal{N}(A,3)$

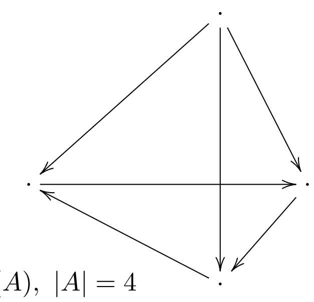
q	$\mathfrak{c}_0(q)$	$\mathfrak{c}_1(q)$	$\mathfrak{c}_2(q)$
$\{a,b,c\}$	a	b	c
$\{a,b,d\}$	b	a	d
$\{a,c,d\}$	c	d	a
$\{b,c,d\}$	d	c	b

f_q	a	b	c	d
a	a	a	c	d
b	b	b	c	d
c	a	b	c	c
d	a	b	d	d

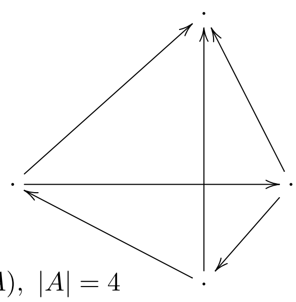
Let $|A| \geq 2$, $a \in A$, $i \in \{0,1\}$ and $\mathfrak{c} \in \mathfrak{C}_r(A)$. Let

$$Z_a^{\mathfrak{c}} = \{b \in A \setminus \{a\} \colon \mathfrak{c}(\{a,b\}) = a\},$$
$$W_i^{\mathfrak{c}} = \{a \in A \colon |Z_a^{\mathfrak{c}}| = i \pmod 2\},$$
$$\mathfrak{C}_2^i(A) = \{\mathfrak{c} \in \mathfrak{C}_2(A) \colon W_{(1-i)}^{\mathfrak{c}} = \varnothing\}.$$

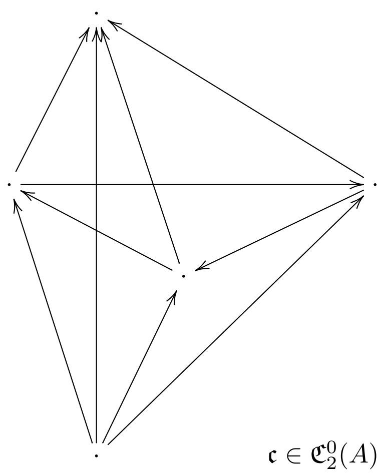
Remark 2.1. Any function $\mathfrak{c} \in \mathfrak{C}_2(A)$ may be represented by the *tournament* $\Gamma_{\mathfrak{c}} = (A, E)$ where $E = \{(a,b) \in A^2 \colon a \neq b \wedge \mathfrak{c}(\{a,b\}) = b\}$. The sets $\mathfrak{C}_2^0(A)$ and $\mathfrak{C}_2^1(A)$ are the sets of all functions $\mathfrak{c} \in \mathfrak{C}_2(A)$ such that the *indegree* of any node of the tournament $\Gamma_{\mathfrak{c}}$ is even (respectively, odd).



$\mathfrak{c} \in \mathfrak{C}_2^0(A)$, $|A| = 4$



$\mathfrak{c} \in \mathfrak{C}_2^1(A)$, $|A| = 4$



$\mathfrak{c} \in \mathfrak{C}_2^0(A)$, $|A| = 5$

We can prove the following proposition.

Proposition 2.2. *1. The sets $\mathfrak{C}_2^0(A)$ and $\mathfrak{C}_2^1(A)$ are symmetric,*

2. $\mathfrak{C}_2^0(A) \neq \varnothing$ iff n equals 0 or $1 \pmod 4$,

3. $\mathfrak{C}_2^1(A) \neq \varnothing$ iff n equals 0 or $3 \pmod 4$,

4. $\mathfrak{C}_2^0(A) \cup \mathfrak{C}_2^1(A) \neq \mathfrak{C}_2(A)$.

Each of the set $\mathfrak{C}_2^0(A)$, $\mathfrak{C}_2^1(A)$, $\mathfrak{C}_2^0(A) \cup \mathfrak{C}_2^1(A)$ is preserved, for example, under the (simple) ternary function $\ell \in \mathcal{N}(A,2)$ defined by

$$\ell_q(x,x,y) = \ell_q(x,y,x) = \ell_q(y,x,x) = y \text{ for all } q \in [A]^2 \text{ and } x,y \in p.$$

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Main Theorem. *Let A be a finite set, r a natural number, and $\mathfrak{D}$ a non-empty proper symmetric subset of the set $\mathfrak{C}_r(A)$. Then the set $\mathfrak{D}$ does not have the Arrow property if and only if one of the following conditions holds:*

1. $r = 2$, $|A|$ *equals* 0 *or* $1 \pmod 4$, *and* $\mathfrak{D} = \mathfrak{C}_2^0(A)$,
2. $r = 2$, $|A|$ *equals* 0 *or* $3 \pmod 4$, *and* $\mathfrak{D} = \mathfrak{C}_2^1(A)$,
3. $r = 2$, $|A| = 0 \pmod 4$, *and* $\mathfrak{D} = \mathfrak{C}_2^0(A) \cup \mathfrak{C}_2^1(A)$,
4. $r = 3$, $|A| = 4$, *and* $\mathfrak{D} = \mathfrak{C}_3^K(A)$.

