The birth of social choice theory from the spirit of mathematical logic: Arrow's theorem as a model-theoretic preservation result

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Abstract

Arrow's axiomatic foundation of social choice theory can be understood as an application of Tarski's methodology of the deductive sciences which is closely related to the latter's foundational contribution to model theory. Thus, Arrow's celebrated impossibility theorem can be read as a model-theoretic preservation result "avant la lettre": The preservation of the first-order properties of weak orders under product formation is only possible in the case of ultraproducts, i.e. if the product is reduced over an ultrafilter on the index set of individuals. This perspective does not only shed new light on a well known source of dictatorship results in the finite case, but allows to conceive of the problem of aggregation in general as a model-theoretic preservation problem.

1 Introduction

By generalizing the classical problem of preference aggregation the recent literature on judgment aggregation (surveyed by List, Puppe and Polak (11) and (12)) has established a close relation between logic and collective decision theory. A recent paper by Herzberg and Eckert (6) has proposed a unified framework for aggregation theory (including judgment aggregation) based on the aggregation of model-theoretic structures, thus extending Lauwers and Van Liedekerke's (9) model-theoretic analysis of preference aggregation. This model-theoretic framework for aggregation theory conceives of an aggregation rule as a map $f :$ $\mathrm{dom}(f) \to \Omega$ with $\mathrm{dom}(f) \subseteq \Omega^I$, wherein I is the electorate and Ω is the collection of all models of some fixed universal theory T (in a first-order language $\mathcal{L}$) with a fixed domain A. This map thus assigns to any profile of models of T an $\mathcal{L}$-structure that is also a model of T. Thus, in model-theoretic terms, an aggregation rule is equivalent to an operation on a product of models of some theory T that guarantees that the outcome of this operation is again a model of T, i.e. that all the properties of the factor models described by the theory T are preserved. The fact that this is typically not the case for a direct product consisting in a profile of preference orderings lies at the heart of the problem of preference aggregation since Condorcet's paradox about the possibly cyclical outcome of majority voting.

This framework is sufficiently general to cover both preference and propositional judgment aggregation: For instance, preference aggregation corresponds to the special case where $\mathcal{L}$ has one binary relation R, T is the theory of weak orders, and A is a set of alternatives; propositional judgment aggregation corresponds to the special case where $\mathcal{L}$ has a unary operator (the belief operator) and A is the agenda. In this model-theoretic approach to aggregation theory, basic (im)possibility theorems from preference aggregation and judgment aggregation (see (10), (5), (16) for some seminal contributions) follow directly from general (im)possibility theorems about the aggregation of first-order model-theoretic structures.

The fundamental observation in the model-theoretic analysis of aggregation is that the preservation of certain properties of the individual factor models requires that the outcome be some reduction of the direct product taken over a family of subsets of the electorate. Once this observation has been made, the proof of characterisations of aggregation functions (in the guise of (im)possibility theorems) only requires relatively basic facts from model theory, such as the construction of reduced products, ultraproducts, Łoś's theorem, and the characterisations of filters and ultrafilters on finite sets. Dictatorship then immediately follows in the finite case, if this family is required to be an ultrafilter, because in this case an ultrafilter is the collection of all supersets of some singleton, - the dictator.

In this historical note, we argue that a model-theoretic approach is not only consistent with Arrow's original research program but also that his dictatorship result is a model-theoretic preservation result "avant la lettre", a historical significance that was explicitly recognized by Hodges (7) in his account of the history of model theory. Roughly speaking, this significance consists in the formulation of the problem of the aggregation of preference relations as a typical model-theoretic preservation problem, i.e. as the problem of the preservation of the properties of the individual factor models under product formation, a core problem in the subsequent literature on model theory in the 60s and 70s (see e.g. (4)). The application of model-theoretic results to preference aggregation can already be found in an old unpublished paper by Brown (3). Our paper aims at an explicit presentation of the connection between Arrow's original impossibility theorem and the model-theoretic preservation problem while also building a bridge to the recent papers on model aggregation.

