Some Complexity Results for Distance-Based Judgment Aggregation (extended abstract)

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Abstract

Judgment aggregation is a social choice method for aggregating information on logically related issues. In distance-based judgment aggregation, the collective opinion is sought as a compromise that satisfies several structural properties. It would seem that the standard conditions on distance and aggregation functions are strong enough to guarantee existence of feasible procedures. In this paper, we show that it is not the case, though the problem becomes easier under some additional assumptions.

1 Introduction

It is often convenient to ascribe information-related stances (such as judgments, opinions, beliefs, etc.) to collectives of agents. External agents may ascribe opinions to institutions and groups in order to simplify their model of the world and reasoning about it. Agents inside the group may need to reach consensus about issues of interest, and in particular to obtain collective decisions that will lead to consistent collective action. One of the formal frameworks that try to explain how collective judgments are formed from individual stances is *judgment aggregation theory* [17].

Distance-based judgment aggregation [20, 18] comprises the largest class of judgment aggregation rules. Inspired by belief merging rules, the idea is to define the collective opinion as a "well-behaved compromise" among the individual opinions of the group members. Distance-based aggregation rules are supposed to satisfy a number of structural constraints (see Section 2 for details) to make sure their output is indeed "well-behaved" in the mathematical sense. It seems – at least at the first glance – that the constraints should lead to computationally well-behaved procedures. In this paper, we show that it is not necessarily true.

Why is computational complexity important for aggregating judgments? Essentially, judgment aggregation provides an intuitive representation for decision problems in collective reasoning. In this context, its computational complexity is crucial. More specifically, judgment aggregation rules are *procedures* that determine the collective view based on individual inputs. The procedure is only useful if it returns the result in reasonable time. Consider, a team of 100 robots reaching a collective decision based on the input from 400 sensors with different (but overlapping) range, or 500 stakeholders trying to agree on a company agenda. Scalability of the procedure becomes clearly of utmost importance.

Complexity analysis of distance-based judgment aggregation has, to the best of our knowledge, been focused on analysis of *particular* aggregation rules. The complexity of the most "typical" aggregation rule, based on the sum of Hamming distances, was studied in [2, 9, 8, 10]. In contrast to studies of particular "natural" distance-based judgment aggregators, *we take the opposite approach and explore the bounds of the framework.* That is, we investigate what kind of complexity can be expected from *arbitrary* distance-based aggregation rules.

Relation to research on preference aggregation. Besides the papers cited above, works on complexity of distance-based belief merging [14] and especially distance-based preference aggregation [1, 7, 6] must be also mentioned. In particular, the complexity of winner determination problem for the Kemeny rule has been studied in [1] and [13], the latter proving it to be Θ_2^P complete. This strand connects to the research on complexity of distance-based judgment aggregation through the result of [7] where it was shown that the Kemeny rule of voting coincides, for strict preference orders, with judgment aggregation based on the sum of Hamming distances.

It has been shown that judgment aggregation is related to preference aggregation by showing when a preference aggregation problem can be translated to a judgment aggregation problem and vice versa [16, 4, 12]. Still, the relationship between judgment aggregation rules and voting rules (or preference aggregation rules) has not been formally established on the general level, despite a number of studies on the topic [5, 19, 15]. Without such formalization, it is hard to draw general conclusions on the relationship between the complexity of voting rules and judgment aggregation rules. We present some preliminary intuitions below.

A judgment set can be used to characterize a strict preference order [4] by using a formula φ_b^a to represent that alternative a is preferred to alternative b. In complexity of preference aggregation, one is typically interested in the winner determination problem, that is, the problem of deciding whether an alternative is top ranked in at least one of preference orders produced by the preference aggregation rule. Considering only aggregation of strict preferences and sticking to the analogy that a preference order is a judgment set, an alternative in preference aggregation corresponds to a judgment, and the winner determination problem can be interpreted as that of determining whether a particular judgment φ_b^a is a part of the collective judgment set produced by the judgment aggregation rule. The difficulty lies in the fact that a judgment aggregation rule can produce multiple collective judgment sets, some containing φ_b^a and some not. Therefore two different meaningful questions can be studied: (1) whether a judgment set as a whole can be selected as the collective opinion, corresponding to our definition of the winner verification problem in Section 3, or (2) whether a given judgment is an element of all collective opinions, as in [10]. For preference aggregation, (1) corresponds to checking if a preference order is selected by the preference aggregation rule, while (2) is about determining whether a given alternative is highest ranked in all selected preference orders. Both decision problems are at least as hard as the problem of deciding whether an alternative is a winner of the election. Therefore we can expect decision problems in judgment aggregation to be no easier than their counterparts in preference aggregation.