3 Outline of the proof

Our proof uses some ideas of paper [5] and some results and methods of universal algebra and the theory of closed classes of discrete functions (see [7], [8], [10], [11]). In particular, we largely use the Post classification of closed classes of Boolean functions (see [8] or [9]). We prove some of the auxiliary propositions in a more general form that is needed for the proof of the Main Theorem. This does not make the proof more complicated but can be used for further generalizations. Our proof can be substantially simplified if considering only simple aggregation rules, or only the case of "impossibility": $|A| \geq 5$ and $r \geq 3$.

We use the basic concepts of a *clone*. In universal algebra, a clone $\mathcal{F}$ on a set X is a set of functions $f\colon X^n \to X$, $n < \omega$, such that

1. $\mathcal{F}$ contains all the projections $\pi_i^m\colon X^m \to X$ ($1 \leq m < \omega$, $1 \leq i \leq m$), defined by

$$\pi_i^m(x_1, x_2, \ldots, x_m) = x_i \text{ for all } x_1, x_2, \ldots, x_m \in X$$

2. $\mathcal{F}$ is closed under superposition: if $f, g_1, g_2 \ldots, g_m \in \mathcal{F}$ and f is m-ary, and g_j is n-ary for every j, then the function $h\colon X^n \to X$, defined by
$h(x_1, x_2, \ldots, x_n) =$

$$= f(g_1(x_1, x_2, \ldots, x_n), g_2(x_1, x_2, \ldots, x_n), \ldots, g_m(x_1, x_2, \ldots, x_n))$$

for all $x_1, x_2, \ldots, x_n \in X$, is in $\mathcal{F}$.

Proposition 3.1. *1. The set $\mathcal{N}(A, r)$ is a clone on $\mathfrak{C}_r(A)$.*

2. For any set $\mathbb{D} \subseteq P(\mathfrak{C}_r(A))$ the set $\operatorname{Pol}\mathbb{D}$ (and the set $\operatorname{Pol}\mathbb{D} \cap \mathcal{N}(A, r)$) is a clone on $\mathfrak{C}_r(A)$.

3. For any clone $\mathcal{F} \subseteq \mathcal{N}(A, r)$ and any set $p \in [A]^r$ the set $\{f_p\colon f \in \mathcal{F}\}$ is a clone on p.

4. Let $\mathfrak{D}$ be a symmetrical subset of $\mathfrak{C}(A, r)$ and $\mathcal{F} = \operatorname{Pol}\mathfrak{D} \cap \mathcal{N}(A, r)$. Then the following condition holds:

(∗) for all n-ary function $f \in \mathcal{F}$ and all permutation $\sigma \in S_A$ the function f^σ defined by

$$(\forall p \in [A]^r)\, (\forall \mathbf{a} \in p^n)\, f_p^\sigma(\mathbf{a}) = \sigma^{-1} f_{\sigma p}(\sigma \mathbf{a}),$$

belongs to $\mathcal{F}$.

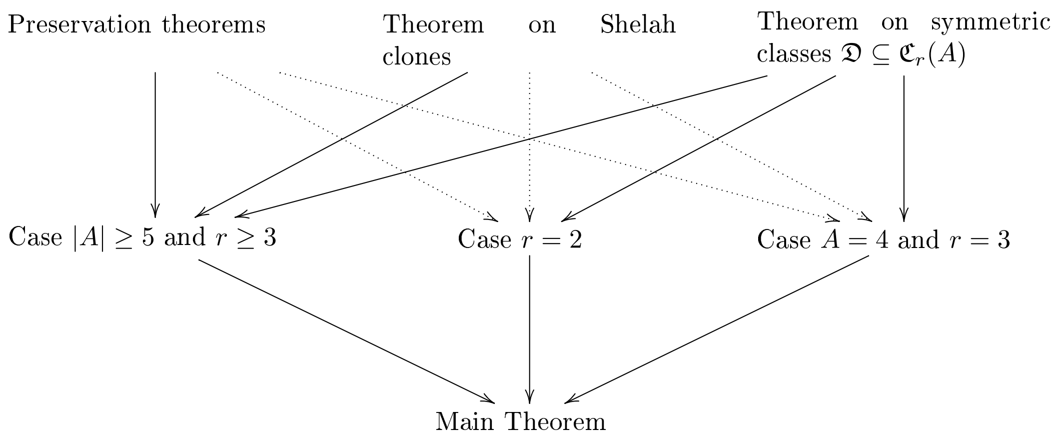
We shall call a *Shelah clone* any clone $\mathcal{F} \subseteq \mathcal{N}(A, r)$ which satisfies condition $(*)$. The main idea of our proof is describing the set $\operatorname{Inv} \mathcal{F}$ for any Shelah clone $\mathcal{F}$.

We prove some "preservation theorems" for clones $\mathcal{F}$ on the set $\mathfrak{C}_r(A)$, satisfying the conditions $\mathrm{L}\Delta^\partial$, $\mathrm{L}\Delta^e_r$ $(r \geq 3)$, $\mathrm{L}\Delta^2$ и $\mathrm{L}\Delta^2_+$ which are defined hereafter; in a "simple" case it will suffice to consider only clones $\mathcal{F}$ on the set A that satisfy simpler conditions Δ^∂, Δ^e_r $(r \geq 3)$ and Δ^2.

Then we prove a theorem on the structure of Shelah clones, which states that "almost all" Shelah clones satisfy one of the above conditions.

Finally we prove a simple theorem on properties of symmetric classes $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$.

