A proof-theoretic view on preference relations and choice functions

Paolo Maffezioli [1]

Abstract

The paper develops a Gentzen-style framework for reasoning about basic notions of social choice theory as preference relations and choice functions. First, it aims at providing an inferentialist account of the meaning of such notions in terms of the inference rules governing their use in formal derivations: to this end it is shown how to formulate the axioms of preference relations (reflexivity, completeness, etc.) and choice functions (Sen's properties α and β, etc.) by means of introduction and elimination rules of natural deduction. The second aim is to provide a logical calculus in which derivations can be automatically generated: a cut and contraction-free sequent system is then introduced and it is shown how the fundamental property of eliminability of the cut rule allows to search systematically for derivations.

Since many properties of order relations have an immediate interpretation in terms of preference, order-theoretic notions have played a central role in social choice theory. Traditionally, such properties are formulated in the language of predicate logic (with or without identity) and presented as axioms of Hilbert-style calculi. Since axiomatic theories are difficult to use in practice to derive theorems from axioms and consequently to regiment the proofs in ordinary mathematics, we present the first-order theory of preference relation as calculus based on rules of inference, following the tradition originated with Genzten [2]. In the first part of the paper we heavily rely on [4] where several calculi of sequents for order theories have been presented. Unlike orders in mathematics, preferences in social choice theory are intrinsically multi-agent notions: although it makes sense to think of an option as preferred to another one, it is far more interesting to say that each agent in a group has a preference and to see what happen whens their preferences are combined into a collective one. Secondly, preference are choice-guiding, i.e they must allow agents to choose the best options. Although we will be focusing mostly (Sections 1 and 2) on the second issue by providing a proof-theoretic account of the connection between preferences and choices, in the last section we show how to extend the framework to collective preference.

Individual Preference Relations

Regardless of whether taken as primitive or defined, there are three most important order-theoretic notions that provide an account of preference relations.

$x > y$	x is strictly preferred to y	(or, x is better than y)
$x \sim y$	x and y are indifferent	(or, x and y are equally good)
$x \geqslant y$	x is weakly preferred to y	(or, x is at least good as y)

If weak preference is taken to be primitive then strict preference and indifference will be defined as follows:

$$x > y \quad =_{df} \quad x \geqslant y \text{ and } y \ngeqslant x$$
$$x \sim y \quad =_{df} \quad x \geqslant y \text{ and } y \geqslant x$$

Alternatively, we could take both $>$ and $\sim$ as primitive and define $\geqslant$ as their union, i.e.

[1] Discussions with N. Tennant have been an invaluable source of inspiration.

$x \sim y \quad =_{df} \quad x > y$ or $x \geqslant y$

Usually, formal simplicity is preferred to conceptual clarity and therefore $\geqslant$ is taken as the only primitive relation. In the standard theory of orders, $\geqslant$ is assumed to be a partial order, i.e. a reflexive ($x \geqslant x$), transitive ($x \geqslant y$ and $y \geqslant z$ implies $x \geqslant z$), and antisymmetric ($x \geqslant y$ and $y \geqslant x$ implies $x = y$) binary relation. However, the latter does not correspond to any intuitively acceptable property of preference relation because it makes perfectly sense to think of two equally good distinct alternatives, i.e. $x \sim$ but $x \neq y$ which contradicts antisymmetry. Therefore, we shall assume that $\geqslant$ is only reflexive and transitive, a preorder. These properties will be given as rules of natural deduction [2].

$$\frac{}{x \geqslant x} \qquad \frac{x \geqslant y \quad y \geqslant z}{x \geqslant z}$$

In most applications it turns out to be useful to assume that $\geqslant$ is complete, i.e. either $x \geqslant y$ or $y \geqslant x$ holds. This corresponds to the following rule (with φ an arbitrary formula)

$$\frac{\begin{array}{c}\overline{x \geqslant y}^{\,(i)} \\ \vdots \\ \varphi\end{array} \quad \begin{array}{c}\overline{y \geqslant x}^{\,(i)} \\ \vdots \\ \varphi\end{array}}{\varphi}\,(i)$$

The label (i) denotes that the formulas below the inference line are discharged, i.e. the conclusion does not depend on them. Also the meanings of the defined relations $>$ and $\sim$ are given inferentially. For each notion, an introduction rule makes explicit the conditions under which $x > y$ or $x \sim y$ can be concluded.

$$\frac{x \geqslant y \quad y \not\geqslant x}{x > y} \qquad \frac{x \geqslant y \quad y \geqslant x}{x \sim y}$$

Conversely, elimination rules specify what can be concluded from $x > y$ or $x \sim y$. Elimination rules presented here are in a general form (also called "parallel"in [7]) . The standard (or "serial"in [7]) rules are special cases when φ and the discharged formula coincide.

$$\frac{x > y \quad \begin{array}{c}\overline{x \geqslant y}^{\,(i)} \\ \vdots \\ \varphi\end{array}}{\varphi}\,(i) \qquad \frac{x > y \quad \begin{array}{c}\overline{y \not\geqslant x}^{\,(i)} \\ \vdots \\ \varphi\end{array}}{\varphi}\,(i) \qquad \frac{x \sim y \quad \begin{array}{c}\overline{x \geqslant y}^{\,(i)} \\ \vdots \\ \varphi\end{array}}{\varphi}\,(i) \qquad \frac{x \sim y \quad \begin{array}{c}\overline{y \geqslant x}^{\,(i)} \\ \vdots \\ \varphi\end{array}}{\varphi}\,(i)$$