2 Arrovian social choice theory and the axiomatic method

From a methodological point of view, Arrow's seminal 1951 monograph *Social Choice and Individual Values* (1) is rightly famous for its introduction of the axiomatic analysis of binary relations into economics and welfare economics in particular (20). While the context of justification of this approach to the modelling of social welfare is the so-called ordinalist revolution of the 1930s, which put into question the measurability and, a fortiori, the interpersonal comparison of utilities ((1), p. 9), its context of discovery is Arrow's exposure as a student to the work of the famous logician Alfred Tarski, in particular to the algebra of relations in the 1940s (for details to this aspect of Arrow's intellectual biography, see (8)). Thus, Arrow explicitly motivates the formal framework of binary relations used for the representation of preferences by its familiarity "in mathematics and particularly in symbolic logic" ((1), p. 11), referring to Tarski's famous *Introduction to Logic and the Methodology of the Deductive Sciences* (21), which he had proofread as a student. More generally, Arrow's analysis of the problem of preference aggregation can be read as an application of the deductive method exposed in Tarski's textbook. Central to Tarski's concept of a deductive theory is not only its derivation from a set of axioms, but the concept of a model of a theory obtained by an interpretation of its terms that makes all the axioms (and thus the theory derived from them) true. The latter can be seen as the conceptual intuition underlying the further development of model theory as well as of its significance for the epistemological analysis of those social sciences that can be counted among the formal sciences, like theoretical economics (19).

While the construction of various types of products with the help of families of sets on some index set would later play a central role in model theory (e.g. in Łoś's (13) fundamental theorem on ultraproducts), Arrow's analysis of collective decision problems in terms of families of winning coalitions can be traced back to another, "semantical" logical strand in the research program of the mathematization of economics. It was the mathematician

Karl Menger (14) who first introduced families of subsets of individuals into the logical analysis of norms, semantically conceiving a norm as the set of individuals accepting it (for a modern reconstruction of Menger's deontic logic see (18)). This approach was then explicitly propagated by Morgenstern in his programmatic paper *Logistics and the Social Science* (15) as a model for the application of formal analysis to the social sciences in general and to economics in particular. In this light, the analysis of games in terms of families of winning coalitions in von Neumann and Morgenstern's foundational *Theory of Games and Economic Behavior* (17), to which Arrow often refers, can be considered a significant step in this logical strand in the mathematization of economics.

Thus Arrow's seminal monograph is located at the confluence of two logical strands, Tarski's model-theoretic approach to the methodology of the deductive sciences and Menger's logical semantics of norms in terms of families of subsets of individuals.

3 Arrow's theorem as a model-theoretic preservation result

In the following we give a reconstruction of Arrow's theorem as a model theoretic preservation result. For the model theoretic analysis of preference aggregation the domain A is interpreted as a set of alternatives and T is the theory of weak orders, which is expressed by the universal sentences:

(i) $\forall x \forall y \ R(x,y) \lor R(y,x)$ (completeness, Axiom I in (1)) and

(ii) $\forall x \forall y \forall z \ R(x,y) \land R(y,z) \rightarrow R(x,z)$ (transitivity, Axiom II in (1)).

Denote by Ω the set of all models of T and by I the (possibly infinite) set of individuals.

A social welfare function is a map f whose domain $\mathrm{dom}(f)$ is contained in Ω^I and whose range is contained in Ω. Under the traditional assumption of universal domain, a social welfare function is then a mapping $f : \Omega^I \rightarrow \Omega$, which assigns to each profile of weak orders a weak order as a social preference. The very definition of a social welfare function, thus, does already imply the requirement of the preservation of the first-order properties of preference relations under product formation.

For the analysis of social welfare functions in terms of families of winning coalitions the property of independence of irrelevant alternatives plays a central role.

Definition 1 *A social welfare function $f : \mathrm{dom}(f) \rightarrow \Omega$ satisfies **independence of irrelevant alternatives** if for any pair of alternatives $x, y \in A$ and all profiles $\underline{\mathfrak{A}}, \underline{\mathfrak{A}}' \in \mathrm{dom}(f)$*

$$\{i \in I : \mathfrak{A}_i \models R(x,y)\} = \{i \in I : \mathfrak{A}_i' \models R(x,y)\} \Rightarrow \left[f(\underline{\mathfrak{A}}) \models R(x,y) \Leftrightarrow f(\underline{\mathfrak{A}}') \models R(x,y) \right]$$

The following proposition establishes the link between the independence property and the analysis of collective decision problems in terms of families of winning coalitions.