The Main Theorem will then be proven in the "impossibility" case ($|A| \geq 5$ and $r \geq 3$) as an immediate consequence of the preceding theorems. The remaining cases are dealt with separately. In addition to those previously proved theorems, we exploit some of the intermediate results and additional considerations.



Preservation theorems. These theorems relate and is of importance to the theory of closed classes of discrete functions.

We will need the following definitions and notations. Let A be a finite non-empty set. For every natural n, elements $\mathbf{a}$ of the cartesian product A^n are considered as functions $\mathbf{a}\colon \{0, 1, \dots, n-1\} \to A$. We denote, as usually, the domain and the range of a function $\mathbf{a} \in A^n$ by $\operatorname{dom} \mathbf{a}$, respectivelly $\operatorname{ran} \mathbf{a}$.

For any natural numbers n, m the set $\{\mathbf{a} \in A^n : |\operatorname{ran} \mathbf{a}| = m\}$ is denoted by A^n_m. Let

$$A^n_{<m} = \bigcup_{k<m} A^n_k, \; A^{<n}_{<m} = \bigcup_{k<m, l<n} A^l_k$$

for any $n, m \in \omega + 1$. The set of all finitary functions on a set X is denoted by the symbol $\mathcal{O}(X)$. For any clone $\mathcal{F} \subseteq \mathcal{O}(A)$ the symbol $\mathcal{F}_{[n]}$ denote the set of all n-ary functions $f \in \mathcal{F}$.

Let r be a natural number. We say that a clone $\mathcal{F} \subseteq \mathcal{O}(A)$ satisfies condition

Δ^e_r, if there is a natural number $i < r$ such that for any sequence $\mathbf{a} \in A^r_r$ and element $a \in \operatorname{ran} \mathbf{a}$ there is a function $w \in \mathcal{F}_{[r]}$ such that

$$w(\mathbf{a}) = a \text{ and } w(\mathbf{b}) = b_i \text{ for any sequence } \mathbf{b} = (b_0, b_1, \dots, b_{r-1}) \in A^r_{<r};$$

Δ^∂, if for any sequence $\mathbf{a} \in A^3_3$ and element $a \in \operatorname{ran} \mathbf{a}$ there is a function $w \in \mathcal{F}_{[3]}$ such that

$$w(\mathbf{a}) = a \text{ and } w(x, x, y) = w(x, y, x) = w(y, x, x) = x \text{ for all } x, y \in A;$$

Δ^2, if for any sequences $\mathbf{a}, \mathbf{b} \in A^2_2$, $\operatorname{ran} \mathbf{a} \neq \operatorname{ran} \mathbf{b}$, and elements $a \in \operatorname{ran} \mathbf{a}$, $b \in \operatorname{ran} \mathbf{b}$ there is a function $w \in \mathcal{F}_{[2]}$ such that

$$w(\mathbf{a}) = a, w(\mathbf{b}) = b \text{ and } w(x, x) = x \text{ for all } x \in A.$$

Let also Q be a finite non-empty set. The set of all function $h\colon Q \to A$ is denoted by the symbol ${}^QA$. Let $H \subseteq {}^QA$ and $f \in \mathcal{O}(A)_{[n]}$. We say that f *preserves* H and H is *preserved* under f, if for all functions $h_0, h_1, \dots, h_{n-1} \in H$ the function h defined by

$$h(q) = f(h_1(q), h_2(q), \dots, h_n(q)) \text{ for all } q \in Q,$$

belongs to H. We say that a set $\mathcal{F} \subseteq \mathcal{O}(A)$ *preserves* H and H is *preserved* under $\mathcal{F}$, if H is *preserved* under any function $f \in \mathcal{F}$.

We show that if a clone $\mathcal{F} \subseteq \mathcal{O}(A)$ satisfies one of conditions Δ^e_r, Δ^∂, Δ^2 and preserves some set $H \subseteq {}^QA$, then the set H is a quite simple one.

For any elements $p, q \in Q$, $a, b \in A$ and permutation $\sigma \in S_A$ we denote

$H_0(p, q, \sigma) = \{h \in {}^QA\colon h(q) = \sigma h(p)\}$;

$H_1(p, q, a, b) = \{h \in {}^QA\colon h(p) = a \vee h(q) = b\}$;

$\mathbb{H}_\leftrightarrow = \{H_0(p, q, \sigma)\colon p, q \in Q, p \neq q, \sigma \in S_A\}$;

$\mathbb{H}_= = \{H_0(p, q, \mathrm{Id})\colon p, q \in Q, p \neq q\}$, where Id is the identity permutation;

$\mathbb{H}_\vee = \{H_1(p, q, a, b)\colon p, q \in Q, p \neq q, a, b \in A\}$.

For any set $H \subseteq {}^QA$, set $Q' \subseteq Q$, set $B \subseteq A$, element $q \in Q$ and natural number r we denote

$H^+_{Q'} = \{h \in {}^QA\colon h \restriction Q' \in H \restriction Q'\}$;

$H(q) = \{h(q)\colon h \in H\}$;

$H^+ = \{h \in {}^QA\colon (\forall q \in Q)\, h(q) \in H(q)\}$ $\left(= \bigcap\{H^+_{\{q\}}\colon q \in Q\}\right)$;

$H^{-1}(B) = \{q \in Q\colon H(q) \in B\}$;

$H^{-1}(< r) = \{q \in Q\colon |H(q)| < r\}$.