From the fact that $\geqslant$ is a preorder (together with the definition of $>$ and $\sim$ in terms of $\geqslant$) it follows that (i) $>$ is an irreflexive, asymmetric and transitive; (ii) $\sim$ is an equivalence relation; (iii) $>$ and $\sim$ are incompatible. One of the properties we shall use in the next and that can be easily derived is the equivalence of $x \geqslant y$ and $y \not> x$ which essentially depends on the the completeness of $\geqslant$.

$$\dfrac{\dfrac{x \geqslant y \qquad \dfrac{\overline{y > x}^{\,(2)} \quad \overline{x \not\geqslant y}^{\,(1)}}{x \not\geqslant y}\,(1)}{\bot}}{y \not> x}\,(2) \qquad \dfrac{\overline{x \geqslant y}^{\,(2)} \qquad \dfrac{\dfrac{y \not> x \qquad \dfrac{\overline{y \geqslant x}^{\,(2)} \quad \overline{x \not\geqslant y}^{\,(1)}}{y > x}}{\bot}}{x \geqslant y}\,(1)}{x \geqslant y}\,(2)$$

More relevantly to the interpretation of order relation as preferences, we have the principles isolated by A. Sen [6]. For instance, the derivation below correspond to the validity of Sen's

PI principle.

$$\dfrac{\dfrac{\dfrac{x>y \quad \dfrac{}{x\geqslant y}\,{\scriptstyle(1)}}{x\geqslant y}\,{\scriptstyle(1)} \quad \dfrac{y\sim z \quad \dfrac{}{y\geqslant z}\,{\scriptstyle(2)}}{y\geqslant z}\,{\scriptstyle(2)}}{x\geqslant z} \quad \dfrac{\dfrac{\dfrac{\dfrac{y\sim z \quad \dfrac{}{y\geqslant z}\,{\scriptstyle(3)}}{y\geqslant z} \quad \dfrac{}{z\geqslant x}\,{\scriptstyle(5)}}{y\geqslant x} \quad \dfrac{x>y \quad \dfrac{}{y\ngeqslant x}\,{\scriptstyle(4)}}{y\ngeqslant x}\,{\scriptstyle(4)}}{\bot}}{z\ngeqslant x}\,{\scriptstyle(5)}}{x>z}$$

It is standard to present these properties as axioms of an axiomatic theory based on some standard Hilbert calculus for predicate logic. An attractive feature of Hilbert systems is modularity: axioms can be added or omitted in a component-wise fashion. However, maintaining modularity is problematic when one deals with systems based on rules of inference, since the fundamental properties of the system may be irremediably lost in presence of new axioms. The situation is more clear in calculi of sequents than in natural deduction. In sequent calculus, some extension of a well-behaved system by a new axiom may result into one that fails to satisfy the cut-elimination theorem. Consider Gentzen's **LK** calculus for first order logic [2]. His celebrated theorem of cut elimination establishes that the rule of cut

$$\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta}\ \mathit{Cut}$$

is eliminable in **LK**, i.e. all the sequents provable with cut can be also proved without it. Suppose we now want to extend **LK** with sequents corresponding to the properties of $\sim$ being an equivalence relation. The most natural solution is to allow derivations to start with the following initial sequents

$$\Rightarrow x \sim x \quad , \quad x \sim y \Rightarrow y \sim x \quad \text{and} \quad x \sim y, y \sim z \Rightarrow x \sim z$$

However, such "axiomatic sequents" make cut elimination fail: there is a sequent that is indeed derivable, but not without the use of cut.

$$\dfrac{x \sim y \Rightarrow y \sim x \quad y \sim x, x \sim z \Rightarrow y \sim z}{x \sim y, x \sim z \Rightarrow y \sim z}\ \mathit{Cut}$$

A possible way out is to consider inference rules instead of axiomatic sequents.

$$\dfrac{x \sim x \Rightarrow}{\Rightarrow} \qquad \dfrac{y \sim x \Rightarrow}{x \sim y \Rightarrow} \qquad \dfrac{x \sim z \Rightarrow}{x \sim y, y \sim z \Rightarrow}$$

Rules and axioms are deductively equivalent, i.e. the axiomatic sequents are derivable in the presence of the rules, and the rules are admissible in the presence of the axiomatic sequents. The difference is that using the rules the sequent $x \sim y, x \sim z \Rightarrow y \sim z$ has a cut-free derivation.

$$\dfrac{\dfrac{y \sim z \Rightarrow y \sim z}{y \sim x, x \sim z \Rightarrow y \sim z}}{x \sim y, x \sim z \Rightarrow y \sim z}$$

A natural question is: what class of axioms can be rearranged into inference rules while preserving cut-elimination? In other words, what axiomatic theories can be formulated as cut-free systems of rules? The idea of inferentializing axioms of mathematical theories has been variously developed by many authors, but for the purposes of this paper we are mostly interested in the one presented for order theories in [4]. Let P and Q denote atomic formulas of the form $x \geqslant y$, $x > y$ and $x \sim y$. In classical logic we have that every quantifier-free formula is equivalent to some formula in conjunctive normal form, that is, to a conjunction of disjunctions of atomic formulas or negation of atomic formulas. Each conjunct is then of the form $\neg P_1 \vee \ldots \vee \neg P_m \vee Q_1 \vee \ldots \vee Q_n$ and it is classically equivalent to $P_1 \wedge \ldots \wedge P_m \to Q_1 \vee \ldots \vee Q_n$ Special cases are when $m = 0$, where the implication reduces to $Q_1 \vee \ldots \vee Q_n$, and when $n = 0$ where it is $\neg(P_1 \wedge \ldots \wedge P_m)$. The universal closure of this implication is called a regular formula and corresponds to scheme of the form

$$\frac{Q_1, P_1, \ldots, P_m, \Gamma \Rightarrow \Delta \quad \ldots \quad Q_n, P_1, \ldots, P_m, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_m, \Gamma \Rightarrow \Delta}$$