2 Preliminaries

2.1 Judgment Aggregation

Let $\mathcal{L}$ be a propositional language over a countable set of atomic propositions $Prop$, and let T be a set of truth values such that $1 \in T$ (i.e., it includes the value for "absolutely true"). Any $v : Prop \to T$ is called a propositional valuation; we denote the set of such valuations as PV. We assume that each $v \in PV$ extends to a valuation $val_v : \mathcal{L} \to T$ for all

	p_1	p_2	p_3
robot 1	1	1	0
robot 2	0	0	0
robot 3	0	1	1
majority	0	1	0

Figure 1: Guarding robots. $N = \{1,2,3\}, \mathcal{A} = \{p_1, p_2, p_3\}, \mathcal{C} = \{\neg p_1 \wedge p_2 \rightarrow p_3\}$

formulae of $\mathcal{L}$. Throughout this paper, we will also assume that $\mathcal{L}$ is the language of classical propositional logic, $T = \{0,1\}$, and val_v is defined by the classical Boolean semantics of negation, conjunction, etc.

Judgment aggregation can be defined as follows.[1] Let N be a finite set of *agents*, $\mathcal{A} \subseteq \mathcal{L}$ a finite *agenda* of issues, and $\mathcal{C} \subseteq \mathcal{L}$ a finite set of *admissibility constraints*. A *judgment set* is a consistent and admissible combination of opinions on issues from $\mathcal{A}$, that is, some $js : \mathcal{A} \rightarrow T$ for which there exists a valuation $v \in PV$ such that: (i) $val_v(\varphi) = js(\varphi)$ for every $\varphi \in \mathcal{A}$, and (ii) $val_v(\psi) = 1$ for every $\psi \in \mathcal{C}$. The set of all judgment sets is denoted by JS. Now, a *judgment profile* is a collection of judgment sets, one per agent, i.e., $jp : N \rightarrow JS$. With a slight abuse of notation, we will denote the set of all such profiles by $JS^{|N|}$. Note that we can conveniently represent judgment profiles as $|Agt| \times |\mathcal{A}|$ matrices of elements from T. Finally, a *judgment aggregation rule* $\nabla : JS^{|N|} \rightarrow \mathcal{P}(JS) \setminus \{\emptyset\}$ aggregates opinions from all the agents into a collective judgment set (or sets). We allow for more than one "winning" set to account for nondeterministic or inconclusive aggregation rules.

Example 1 *Consider* 3 *robots guarding a building, that have just observed a person. Each robot must assess whether the person is authorized to be there (proposition* p_1*), if it has malicious intent (*p_2*), and whether to classify the event as dangerous intrusion (*p_3*). Additionally, it is assumed that a non-authorized person with malicious intent implies intrusion:* $\neg p_1 \wedge p_2 \rightarrow p_3$ *(note that the converse does not have to hold). A possible judgment profile is shown in Figure 1. The figure also shows that the most "obvious" aggregation rule (majority) results in an inadmissible judgment set.*

2.2 Distance-Based Aggregation Rules

A distance-based aggregation rule [20, 18] looks for a collective opinion that does not stray too much from the individual judgments:

$$\nabla_{d,aggr}(jp) = \text{argmin}_{js \in JS} \left\{ aggr\big(d(js, jp[1]), \ldots, d(js, jp[|N|])\big) \right\},$$

where d is a *distance function* [3, p.3-4 and 45], and $aggr$ an *aggregation function* [11, p.3], cf. the definitions below.

Definition 1 *An* algebraic aggregation *is a function* $aggr : (\mathbb{R}^+)^n \rightarrow \mathbb{R}^+$ *such that:* ***(minimality)*** $aggr(0^n) = 0$*, and* ***(non-decreasing)*** *if* $x \leq y$*, then* $aggr(x_1, \ldots, x, \ldots, x_n) \leq aggr(x_1, \ldots, y, \ldots, x_n)$.

Well known aggregators are: min, max, sum, and product.