Theorem 3.2. *Let A, Q be a finite sets. Let a clone $\mathcal{F} \subseteq \mathcal{O}(A)$ preserve a non-empty set $H \subseteq {}^QA$. Then*

1. *if $\mathcal{F}$ satisfies condition Δ^e_r for some natural number $r \geq 3$, then there is a set $\mathbb{H} \subseteq \mathbb{H}_\leftrightarrow$ such that*
$$H = H^+ \cap H^+_{H^{-1}(<r)} \cap \bigcap \mathbb{H},$$
2. *if $\mathcal{F}$ satisfies condition Δ^∂, then there is a set $\mathbb{H} \subseteq \mathbb{H}_\leftrightarrow \cup \mathbb{H}_\vee$ such that*
$$H = H^+ \cap \bigcap \mathbb{H},$$
3. *if $\mathcal{F}$ satisfies condition Δ^2, then there is a set $\mathbb{H} \subseteq \mathbb{H}_=$ such that*
$$H = H^+ \cap \bigcap\{H^+_{H^{-1}(B)}\colon B \in [A]^2\} \cap \bigcap \mathbb{H}.$$

Then we consider clones on subsets of the set ${}^{Q}A$ (this portion of the proof can be skipped in the "simple" case).

Let $C \subseteq {}^{Q}A$. We say that a function $f \in \mathcal{O}(C)$ *preserves* a set $H \subseteq C$ and a set H is *preserved* under a function f, if for all functions $h_0, h_1, \dots, h_{n-1} \in H$ the function

$$h = f(h_1, h_2, \dots, h_n)$$

belong to H. We say that a set $\mathcal{F} \subseteq \mathcal{O}(C)$ *preserves* a set H and a set H is *preserved* under a set $\mathcal{F}$, if H is *preserved* under any function $f \in \mathcal{F}$. In other terms, $\mathcal{F}$ preserves H iff H is a support of some subalgebra of algebra $(C; \mathcal{F})$.

A function $f \in \mathcal{O}(C)_{[n]}$ is *almost normal* if for any $q \in Q$ there is a function $f_q \colon C(q) \to C(q)$ such that

$$f(h_0, h_1, \dots, h_{n-1})(q) = f_q(h_0(q), h_1(q), \dots, h_{n-1}(q))$$

for all $h_0, h_1, \dots, h_{n-1} \in C$ and $q \in Q$.

We denote the set of all almost normal functions $f \in \mathcal{O}(C)$ by the symbol $\mathcal{AN}(C)$. Obviously, $\mathcal{AN}(C)$ is a clone on C. For any clone $\mathcal{F} \subseteq \mathcal{AN}(C)$ and $q \in Q$ the set $\{f_q \colon f \in \mathcal{F}\}$ is denoted by $\mathcal{F}_q$. The set $\mathcal{F}_q$ is a clone on $C(q)$.

A function $f \in \mathcal{AN}(C)_{[n]}$ is *simple* if

$$f_p(\mathbf{a}) = f_q(\mathbf{a}) \text{ for all } p, q \in Q \text{ and } \mathbf{a} \in (C(p))^n \cap (C(q))^n.$$

We say that a clone $\mathcal{F} \subseteq \mathcal{AN}(C)$ is *simple* if any function $f \in \mathcal{F}$ simple. We denote by the symbol SIM the condition "clone $\mathcal{F}$ is simple".

For clones $\mathcal{F} \subseteq \mathcal{AN}(C)$ we can define "local analogs" of the definitions introduced above for clones $\mathcal{F} \subseteq \mathcal{O}(A)$. We say that a clone $\mathcal{F} \subseteq \mathcal{AN}(C)$ satisfies condition

$\mathrm{L}\Delta^e_r$, if there is a natural number $i < r$ such that for any $q \in Q$, $\mathbf{a} \in (C(q))^r_r$ and $a \in \operatorname{ran} \mathbf{a}$ there is a function $w \in \mathcal{F}_{[r]}$ such that

$$w_q(\mathbf{a}) = a \text{ and } w_p(\mathbf{b}) = b_i \text{ for any } p \in Q \text{ and } \mathbf{b} = (b_0, b_1, \dots, b_{r-1}) \in (C(p))^r_{<r};$$

$\mathrm{L}\Delta^\partial$, if for any $q \in Q$, $\mathbf{a} \in (C(q))^3_3$ and $a \in \operatorname{ran} \mathbf{a}$ there is a function $w \in \mathcal{F}_{[3]}$ such that

$$w_q(\mathbf{a}) = a \text{ and } w_p(x, x, y) = w_p(x, y, x) = w_p(y, x, x) = x \text{ for all } p \in Q \text{ and } x, y \in C(p);$$

$\mathrm{L}\Delta^2$, if for any $p, q \in Q$, $\mathbf{a} \in (C(p))^2_2$, $\mathbf{b} \in (C(q))^2_2$, $\operatorname{ran} \mathbf{a} \neq \operatorname{ran} \mathbf{b}$, $a \in \operatorname{ran} \mathbf{a}$, $b \in \operatorname{ran} \mathbf{b}$ there is a function $w \in \mathcal{F}_{[2]}$ such that

$$w_p(\mathbf{a}) = a, w_q(\mathbf{b}) = b \text{ and } w_s(x, x) = x \text{ for all } s \in Q \text{ and } x \in C(s).$$