Notice that the rule scheme is cumulative in the sense that it has the principal formulas $P_1, \ldots, P_m$ repeated in the premises. We assume that rules following the regular rule scheme are added to a variant of the system **LK**, known in [8] as **G3c**, in which all the structural rules, including weakening and contraction, are admissible. It is possible to prove that the resulting system is still cut free and all the structural rules are admissible (see [4] for details). In practice, the regular rule scheme specializes into many rules, depending on the context of application. In our case, all the properties of preference relations we encountered so far are regular formulas; reflexivity of $\geqslant$ is just a regular formula with $m = 0$, whereas for irreflexivity of $>$ we have $n = 0$. Therefore the fact that $\geqslant$ is a complete preorder is formulated as

$$\frac{x \geqslant x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Ref}_{\geqslant} \qquad \frac{x \geqslant z, x \geqslant y, y \geqslant z, \Gamma \Rightarrow \Delta}{x \geqslant y, y \geqslant z, \Gamma \Rightarrow \Delta}\ \mathit{Trans}_{\geqslant}$$

$$\frac{x \geqslant y, \Gamma \Rightarrow \Delta \quad y \geqslant x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Compl}_{\geqslant}$$

Notice that the set of rules is redundant since reflexivity can be derived from completeness. However, we assume it as primitive. The same system for the first-order theory of linear orders has been introduced by S. Negri and J. von Plato and called in [4] **GLO**. Since many concepts of social choice theory, like Pareto optimality, are traditionally expressed by using both weak and strict preference, we depart from [4] and introduce rules for $>$ and $\sim$ as well. Consider the definition of $>$ first. In one direction of the equivalence we have that if $x > y$ then $x \geqslant y$ and $y \not\geqslant x$. By propositional reasoning we can arrange this axiom in such a way that it corresponds to two inference rules following the regular rule scheme. In greater detail we have

$$\begin{array}{ll} x > y \to (x \geqslant y \wedge y \not\geqslant x) & \equiv \\ (x > y \to x \geqslant y) \wedge (x > y \to y \not\geqslant x) & \equiv \\ (x > y \to x \geqslant y) \wedge (x > y \to (y \geqslant x \to \bot)) & \equiv \\ (x > y \to x \geqslant y) \wedge ((x > y \wedge y \geqslant x) \to \bot) & \end{array}$$

Each conjunct is an instance of a regular formula and thus it gets converted by applying the method into the following inference rules.

$$\frac{x \geqslant y, x > y, \Gamma \Rightarrow \Delta}{x > y, \Gamma \Rightarrow \Delta}\ >_1 \qquad \frac{}{x > y, y \geqslant x, \Gamma \Rightarrow \Delta}\ >_2$$

In the other direction of the definition fo $>$, we have that if both $x \geqslant y$ and $y \not\geqslant x$ then $x > y$.

$$\begin{array}{ll} (x \geqslant y \wedge y \not\geqslant x) \to x > y & \equiv \\ x \geqslant y \to (y \not\geqslant x \to x > y) & \equiv \\ x \geqslant y \to (\neg y \not\geqslant x \vee x > y) & \equiv \\ x \geqslant y \to (y \geqslant x \vee x > y) & \end{array}$$

And this gives the rule

$$\frac{y \geqslant x, x \geqslant y, \Gamma \Rightarrow \Delta \quad x > y, x \geqslant y, \Gamma \Rightarrow \Delta}{x \geqslant y, \Gamma \Rightarrow \Delta}\ {>_3}$$

Once again, we can prove that $>$ is a strict order, i.e. a irreflexive, asimmetric and transitive. The derivations are better understood if read from the root to the leaves since they have been specifically designed to systematically search for formal derivations.

$$\cfrac{\cfrac{\cfrac{}{x \geqslant x, x > x \Rightarrow}\ {>_2}}{x > x \Rightarrow}\ {>_1}}{\Rightarrow x \not> x} \qquad \cfrac{\cfrac{\cfrac{}{y \geqslant x, x > y, y > x \Rightarrow}\ {>_2}}{x > y, y > x \Rightarrow}\ {>_1}}{x > y \Rightarrow y \not> x}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{z \geqslant y, x > z, x \geqslant y, y \geqslant z, x > y, y > z \Rightarrow x > z}\ {>_2}}{z \geqslant x, x \geqslant z, x \geqslant y, y \geqslant z, x > y, y > z \Rightarrow x > z} \quad x > z, x \geqslant y, y \geqslant z, x > y, y > z \Rightarrow x > z}{x \geqslant z, x \geqslant y, y \geqslant z, x > y, y > z \Rightarrow x > z}\ {>_3}}{x \geqslant y, y \geqslant z, x > y, y > z \Rightarrow x > z}}{x > y, y > z \Rightarrow x > z}\ {>_1}$$

The defintion of $\sim$ does not involve negation and it is rather easy to find regular sequent rules for that.

$$\frac{x \sim y, x \geqslant y, y \geqslant x, \Gamma \Rightarrow \Delta}{x \geqslant y, y \geqslant x, \Gamma \Rightarrow \Delta}\ {\sim_1} \qquad \frac{x \geqslant y, y \geqslant x, x \sim y, \Gamma \Rightarrow \Delta}{x \sim y, \Gamma \Rightarrow \Delta}\ {\sim_2}$$