Definition 2 *A* distance *over set* X *is a function* $d : X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$ *such that:* ***(minimality)*** $d(x,y) = 0$ *iff* $x = y$*,* ***(symmetry)*** $d(x,y) = d(y,x)$*, and* ***(triangle inequality)*** $d(x,y) + d(y,z) \geq d(x,z)$.

[1] Our definition of judgment aggregation combines features of logic-based aggregation [17] and algebraic aggregation [21]. It is easy to see that both formulations can be expressed in our notation.

Two well known distances over $\{0,1\}^m$ are: the Hamming distance $d_H(x,y) = \sum_{i=1}^{m} \delta_H(x[i],y[i])$, and the drastic distance $d_D(x,y) = \max_{j=1,\ldots,m} \delta_H(x[i],y[i])$, where $\delta_H(x,y) = 0$ if $x = y$ and 1 otherwise.

Example 2 *Consider the robots from Example 1, and let us use d_H as the distance and $\sum$ as the aggregator. Then, the winners are $\{000, 011, 110\}$, all with score (i.e., aggregate distance) 3. In other words, the agents cannot do better than to accept one of their individual opinions.*

3 Complexity of Distance-Based Winner Verification

There are two natural computational problems related to judgment aggregation: the function problem of computing a "winning" judgment set, and the decision problem of verifying that a given judgment set is one of the winners. We look closer at the latter.

Definition 3 WINVER$_\nabla$ *is the decision problem defined as follows:*
Input: *Agents N, agenda $\mathcal{A}$, constraints $\mathcal{C}$, judgment profile $jp \in JS^{|N|}(\mathcal{A},\mathcal{C})$, and judgment set $js \in JS(\mathcal{A},\mathcal{C})$.*
Output: *true if $js \in \nabla(jp)$, else false.*

What is the complexity of WINVER? One could expect that, under the assumptions in Definitions 1 and 2, distance-based aggregation should behave reasonably in computational terms. Unfortunately, it is not the case.

3.1 Bad News

Theorem 1 *There is a distance which is not Turing computable.*

Proof. We construct the *Turing distance* d_{TR} as follows. First, we assume a standard encoding of Turing machines in binary strings; we use $TM(X)$ to refer to the machine represented by the string of bits $X \in \{0,1\}^m$. We also assume by convention that strings starting with 0 or ending with 1 represent only machines that always halt (e.g., they can represent various TM's with only accepting states).

Let $halts(X) = 0$ if the $TM(X)$ halts, and 1 otherwise. Now, for any $js, js' \in \{0,1\}^m$, we take

$$d_{TR}(js,js') = d_D(js,js') + halts(h(js,js')),$$

where d_D is the drastic distance (i.e., $d_D(js,js') = 0$ if $js = js'$ and 1 otherwise), and $h(js,js') = \big(\delta_H(js[1],js'[1]),\ldots,\delta_H(js[m],js'[m])\big)$ is the Hamming sequence for (js,js').

We check that d_{TR} is a distance:

1. $d_{TR}(js,js) = d_D(js,js) + halts(0^m) = 0$;
2. $d_{TR}(js,js') = 0 \Rightarrow d_D(js,js') = 0 \Rightarrow js = js'$;
3. $d_{TR}(js,js') = d_{TR}(js',js)$: straightforward;
4. Triangle inequality: the nontrivial case is $js \neq js' \neq js''$, then $d_{TR}(js,js') + d_{TR}(js',js'') \geq 2 \geq d_{TR}(js,js'')$.

For incomputability, we observe that $TM(X)$ halts iff $d_{TR}(X, 0^{|X|}) \leq 1$. □

Theorem 2 *There is a distance and an aggregation function for which* WINVER *is undecidable.*

Proof. We construct a Turing reduction from the halting problem. Given is a representation $X \in \{0,1\}^m$ of a Turing machine (same assumptions as in Theorem 1). We take d_{TR} as the distance, and $aggr = \sum$. Let $\mathcal{A} = \{p_1, \ldots, p_m\}$ consist of n unrelated atomic propositions, $\mathcal{C} = \emptyset$, and $jp = \{0^m, X\}$. Now, for $X = 1 \ldots 0$ (the other cases of X trivially halt), we have that $TM(X)$ halts iff $js = 0^m, X$ are the only winners. This is because the aggregate scores of 0^m and X are 1 if $TM(X)$ halts and 2 otherwise, and no score can be less than 1. Moreover, for all other candidates $Y \in \{0,1\}^m$ the score is at least 2, and in particular for $Y = (1)^m$ it is always 2.