We will also formulate a condition in some sense dual to the condition $\mathrm{L}\Delta^2_2$. For any natural number n the *disjoint union* of the family $\{(C(q))^n \colon q \in Q\}$ is denoted by the symbol $C^{\langle n \rangle}$, i.e.

$$C^{\langle n \rangle} = \bigcup_{q \in Q} (\{q\} \times (C(q))^n).$$

For any clone $\mathcal{F} \subseteq \mathcal{AN}(C)$ and sequences $(p, \mathbf{a}), (q, \mathbf{b}) \in C^{\langle n \rangle}$ we will say that $\mathcal{F}$

– *separates* $(p, \mathbf{a})$ *from* $(q, \mathbf{b})$, if there is a natural number k and sequence $\mathbf{c} \in (C(p))^k$ such that for any sequence $\mathbf{d} \in (C(q))^k$ and elements $a \in \operatorname{ran} \mathbf{a}$ and $b \in \operatorname{ran} \mathbf{b}$ there is a function $w \in \mathcal{F}_{[n+k]}$ such that

$$w_p(\mathbf{ac}) = a, w_q(\mathbf{bd}) = b \text{ and } w_s(xx\dots x) = x \text{ for any } s \in Q,\ x \in C(s)$$

– *separates* $(p, \mathbf{a})$ *and* $(q, \mathbf{b})$ if $\mathcal{F}$ separates $(p, \mathbf{a})$ from $(q, \mathbf{b})$ or $(q, \mathbf{b})$ from $(p, \mathbf{a})$.

We say that a clone $\mathcal{F} \subseteq \mathcal{AN}(C)$ satisfies condition

$\mathrm{L}\Delta^2_+$, if there is a set $\Lambda \subseteq Q^2$ such that
1. for any non-empty $P \subsetneq Q$ there is a pair $(p, q) \in \Lambda$ such that $p \in P$ and $q \notin P$,
2. for any pair $(p, q) \in \Lambda$ and sequences $\mathbf{a} \in (C(p))^2_2$, $\mathbf{b} \in (C(q))^2_2$ the clone $\mathcal{F}$ separates $p\mathbf{a}$ and $q\mathbf{b}$.

Theorem 3.3. *Let A, Q be a finite sets and $C \subseteq {}^QA$. Let a clone $\mathcal{F} \subseteq \mathcal{AN}(C)$ preserve a non-empty set $H \subseteq C$. Then*

1. *if $\mathcal{F}$ satisfies condition $\mathrm{L}\Delta^e_r$ for some natural number $r \geq 3$, then there is a set $\mathbb{H} \subseteq \mathbb{H}_\leftrightarrow$ such that*
$$H = H^+ \cap H^+_{H^{-1}(<r)} \cap \bigcap \mathbb{H},$$
2. *if $\mathcal{F}$ satisfies condition $\mathrm{L}\Delta^\partial$, then there is a set $\mathbb{H} \subseteq \mathbb{H}_\leftrightarrow \cup \mathbb{H}_\vee$ such that*
$$H = H^+ \cap \bigcap \mathbb{H},$$
3. *if $\mathcal{F}$ satisfies condition $\mathrm{L}\Delta^2 \wedge \mathrm{SIM}$, then there is a set $\mathbb{H} \subseteq \mathbb{H}_=$ such that*
$$H = H^+ \cap \bigcap \{H^+_{H^{-1}(B)} \colon B \in [A]^2\} \cap \bigcap \mathbb{H}.$$
4. *if $\mathcal{F}$ satisfies condition $\mathrm{L}\Delta^2_+$, then $H = H^+$.*

Theorem on Shelah clones. We will use the following notations.

The set of all projections on a set X is denoted by $\mathcal{E}(X)$. For any clone $\mathcal{F} \subseteq \mathcal{O}(X)$, $\mathcal{F} \neq \mathcal{E}(X)$, the natural number $\mathrm{r}(\mathcal{F})$ is defined by

$$\mathrm{r}(\mathcal{F}) = \min\{n < \omega \colon \mathcal{F}_{[n]} \neq \mathcal{E}_{[n]}\}.$$

Let $\mathrm{r}(\mathcal{F}) = \omega$ if $\mathcal{F} = \mathcal{E}(X)$.

Let $C \subseteq {}^QA$. For any clone $\mathcal{F} \subseteq \mathcal{AN}(C)$ the natural number $\mathrm{r}^+(\mathcal{F})$ is defined by

$$\mathrm{r}^+(\mathcal{F}) = \min\{\mathrm{r}(\mathcal{F}_q) \colon q \in Q\}.$$

Obviously, $\mathrm{r}(\mathcal{F}) \leq \mathrm{r}^+(\mathcal{F})$ and $\mathrm{r}(\mathcal{F}) = \mathrm{r}^+(\mathcal{F})$ if $\mathcal{F}$ is simple.

The function $f \in \mathcal{AN}(C)_{[n]}$ is called *normal* if

$$f_q(\mathbf{a}) \in \operatorname{ran} \mathbf{a} \text{ for all } q \in Q \text{ and } \mathbf{a} \in (C(q))^n.$$

We denote by the symbol $\mathcal{N}(C)$ the set of all normal function $f \in \mathcal{AN}(C)$. We shall write $\mathcal{N}(A, r)$ instead of $\mathcal{N}(\mathfrak{C}_r(A))$ (see Section 1).