Thus, $\sim$ is easily proved to be an equivalence relation

$$\cfrac{\cfrac{\cfrac{x \sim x, x \geqslant x, x \geqslant x \Rightarrow x \sim x}{x \geqslant x, x \geqslant x \Rightarrow x \sim x}\ {\sim_1}}{x \geqslant x \Rightarrow x \sim x}}{\Rightarrow x \sim x} \qquad \cfrac{\cfrac{y \sim x, x \geqslant y, y \geqslant x, x \sim y \Rightarrow y \sim x}{x \geqslant y, y \geqslant x, x \sim y \Rightarrow y \sim x}\ {\sim_1}}{x \sim y \Rightarrow y \sim x}\ {\sim_2}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{x \sim z, z \geqslant x, x \geqslant z, x \geqslant y, y \geqslant x, y \geqslant z, z \geqslant y, x \sim y, y \sim z \Rightarrow x \sim z}{z \geqslant x, x \geqslant z, x \geqslant y, y \geqslant x, y \geqslant z, z \geqslant y, x \sim y, y \sim z \Rightarrow x \sim z}\ {\sim_1}}{x \geqslant z, x \geqslant y, y \geqslant x, y \geqslant z, z \geqslant y, x \sim y, y \sim z \Rightarrow x \sim z}}{x \geqslant y, y \geqslant x, y \geqslant z, z \geqslant y, x \sim y, y \sim z \Rightarrow x \sim z}}{x \sim y, y \sim z \Rightarrow x \sim z}\ {\sim_2}$$

To complete the picture, we also have that $>$ and $\sim$ are incompatible, i.e. if $x > y$ then $x \not\sim y$.

$$\cfrac{\cfrac{\cfrac{}{x \geqslant y, y \geqslant x, x > y, x \sim y \Rightarrow}}{x > y, x \sim y \Rightarrow}}{x > y \Rightarrow x \not\sim y}$$

Let **GPR** be the sequent calculus **GPO** of [4] for the first-order theory of linear order plus the rules corresponding to the definitions of $>$ and $\sim$.

Theorem. *In* **GPR** *weakening, contraction and cut are admissible.*

Proof. Since all the new rules of **GPR** follow the regular rule scheme, the result is a corollary of Theorem 2.1. in [4]. □

When the structural rules are admissible, it is possible to search systematically formal derivations in **GPR**. To see whether a given sequent $\Gamma \Rightarrow \Delta$ is derivable it is enough to put $\Gamma \Rightarrow \Delta$ at the root of a derivation tree and apply systematically all the rules that may have $\Gamma \Rightarrow \Delta$ as the conclusion. Therefore, cut elimination paves the way to automated deduction and this might be significant to the project of computational choice theory (see [1]). Moreover, the system **GPR** can be used to prove decidability results. Indeed, the first-order theory of linear orders is decidable as proved in [3] by showing that the set of non-theorems is recursively enumerable. A decidability result for the universal fragment is given in [4], Cor. 6.4. using sequent calculi and it is conjectured that the general decision problem reduces to that of the universal fragment. Besides the computational and decidability issues, the rules of **GPR** allows allow to isolate the purely logical part of a derivation from the mathematical one. Indeed, the rules for preference relations are logic-free, in the sense that only atomic preference formulas occur as principal in them. Thus, at each step of a derivation in **GPR** one knows whether the conclusion has been derived by purely logical reasoning or whether it follows from the properties of preference.

Choice Functions

While preferences are usually represented as binary relations, choices are represented by functions. Although conceptually independent from preferences, choice functions must be defined in such a way that agents confronted with multiple options choose according to their preferences. On the other hand, it is also natural to think that most preferred alternatives will presumably chosen. Formally, let X be a finite and $\wp(X)$ be the power-set of X. A function $c : \wp(X) \longrightarrow \wp(X)$ is said to be a choice function whenever for all $A \in \wp(X)$,

(a) $c(A) \subseteq A$

(b) if $A \neq \emptyset$ then $c(A) \neq \emptyset$

These requirements seem to be minimal. To guarantee that choice will be not only possible but also rational, two extra constraints are often considered in the literature and are known as Sen's α and β principles (see [6], p. 17):

(α) If $x \in B \subseteq A$ and $x \in c(A)$, then $x \in c(B)$

(β) If $B \subseteq A$ and $x, y \in c(B)$ then $x \in c(A)$ if and only if $y \in c(A)$.

Intuitively, according to α the choice does not change when not chosen options are removed, whereas β says that if two options are chosen from a set then when we move to a larger set of options it must not be the case that one is chosen and the other one is not. Thus, choices can be defined without referring to preference. An alternative approache consists into the definition of choices in terms of preferences. Given a set A of alternatives, a choice function c and a preference relation $\geqslant$ it is possible to consider the choice set induced by $\geqslant$ and to write $c_{\geqslant}$ when

$$c_{\geqslant}(A) \quad = \quad \{x \in A \mid x \geqslant y \text{ for all } y \in A\}$$

Inferentially, the corresponding rules are

$$\dfrac{x \in A \qquad \begin{array}{c}\overline{y \in A}^{\,(i)}\\ \vdots \\ x \geqslant y\end{array}}{x \in c_{\geqslant}(A)}\,(i) \qquad \dfrac{x \in c_{\geqslant}(A)}{x \in A} \qquad \dfrac{x \in c_{\geqslant}(A) \quad y \in A}{x \geqslant y}$$