Suppose now that deciding WINVER terminates in finite time. Then, the halting of $TM(X)$ could be verified by 2^m WINVER checks, i.e., also in finite time – which is a contradiction. □

Thus, it turns out that the standard requirements on distance metrics and aggregation function are not sufficient to guarantee even decidability of the winner verification problem. Of course, the metric that we used to prove this is utterly artificial, and unlikely to appear in any realistic context. Distance-based aggregation rules that are actually used have much better computational properties, as we demonstrate in Section 3.2. Still, Theorem 2 shows the *bounds* of the framework: in principle, the complexity of related decision problems can be very bad. This means that, when trying a *new* variant of distance-based aggregation, one should be cautious, and carefully examine its computational characteristic beforehand.

3.2 Positive Results

We now prove that, under reasonable conditions, winner verification sits in the first level of the polynomial hierarchy. We recall that $\mathbf{P}^{\mathbf{NP}[k]}$ is the class of problems solvable by a polynomial-time deterministic Turing machine asking at most k adaptive queries to an **NP** oracle. It is easy to see that $\mathbf{NP} \subseteq \mathbf{P}^{\mathbf{NP}[k]} \subseteq \mathbf{\Delta_2^P} = \mathbf{P}^{\mathbf{NP}}$.

Theorem 3 *If aggr and d are computable in polynomial time then* WINVER *for* $\nabla_{d,aggr}$ *is in* $\mathbf{P}^{\mathbf{NP}[2]}$.

Proof. We prove the inclusion by showing an algorithm for WINVER.

Algorithm: $\underline{Winver(js, jp, N, \mathcal{A}, \mathcal{C}, d, aggr)}$

1. if $Consistent(js, \mathcal{A}, \mathcal{C})$ and not $ExistsBetter(js, jp, N, \mathcal{A}, \mathcal{C}, d, aggr)$ then $return(true)$ else $return(false)$;

Oracle: $\underline{Consistent(js, \mathcal{A}, \mathcal{C})}$

1. guess a valuation $v \in PV$ for the atomic propositions in $\mathcal{A}$;
2. if $val_v(\varphi) = js(\varphi)$ for every $\varphi \in \mathcal{A}$ and $val_v(\psi) = 1$ for every $\psi \in \mathcal{C}$ then $return(true)$ else $return(false)$;

Oracle: $\underline{ExistsBetter(js, jp, N, \mathcal{A}, \mathcal{C}, d, aggr)}$

1. guess $js' \in JS$;
2. guess a valuation $v' \in PV$ for the atomic propositions in $\mathcal{A}$;
3. if $val_{v'}(\varphi) = js'(\varphi)$ for every $\varphi \in \mathcal{A}$ and $val_{v'}(\psi) = 1$ for every $\psi \in \mathcal{C}$ and $aggr\big(d(js', jp[1]), \ldots, d(js', jp[|N|])\big) < aggr\big(d(js, jp[1]), \ldots, d(js, jp[|N|])\big)$ then $return(true)$ else $return(false)$; □

So, the idea is to ask an oracle if the judgment set is consistent and admissible and whether there is no set with a better score. For combinations of the most typical distances and aggregation functions, we get the following as a straightforward consequence.

Corollary 4 *If* $aggr \in \{\min, \max, \sum, \prod\}$ *and* $d \in \{d_H, d_D\}$ *then* WINVER *for* $\nabla_{d,aggr}$ *is in* $\mathbf{P}^{\mathbf{NP}[2]}$.

Note also that the problem is already known to be **NP**-complete for $d = d_H, aggr = \sum$ [9].

4 Conclusions

Complexity-theoretic properties of voting procedures are a frequent topic of study in computational social choice. In contrast, the complexity of judgment aggregation has drawn attention only recently. In this paper, we explore the complexity bounds of an important family of judgment aggregation rules, namely those based on minimization of aggregate distance. More precisely, we study the decision problem of verifying if a given judgment set can be selected as the collective opinion. It turns out that feasibility of distance-based aggregation in general cannot be guaranteed. However, by assuming some requirements on the possible outcomes of the distance and aggregation functions, we can tame the complexity reasonably.

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