Theorem 3.4. *Let A be a finite set, $|A| \geq 2$, and let r be a natural number, $2 \leq r \leq |A|$. Let $\mathcal{F}$ be a Shelah clone on $\mathfrak{C}_r(A)$ and $\mathrm{r}(\mathcal{F}) < \omega$. Then $2 \leq \mathrm{r}(\mathcal{F}) \leq \max\{r, 3\}$, and*

1. *if* $\mathrm{r}(\mathcal{F}) \geq 4$*, then the clone* $\mathcal{F}$ *satisfies condition* $\mathrm{L}\Delta^{e}_{\mathrm{r}(\mathcal{F})}$*;*

2. *if* $\mathrm{r}(\mathcal{F}) = 3$*, then the clone* $\mathcal{F}$ *satisfies one of conditions* $\mathrm{L}\Delta^{e}_{3}$*,* $\mathrm{L}\Delta^{\partial}$*;*

3. *if* $\mathrm{r}(\mathcal{F}) = 2$ *and* $\mathrm{r}^{+}(\mathcal{F}) \geq 3$*, then the clone* $\mathcal{F}$ *satisfies conditions* $\mathrm{L}\Delta^{2}_{+}$*;*

4. *if* $\mathrm{r}(\mathcal{F}) = \mathrm{r}^{+}(\mathcal{F}) = 2$ *and* $|A| \geq 5$*, then the clone* $\mathcal{F}$ *satisfies one of conditions* $\mathrm{L}\Delta^{2}_{+}$*,* $\mathrm{L}\Delta^{2} \wedge \mathrm{SIM}$.

For the case $\mathrm{r}(\mathcal{F}) \geq 3$ our proof is based on a lemma that relates the clone $\mathcal{F}$ to some Post class. Here is a simple variation of the lemma.

The Post class of all **0**- *and* **1**-*preserving self-dual* Boolean functions is denoted by D_1 (see [8] or [9]). Let X is a set and Π is a subclass of D_1. For every natural number n and function $\pi \in \Pi_{[n]}$ the function $\pi_X \colon X^n_{\leq 2} \to X$ is defined by

$$\pi_X(\mathbf{a}) = \sigma^{-1}\pi(\sigma\mathbf{a}) \ \text{ for all } \mathbf{a} \in A^n_{\leq 2},$$

where σ is an arbitrary injective mapping from $\operatorname{ran}\mathbf{a}$ to $\{0,1\}$. We denote

$$\Pi_X = \{\pi_X \colon \pi \in \Pi\}.$$

Let $\mathcal{F}$ is a clone on X. For any natural number r we denote

$$F_{\langle r\rangle} = \bigcup_{n<\omega} \{f \restriction X^n_{<r} \colon f \in \mathcal{F}_{[n]}\}.$$

Lemma 3.5. *Let* $\mathcal{F}$ *be a clone on* X *and* $\mathrm{r}(\mathcal{F}) \geq 3$*. Then*

1. *there is a Post class* $\Pi \subseteq D_1$ *such that* $\mathcal{F}_{\langle 3\rangle} = \Pi_X$*;*

2. *if* $\mathrm{r}(\mathcal{F}) \geq 4$ *then* $\mathcal{F}_{\langle \mathrm{r}(\mathcal{F})\rangle} = \mathcal{E}(X)_{\langle \mathrm{r}(\mathcal{F})\rangle}$.

We then use some simple properties of Post classes $\Pi \subseteq D_1$. There are only four Post classes O_1, D_1, D_2 and L_4 (in Post's notation) of **0**- and **1**-preserving self-dual Boolean functions. They are respectively generated by the functions x, $\overline{x}y \vee \overline{x}z \vee yz$, $xy \vee yz \vee xz$ and $x \oplus y \oplus z$.

The cases $\mathrm{r}(\mathcal{F}) = 2 \wedge \mathrm{SIM}$ and $\mathrm{r}(\mathcal{F}) = 2 \wedge \mathrm{r}^{+}(\mathcal{F}) \geq 3$ present no difficulties.

In case $\mathrm{r}(\mathcal{F}) = \mathrm{r}^{+}(\mathcal{F}) = 2 \wedge \neg\mathrm{SIM}$ we prove that $\mathcal{F}$ separates $p\mathbf{a}$ and $q\mathbf{b}$ for all $p, q \in [A]^r$, $p \neq q$, $\mathbf{a} \in p^2_2$.

Theorem on symmetric sets $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$.

Theorem 3.6. *Let* A *be a finite set and* r *a natural number,* $2 \leq r < |A|$*. Let* $\mathfrak{D}$ *be a symmetric subset of* $\mathfrak{C}_r(A)$*. Then one of the following conditions holds:*

1. $|A| = 3$*,* $r = 2$ *and* $\mathfrak{D} = \mathfrak{C}^1_2(A)$*;*

2. $|A| = 4$*,* $r = 3$ *and* $\mathfrak{D} = \mathfrak{C}^K_3(A)$*;*

3. $\mathfrak{D} \cap H = \varnothing$ *for any* $H \in \mathbb{H}_{\leftrightarrow} \cup \mathbb{H}_{\vee}$.

From theorems 3.6, 3.4 and 3.3 the case of "impossibility" of the Main Theorem can be easily inferred.

Theorem 3.7. *Let* A *be a finite set and* r *a natural number. Let* $\mathfrak{D}$ *be a non-empty proper symmetric subset of the set* $\mathfrak{C}_r(A)$*. Then if* $|A| \geq 5$ *and* $r \geq 3$*, the set* $\mathfrak{D}$ *have the Arrow property.*

Case $r = 2$. The proof starts with the following proposition.