In the introduction rule the variable y must not occur free in any assumption on which $y \in A$ depends. Choice sets must not be confused with maximal sets, defined as the set of alternatives in A such that there does not exist any better alternative. A set is then maximal when

$$m_{\geqslant}(A) \quad = \quad \{x \in A \mid y \not> x \text{ for all } y \in A\}$$

Consequently, the rules for $m_{\geqslant}$ are

$$\dfrac{x \in A \quad \begin{matrix} \overline{y \in A}^{\,(i)} \\ \vdots \\ y \not> x \end{matrix}}{x \in m_{\geqslant}(A)}\,(i) \qquad \dfrac{x \in m_{\geqslant}(A)}{x \in A} \qquad \dfrac{x \in c_{\geqslant}(A) \quad y \in A}{y \not> x}$$

As above, the introduction rule must satisfy the variable condition y must not occur free in any assumption on which $y \in A$ depends. Although it is immediately true that if a set of alternatives in A is chosen then it is also maximal, in general it is not true that what we choose is a maximal set. For instance if $B = \{x, y\}$ with $x \not\geqslant y$ and $y \not\geqslant x$, there is no reason to choose x over y, or *viceversa*, therefore $c_{\geqslant}(B) = \emptyset$. However, when $\geqslant$ is assumed to be complete then maximal and choice sets coincide. To see this consider the following derivation where the steps marked by $\star$ from $x \geqslant y$ to $y \not> x$, and from $y \not> x$ to $x \geqslant y$ has been proved to be derivable at the beginning.

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{x \in c_{\geqslant}(A)}^{\,(2)}}{x \in A}\,(2) \quad \dfrac{\dfrac{\overline{x \in c_{\geqslant}(A)}^{\,(2)} \quad \overline{y \in A}^{\,(1)}}{x \geqslant y}}{y \not> x}\,\star}{x \in m_{\geqslant}(A)}\,(1)}{c_{\geqslant}(A) \subseteq m_{\geqslant}(A)}\,(2) \qquad \dfrac{\dfrac{\dfrac{\overline{x \in m_{\geqslant}(A)}^{\,(2)}}{x \in A} \quad \dfrac{\dfrac{\overline{x \in m_{\geqslant}(A)}^{\,(2)} \quad \overline{y \in A}^{\,(1)}}{y \not> x}}{x \geqslant y}\,\star}{x \in c_{\geqslant}(A)}\,(1)}{m_{\geqslant}(A) \subseteq c_{\geqslant}(A)}\,(2)}{c_{\geqslant}(A) \subseteq\supseteq m_{\geqslant}(A)}$$

We now want to introduce maximal sets and choice sets into the sequent calculus in a more regimented way. We extend the language with two binary predicates $\mathsf{M}(x, A)$ and $\mathsf{C}(x, A)$. Intuitively, $\mathsf{M}(x, A)$ denotes that x is maximal in A, whereas $\mathsf{C}(x, A)$ says that x is best in A. We have

$$\begin{array}{lll} \mathsf{M}(x, A) & =_{df} & x \in A \wedge \forall y (y \in A \rightarrow y \not> x) \\ \mathsf{C}(x, A) & =_{df} & x \in A \wedge \forall y (y \in A \rightarrow x \geqslant y) \end{array}$$

From the definition of $\mathsf{M}(x, A)$ we have that a formula of the form $\mathsf{M}(x, A)$ can be derived if $x \in A$ can be derived and $y \not> x$ can be derived from $y \in A$, for an arbitrary y. This gives a rule of the form

$$\dfrac{\Gamma \Rightarrow \Delta, x \in A \quad y \in A, y > x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \mathsf{M}(x, A)}$$

where y does not occur in the conclusion, i.e. is not in Γ, Δ and it is different from x. This *eigenvariable* condition reflects the role of the universal quantifier in the definition of $\mathsf{M}(x, A)$. The corresponding left rules are found consequently,

$$\dfrac{x \in A, \mathsf{M}(x, A), \Gamma \Rightarrow \Delta}{\mathsf{M}(x, A), \Gamma \Rightarrow \Delta} \qquad \overline{\mathsf{M}(x, A), y \in A, y > x, \Gamma \Rightarrow \Delta}$$

The reasoning to find rule for $\mathsf{C}(x, A)$ is analogous and we obtain

$$\frac{\Gamma \Rightarrow \Delta, x \in A \quad y \in A, \Gamma \Rightarrow \Delta, x \geqslant y}{\Gamma \Rightarrow \Delta, \mathsf{C}(x, A)} \qquad \frac{x \in A, \mathsf{C}(x, A), \Gamma \Rightarrow \Delta}{\mathsf{C}(x, A), \Gamma \Rightarrow \Delta} \qquad \frac{x \geqslant y, \mathsf{C}(x, A), y \in A, \Gamma \Rightarrow \Delta}{\mathsf{C}(x, A), y \in A, \Gamma \Rightarrow \Delta}$$

We can prove that all the best elements are maximal. i.e.

$$\dfrac{\dfrac{x \in A \Rightarrow x \in A}{\mathsf{C}(x, A) \Rightarrow x \in A} \qquad \dfrac{\overline{x \geqslant y, \mathsf{C}(x, A), y \in A, y > x \Rightarrow}}{\mathsf{C}(x, A), y \in A, y > x \Rightarrow}}{\mathsf{C}(x, A) \Rightarrow \mathsf{M}(x, A)}$$

The other direction requires the completeness of $\geqslant$.