Proposition 3.8. *Let a non-empty proper symmetric subset $\mathfrak{D}$ of the set $\mathfrak{C}_2(A)$, $|A| < \omega$, do not have the Arrow property. Then the set $\mathfrak{D}$ is preserved under the normal (simple) function $\ell\colon (\mathfrak{C}_2(A))^3 \to \mathfrak{C}_2(A)$ defined by*

$$\ell_q(x, x, y) = \ell_q(x, y, x) = \ell_q(y, x, x) = y \text{ for all } q \in [A]^2 \text{ and } x, y \in p.$$

.

If $|A| \geq 5$, this Proposition follow from Theorems 3.6, 3.4, 3.3, Lemma 3.5 and the structure of classes $\Pi \subseteq D_1$ (in case $|A| \in \{3, 4\}$, additional arguments).

Then we use the properties of linear structures over $\mathbb{Z}_2$.

Case $|A| = 4$ **and** $r = 3$. This case is rather difficult (though it can be investigated by using an appropriate look-up computer program).

The set $\mathfrak{C}_3^K(A)$ is preserved under any normal simple binary function $f \in \mathcal{N}(A, 3)$ satisfying the condition

$$\sigma f_q(\mathbf{a}) = f_{\sigma q}(\sigma \mathbf{a}) \text{ for all } q \in [A]^3,\ \mathbf{a} \in q^2 \text{ and } \sigma \in K.$$

To prove that no other cases, we show that clone $\mathcal{F}$ satisfies one of the conditions $\mathrm{L}\Delta_3^{\mathfrak{e}}$, $\mathrm{L}\Delta^{\partial}$, $\mathrm{L}\Delta_+^2$ and $\mathrm{L}\Delta^2 \wedge \mathrm{SIM}$. Then it remains to use Theorems 3.6 and 3.3.

4 Concluding remarks

The Main Theorem and impossibility theorems. Our Main Theorem is not formulated as an "impossibility theorem", because it demonstrates that some of the sets of choice functions do not satisfy the Arrow property. However, by considering aggregation rules that satisfy some additional condition, we can formulate a corollary that is an impossibility theorem for all non-empty proper symmetric subsets $\mathfrak{D}$ of the set $\mathfrak{C}_r(A)$.

Proposition 4.1. *Let A be a finite set. Let $\mathfrak{D}$ be a non-empty proper symmetric subset of the set $\mathfrak{C}_2(A)$. Let $\mathfrak{D}$ do not have the Arrow property. Then if $|A| \geq 5$, the clone* $\mathrm{Pol}\,\mathfrak{D} \cap \mathcal{N}(A, r)$ *is generated by the normal (simple) function $\ell\colon (\mathfrak{C}_2(A))^3 \to \mathfrak{C}_2(A)$ defined by*

$$\ell_p(x, x, y) = \ell_p(x, y, x) = \ell_p(y, x, x) = y \text{ for all } p \in [A]^2 \text{ and } x, y \in p.$$

(A clone $\mathcal{F}$ is *generated* by a function $f \in \mathcal{O}(X)$ if $\mathcal{F}$ is the minimal clone on X which contains f).

We will call a normal aggregation rule $f\colon (\mathfrak{C}_2(A))^n \to \mathfrak{C}_2(A)$ *conjectural*, if there exist a set $I \subseteq \{0, 1, \ldots, n-1\}$, $|I|$ is odd, such that

$$f_q(a_0, a_1, \ldots, a_{n-1}) = a_j \leftrightarrow |\{i \in I\colon a_i = a_j\}| \text{ is odd}$$

for all $q \in [A]^2$, $a_0, a_1, \ldots, a_{n-1} \in q$ and $j \in \{0, 1, \ldots, n-1\}$.

Proposition 4.2. *The clone $\mathcal{F}$ on $\mathfrak{C}_2(A)$ generated by the function ℓ from Proposition 4.1 is the set of all conjectural aggregation rules.*

Therefore, the following impossibility theorem is true:

Theorem 4.3. *Let A be a finite set, $|A| \geq 5$, and let $\mathfrak{D}$ be a non-empty proper symmetric subset of the set $\mathfrak{C}_r(A)$ for some natural number r. Then there exists no normal non-dictatorial and non-conjectural aggregation rule f which preserves the set $\mathfrak{D}$.*

Obviously, this theorem remains true if the condition of being "non-conjectural" is replaced with a stronger condition of being "monotone".

A normal aggregation rules $f\colon (\mathfrak{C}_r)^n \to \mathfrak{C}_r$ is called *monotone* if

$$\{i < n\colon a_i = a\} \subseteq \{i < n\colon b_i = a\} \to (f_q(\mathbf{a}) = a \to f_q(\mathbf{b}) = a)$$

for all $q \in [A]^r$, $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1}), \mathbf{b} = (b_0, b_1, \ldots, b_{n-1}) \in q^n$ and $a \in \operatorname{ran}\mathbf{a} \cap \operatorname{ran}\mathbf{b}$.

We should also note that Main Theorem provides yet another proof of the classical Arrow's impossibility theorem.