$$\dfrac{\dfrac{x \in A, \mathsf{M}(x, A) \Rightarrow x \in A}{\mathsf{M}(x, A) \Rightarrow x \in A} \qquad \dfrac{x \geqslant y, \mathsf{M}(x, A), y \in A \Rightarrow x \geqslant y \qquad \dfrac{x \geqslant y, \mathsf{M}(x, A), y \in A \Rightarrow x \geqslant y \quad \overline{y > x, \mathsf{M}(x, A), y \in A \Rightarrow x \geqslant y}}{y \geqslant x, \mathsf{M}(x, A), y \in A \Rightarrow x \geqslant y}}{\mathsf{M}(x, A), y \in A \Rightarrow x \geqslant y}}{\mathsf{M}(x, A) \Rightarrow \mathsf{C}(x, A)}$$

At this point we need to be sure that properties of choice functions are satisfied. Notice that set-theoretic operations are implicitly taken as abbreviations for the standard logical ones.

$$\begin{array}{lll} B \subseteq A & =_{df} & \forall y (y \in B \to y \in A) \\ x \in B \subseteq A & =_{df} & x \in B \wedge B \subseteq A \end{array}$$

Consequently, the rules for $\subseteq$ are derivable from the logical ones and they will be freely used to shorten the derivations. The main result is to prove that $c_{\geqslant}$ satisfies α and β. In our system of natural deduction that means that the following inference rules must be derivable.

$$\frac{x \in B \subseteq A \quad x \in c_{\geqslant}(A)}{x \in c_{\geqslant}(B)}\ \alpha_{\geqslant} \qquad \frac{B \subseteq A \quad x \in c_{\geqslant}(B) \quad y \in c_{\geqslant}(B)}{x \in c_{\geqslant}(A) \leftrightarrow y \in c_{\geqslant}(A)}\ \beta_{\geqslant}$$

Consider the three following derivations of $\alpha_{\geqslant}$ and $\beta_{\geqslant}$.

$$\dfrac{\dfrac{x \in B \subseteq A}{x \in B} \qquad \dfrac{x \in c_{\geqslant}(A) \qquad \dfrac{\dfrac{x \in B \subseteq A}{B \subseteq A} \quad \overline{y \in B}^{\,(1)}}{y \in A}}{x \geqslant y}}{x \in c_{\geqslant}(B)}\ (1)$$

$$\dfrac{\dfrac{\dfrac{B \subseteq A \quad \dfrac{y \in c_{\geqslant}(B)}{y \in B}}{y \in A} \qquad \dfrac{\dfrac{y \in c_{\geqslant}(B) \quad \dfrac{x \in c_{\geqslant}(B)}{x \in B}}{y \geqslant x} \qquad \dfrac{\overline{x \in c_{\geqslant}(A)}^{\,(2)} \quad \overline{z \in A}^{\,(1)}}{x \geqslant z}}{y \geqslant z}}{y \in c_{\geqslant}(A)}}{x \in c_{\geqslant}(A) \to y \in c_{\geqslant}(A)}\ (2)$$

and

$$\dfrac{\dfrac{\dfrac{B \subseteq A \quad x \in c_{\geqslant}(B)}{x \in A} \qquad \dfrac{\dfrac{x \in c_{\geqslant}(B) \quad \dfrac{y \in c_{\geqslant}(B)}{y \in B}}{x \geqslant y} \qquad \dfrac{\overline{y \in c_{\geqslant}(A)}^{\,(4)} \quad \overline{z \in A}^{\,(3)}}{y \geqslant z}}{x \geqslant z}}{x \in c_{\geqslant}(A)}}{y \in c_{\geqslant}(A) \to x \in c_{\geqslant}(A)}\ (4)$$

In sequent calculus **GP** with the new rule for choice and maximal sets, the properties α and β can be easily formulated as sequents and proved to be derivable.

(α) $x \in B \subseteq A, \mathsf{C}(x, A) \Rightarrow \mathsf{C}(x, B)$

(β') $B \subseteq A, \mathsf{C}(x, B), \mathsf{C}(y, B), \mathsf{C}(x, A) \Rightarrow \mathsf{C}(y, A)$

(β'') $B \subseteq A, \mathsf{C}(x, B), \mathsf{C}(y, B), \mathsf{C}(y, A) \Rightarrow \mathsf{C}(x, A)$

We show that α, β' and $\beta''$are derivable using the rules for C. For α consider

$$\dfrac{\dfrac{x \in B, B \subseteq A, \mathsf{C}(x, A) \Rightarrow x \in B \qquad \dfrac{x \in B, \mathsf{C}(x, A), y \in B \Rightarrow x \geqslant y, y \in B \qquad \dfrac{x \geqslant y, x \in B, B \subseteq A, \mathsf{C}(x, A), y \in B, y \in A \Rightarrow x \geqslant y}{x \in B, B \subseteq A, \mathsf{C}(x, A), y \in B, y \in A \Rightarrow x \geqslant y}}{x \in B, B \subseteq A, \mathsf{C}(x, A), y \in B \Rightarrow x \geqslant y}}{x \in B, B \subseteq A, \mathsf{C}(x, A) \Rightarrow \mathsf{C}(x, B)}}{x \in B \subseteq A, \mathsf{C}(x, A) \Rightarrow \mathsf{C}(x, B)}$$

The property β is derivable similarly. We decompose the derivations for readability.

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ B \subseteq A, \mathsf{C}(x, B), \mathsf{C}(y, B), \mathsf{C}(x, A) \Rightarrow y \in A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ z \in A, B \subseteq A, \mathsf{C}(x, B), \mathsf{C}(y, B), \mathsf{C}(x, A) \Rightarrow y \geqslant z\end{array}}{B \subseteq A, \mathsf{C}(x, B), \mathsf{C}(y, B), \mathsf{C}(x, A) \Rightarrow \mathsf{C}(y, A)}$$

The subderivation $\mathcal{D}_1$ is

$$\dfrac{\dfrac{y \in B, B \subseteq A, \mathsf{C}(x, B), \mathsf{C}(y, B), \mathsf{C}(x, A) \Rightarrow y \in A, y \in B \qquad y \in A, y \in B, B \subseteq A, \mathsf{C}(x, B), \mathsf{C}(y, B), \mathsf{C}(x, A) \Rightarrow y \in A}{y \in B, B \subseteq A, \mathsf{C}(x, B), \mathsf{C}(y, B), \mathsf{C}(x, A) \Rightarrow y \in A}}{B \subseteq A, \mathsf{C}(x, B), \mathsf{C}(y, B), \mathsf{C}(x, A) \Rightarrow y \in A}$$

And $\mathcal{D}_2$ is

$$\dfrac{\dfrac{\dfrac{\dfrac{y \geqslant z, x \geqslant z, y \geqslant x, x \in B, z \in A, B \subseteq A, \mathsf{C}(x, B), \mathsf{C}(y, B), \mathsf{C}(x, A) \Rightarrow y \geqslant z}{x \geqslant z, y \geqslant x, x \in B, z \in A, B \subseteq A, \mathsf{C}(x, B), \mathsf{C}(y, B), \mathsf{C}(x, A) \Rightarrow y \geqslant z}}{y \geqslant x, x \in B, z \in A, B \subseteq A, \mathsf{C}(x, B), \mathsf{C}(y, B), \mathsf{C}(x, A) \Rightarrow y \geqslant z}}{x \in B, z \in A, B \subseteq A, \mathsf{C}(x, B), \mathsf{C}(y, B), \mathsf{C}(x, A) \Rightarrow y \geqslant z}}{z \in A, B \subseteq A, \mathsf{C}(x, B), \mathsf{C}(y, B), \mathsf{C}(x, A) \Rightarrow y \geqslant z}$$

The system **GP** extended by the rules for choice set and maximal set is cut-free and preserves the structural properties.

Theorem. *In* **GPR** *weakening, contraction and cut are admissible.*

Proof. We only consider the case of an application of cut on $\mathsf{C}(x, A)$ principal on both the premises of cut.

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x \in A \qquad y \in A, \Gamma \Rightarrow \Delta, x \geqslant y}{\Gamma \Rightarrow \Delta, \mathsf{C}(x, A)} \qquad \dfrac{x \geqslant y, \mathsf{C}(x, A), y \in A, \Gamma' \Rightarrow \Delta'}{\mathsf{C}(x, A), y \in A, \Gamma' \Rightarrow \Delta'}}{\Gamma, y \in A, \Gamma' \Rightarrow \Delta', \Delta}$$

The derivation is converted into one with two cuts, the former on a shorter derivation, the latter on a smaller formula.

$$\dfrac{\dfrac{y \in A, \Gamma \Rightarrow \Delta, x \geqslant y \qquad \dfrac{\Gamma \Rightarrow \Delta, \mathsf{C}(x, A) \qquad x \geqslant y, \mathsf{C}(x, A), y \in A, \Gamma' \Rightarrow \Delta'}{x \geqslant y, y \in A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}}{y \in A, y \in A, \Gamma, \Gamma, \Gamma' \Rightarrow \Delta', \Delta, \Delta}}{y \in A, \Gamma, \Gamma' \Rightarrow \Delta', \Delta}$$

□

Collective Preference

We now show how to extend individual preference to collective preference. Let $B = \{1 \dots n\}$ be a set of n agents. We consider the language of first-order logic with n binary predicates $\geqslant_1, \dots, \geqslant_n$; thus $x \geqslant_i y$ stands for preference of the ith element of B, and similarly for $>_i$ and $\sim_i$. It is also assumed that each $x \geqslant_i y$ is reflexive, transitive and complete. The rules for preference relation indexed by agents are similar to those for $\geqslant$, $>$ and $\sim$. The role of index i becomes relevant when individual preferences are aggregate into a collective one R. Intuitively, R is the preference of the group, or more generally denotes what the entire society prefers. We may think of a collective choice rule as an inference rule of **GPR** that allows one to derive as a conclusion properties of R assuming as premises properties of $\geqslant_i$. Following [6], traditional welfare economics is based on "Paretian" collective preference relation according to which an alternative x is collectively better than y, denoted xPy when all agent consider x as good as y but some strictly prefers x to y.

$$xRy \quad =_{df} \quad \bigwedge_{i=1}^{n} x \geqslant_i y$$

$$xPy \quad =_{df} \quad xRy \wedge \neg yRx$$

The above definitions may be formulated as inference rules of **GPR** as follows

$$\frac{\Gamma \Rightarrow \Delta, x \geqslant_1 y \quad \cdots \quad \Gamma \Rightarrow \Delta, x \geqslant_n y}{\Gamma \Rightarrow \Delta, xRy}\, RR \qquad \frac{x \geqslant_1 y, \dots, x \geqslant_n y, \Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta}\, LR$$

$$\frac{\Gamma \Rightarrow \Delta, xRy \quad yRx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, xPy}\, RP \qquad \frac{xRy, \Gamma \Rightarrow \Delta, yRx}{xPy, \Gamma \Rightarrow \Delta}\, LP$$