Applications and further development of the method. The Shelah's *clonal approach* can be used for solving further problems of the social choice theory. We believe it is appropriate for researching into the following type of questions:

- *Given a non-empty symmetric set of preferences $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ and an aggregation rule $f\colon (\mathfrak{C}_r(A))^n \to \mathfrak{C}_r(A)$, find all sets $\mathfrak{D}' \subseteq \mathfrak{D}$ belonging to* $\operatorname{Inv} f$ *(including non-symmetric ones).*

To solve this problem means to answer the following question: what kind of "agreements" should be reached by the vote participants in order for their collective choice to be kept within the limits of the set $\mathfrak{D}$?

We can suggest one good example. Let$A = \{a, b, c\}$; consider the set $\mathfrak{R}_2(A)$ of rational choice functions $\mathfrak{c} \in \mathfrak{C}_2(A)$, and a function $maj\colon (\mathfrak{C}_2(A))^{2k+1} \to \mathfrak{C}_2(A)$ which is the *majority rule*, i.e.

$$maj_q(\mathbf{a}) = a \leftrightarrow |\{i < 2k+1\colon a_i = a\}| > k$$

for all $q \in [A]^2$, $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1}) \in q^n$ and $a \in \operatorname{ran}\mathbf{a}$. Then the set $\mathfrak{R}_{a<b} = \{\mathfrak{c} \in \mathfrak{R}\colon \mathfrak{c}(\{a,b\})\} = b\}$ and the set $\mathfrak{R}_{b<a} = \{\mathfrak{c} \in \mathfrak{R}\colon \mathfrak{c}(\{a,b\})\} = a\}$ are preserved by the rule maj.

Really, $\mathfrak{R}_{a<b} = \{\mathfrak{r}_0, \mathfrak{r}_1, \mathfrak{r}_2\}$ where

q	$\mathfrak{r}_0(q)$	$\mathfrak{r}_1(q)$	$\mathfrak{r}_2(q)$
$\{a,b\}$	b	b	b
$\{b,c\}$	c	b	b
$\{a,c\}$	c	c	a

Let $(\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{2k}) \in (\mathfrak{R}_{a<b})^{2k+1}$ and $I_j = \{i < 2k+1\colon \mathfrak{c}_i = \mathfrak{r}_j\}$, $j \in \{0,1,2\}$.

If $|I_j| > k$ for some $j \in \{0,1,2\}$, we have

$$maj(\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{2k}) = \mathfrak{r}_j.$$

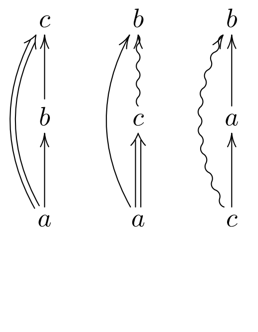
Conversely, $|I_j \cup I_{j'}| > k$ for all $j, j' \in \{0,1,2\}$, $j \neq j'$, and we have

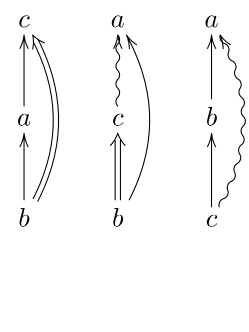
$$|\{i < 2k+1\colon \mathfrak{c}_i(\{b,c\}) = b\}| > k,\ |\{i < 2k+1\colon \mathfrak{c}_i(\{a,c\}) = c\}| > k,$$

and, so,

$$maj(\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{2k}) = \mathfrak{r}_1.$$

For $\mathfrak{R}_{b<a}$ analogously.

$\mathfrak{R}_{a<b}$ 

 $\mathfrak{R}_{b<a}$

By applying our above-formulated theorems, we can generalize this example in order to obtain some "anti-Arrow" theorem.

Application of the method in a more general context may consist in the following. We could hopefully expand our Main Theorem to the case when preferences are functions $h \colon Q \to A$ on some set Q of "Situations" (see Theorems 3.2 and 3.3). Given that, "Situations" here could be understood as sets $B \in [A]^r$ bearing some additional structure, e.g. sequences $\mathbf{b} \in A^{<k}$ with $\operatorname{ran} \mathbf{b} = B$, or *multisets*, or graphs with vertices from B, or *weighting functions* $\mu \colon B \to R$, where R is the set of real numbers, etc.

Yet other generalization of our method would consist in studying "imprecise" preferences, represented as functions $\mathfrak{d} \colon [A]^r \to [A]^t$, where $\mathfrak{d}(q) \subseteq q$ for all $q \in [A]^r$. Here, (simple) aggregation rules could be functions $f \colon ([A]^t)^n \to [A]^t$ for which

$$f(B_0, B_1, \ldots B_{n-1}) \subseteq \bigcup_{0 \le i < n} B_i$$

for all $B_0, B_1, \ldots, B_{n-1} \in [A]^t$.

The definition of "*aggregation rule f preserving a set of preferences $\mathfrak{D}$*" can be introduced here in a natural way, and the clone effect takes place.

Question. Let $\mathcal{PN}(A, r)$ denote the set of all almost normal aggregation rules $f \in \mathcal{O}(A, r)$ (see Remark 1.3) that satisfy the unanimity condition:

2′. $f_p(a, a, \ldots, a) = a$ for all $p \in [A]^r$ and $a \in p$.

Would then the Main Theorem (or at least the theorem 3.7) hold, if the Arrow property is understood as the condition $\operatorname{Pol} \mathfrak{D} \cap \mathcal{PN}(A, r) = \mathcal{M}(A, r)$)?

The "simple" case follows by Main Theorem.

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