It is known that if there are at least two agents, the collective preference relation R derived by the Pareto rules may fail to be complete, even if the all the corresponding individual are so. In **GPR** this corresponds to the underivability of the sequent $\Rightarrow xRy \vee yRx$. Moreover, a counter example can read off from the failed proof-search procedure by considering all the leaves that are at the same time not initial and that cannot by analyzed any further. Moreover, with at least two agents, the decidability is lost, see [5]. In **GPR** it easy to see the the proof-search procedure for $\Rightarrow xRy \vee yRx$ does not terminate. Weaker versions of the Paretian collective preference may be obtained when P is defined as

$$xP'y \quad =_{df} \quad \bigwedge_{i=1}^{n} x >_i y$$

The rules are similar to those for R. For readability we limit ourselves to a two-agent set $B = \{1, 2\}$ and we derive that $P' \subseteq P$ as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{x \geqslant_1 y, x >_1 y, x >_2 y \Rightarrow x \geqslant_1 y}{x >_1 y, x >_2 y \Rightarrow x \geqslant_1 y} \quad \dfrac{x \geqslant_2 y, x >_1 y, x >_2 y \Rightarrow x \geqslant_2 y}{x >_1 y, x >_2 y \Rightarrow x \geqslant_2 y}}{x >_1 y, x >_2 y \Rightarrow xRy} \quad \dfrac{\overline{x >_1 y, x >_2 y, y \geqslant_1 x, y \geqslant_2 x \Rightarrow}}{x >_1 y, x >_2 y, yRx \Rightarrow}}{x >_1 y, x >_2 y \Rightarrow xPy}}{xP'y \Rightarrow xPy}$$

It is also possible to formalize properties of voting procedure. In this case agents are assumed to be individual voters and the alternatives they can vote for are thought as candidates. In [5] the following formula captures the fact that the majority of agents prefer an alternative x to y.

$$xMy \quad =_{df} \quad \bigvee_{A\subseteq B:|A|\geq \frac{n}{2}} \bigwedge_{i\in A} x \geqslant_i y$$

Suppose now to have three agent, $B = \{1, 2, 3\}$. Then M is defined by

$$x \geqslant y \quad =_{df} \quad (x \geqslant_1 y \wedge x \geqslant_2 y) \vee (x \geqslant_3 y \wedge x \geqslant_1 y) \vee (x \geqslant_2 y \wedge x \geqslant_3 y)$$

It is convenient to write $x \geqslant_{ij} y$ for $x \geqslant_i y \wedge x \geqslant_j y$; thus we have the following rules for majority voting

$$\frac{\Gamma \Rightarrow \Delta, x \geqslant_1 y \quad \Gamma \Rightarrow \Delta, x \geqslant_2 y}{\Gamma \Rightarrow \Delta, x \geqslant y} \quad \frac{\Gamma \Rightarrow \Delta, x \geqslant_3 y \quad \Gamma \Rightarrow \Delta, x \geqslant_1 y}{\Gamma \Rightarrow \Delta, x \geqslant y} \quad \frac{\Gamma \Rightarrow \Delta, x \geqslant_2 y \quad \Gamma \Rightarrow \Delta, x \geqslant_3 y}{\Gamma \Rightarrow \Delta, x \geqslant y}$$

$$\frac{x \geqslant_{12} y, \Gamma \Rightarrow \Delta \quad x \geqslant_{31} y, \Gamma \Rightarrow \Delta \quad x \geqslant_{23} y, \Gamma \Rightarrow \Delta}{x \geqslant y, \Gamma \Rightarrow \Delta}$$

Conclusion

The paper is a report on a work in progress and consequently there are many problems for which a satisfactory solution is still unknown. Most notably, Section 2 left open the question of decidability of **GPR** with rules for C and M operators, whereas Section 3 only outlines the idea of a fully formal analysis of impossibility theorems in social choice theory. Besides the issue of formalization (on which many approaches have been already developed, see [1] and the bibliography therein), we are mostly concerned with a foundational issue. We have seen that our rules make use of set-theoretic notions without assuming any particular theory of sets. Therefore, we tacitly assumed that operations of sets can be reduced to the logical ones. To what extent this reduction is indeed possible? What part of social choice theory is genuinely set-theoretic and how to distinguish it from the purely logical one? Since among the advantages of a proof-theoretic systematization of axiomatic theories there is the possibility of keeping separated the mathematical content from the analytical one, it would be interesting to see whether the method of proof analysis allow one the achieve this separation for social choice theory.

References

[1] U. Endriss. Logic and Social Choice Theory. In A. Gupta and J. van Benthem (Eds.) *Logic and Philosophy Today*, College Publications, pp. 333–377, 2011.

[2] G. Gentzen. Investigations into Logical Deduction. In M. Szabo (Ed.) *The Collected Papers of Gerhard Gentzen*, North-Holland, Amsterdam, pp. 68–131, 1969.

[3] H. Läuchli, J. Leonard. On the elementary theory of linear order. *Fundamenta Mathematicae*, 59:109–116, 1966.

[4] S. Negri, J. von Plato. Proof-theoretical analysis of order relations. *Arch. Math. Logic*, 43:297–309, 2004.

[5] M. Pauly. Social Choice in First-Order Logic: Investigating Decidability & Definability. In *5th International Conference on Logic, Game Theory and Social Choice*. Extended Abstract.

[6] A. Sen. *Collective Choice and Social Welfare*, Holden-Day, San Francisco, 1970.

[7] N. Tennant. *Autologic*. Edinburgh University Press, Edinburgh, 1992.

[8] A. S. Troelstra, H. Schwichtenberg. *Basic Proof Theory*, CUP, Cambridge, 2000.

Paolo Maffezioli
Faculty of Philosophy
University of Groningen
Oude Boteringestraat 52, 9712 GL Groningen, the Netherlands
Email: P.Maffezioli@rug.nl