Proposition 2 *A social welfare function $f : \mathrm{dom}(f) \rightarrow \Omega$ satisfies independence of irrelevant alternatives if and only if for any pair of alternatives $x, y \in A$ there exists a family of winning coalitions $\mathcal{W}_{(x,y)}^f \subset 2^I$ such that for any profile $\underline{\mathfrak{A}} \in \mathrm{dom}(f)$*

$$f(\underline{\mathfrak{A}}) \models R(x,y) \Leftrightarrow \{i \in I : \mathfrak{A}_i \models R(x,y)\} \in \mathcal{W}_{(x,y)}^f$$

Proof. The "if" part of the Proposition is trivial.

For the proof of the "only if" part, suppose f satisfies independence of irrelevant alternatives and let $x, y \in A$. For all $\underline{\mathfrak{A}} \in \mathrm{dom}(f)$, define

$$C(\underline{\mathfrak{A}}, x, y) = \{i \in I : \mathfrak{A}_i \models R(x,y)\}.$$

and let
$$\mathcal{C}^f_{(x,y)} = \{C(\underline{\mathfrak{A}}, x, y) : \underline{\mathfrak{A}} \in \mathrm{dom}(f)\}$$

Next let $g^f_{(x,y)} : \mathcal{C}^f_{(x,y)} \to \{0, 1\}$ be such that

$$g^f_{(x,y)} : C(\underline{\mathfrak{A}}, x, y) \mapsto \begin{cases} 1, & f(\underline{\mathfrak{A}}) \models R(x,y) \\ 0, & f(\underline{\mathfrak{A}}) \not\models R(x,y). \end{cases}$$

Since f satisfies independence of irrelevant alternatives, $g^f_{(x,y)}$ is well-defined. If we now put $\mathcal{W}^f_{(x,y)} = g^f_{(x,y)}{}^{-1}\{1\}$, we have found a family of winning coalitions as postulated in the Proposition. ∎

If f has universal domain, then $\mathcal{C}^f_{(x,y)} = 2^I$ and $\mathcal{W}^f_{(x,y)}$ turns out to be unique.

Much of the literature on Arrovian social choice investigates the properties of these families of winning coalitions $\mathcal{W}^f_{(x,y)}$. In particular, these families of winning coalitions can be used to express other properties of social welfare functions, as the Pareto property:

Definition 3 *A social welfare function $f : \Omega^I \to \Omega$ with universal domain which satisfies independence of irrelevant alternatives is **weakly Paretian**, if for any pair of alternatives $x, y \in A$*

$$\varnothing \notin \mathcal{W}^f_{(x,y)}$$

Note that this captures the Arrovian weak Pareto principle as R models a negated *strict* preference ordering.

Similarly, the property of non-dictatorship can be characterized via sets of winning coalitions. The classical definition, which we will also employ in the proof of our main theorem, reads as follows:

Definition 4 *A social welfare function $f : \mathrm{dom}(f) \to \Omega$ is **non-dictatorial**, if there does not exist an individual $k \in I$ such that for any pair of alternatives $x, y \in A$ and all profiles $\underline{\mathfrak{A}} \in \mathrm{dom}(f)$,*

$$f(\underline{\mathfrak{A}}) \models R(x,y) \Leftrightarrow \mathfrak{A}_k \models R(x,y).$$

*(Otherwise, such an individual k is called **dictator**.)*

It is, however, not difficult to establish an alternative description:[1] Provided that f has universal domain and satisfies independence of irrelevant alternatives, k is a dictator if and only if for all alternatives $x, y \in A$,

$$\mathcal{W}^f_{(x,y)} = \{S \subseteq I : k \in S\}.$$

Thus, (non-)dictatorship can be characterized in terms of the set of winning coalitions.

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[1] Let f have universal domain and satisfy independence of irrelevant alternatives. The definitions of $\mathcal{W}^f_{(x,y)}$ and $C(\underline{A}, x, y)$ entail for all $\underline{A} \in \mathrm{dom}(f)$ and all $x, y \in A$,

$$f(\underline{A}) \models R(x,y) \Leftrightarrow C(\underline{A}, x, y) \in \mathcal{W}^f_{(x,y)}$$
$$A_k \models R(x,y) \Leftrightarrow k \in C(\underline{A}, x, y).$$

Since f, in addition to satisfying independence of irrelevant alternatives, has universal domain, k will be a dictator if and only if for all $\underline{A} \in \mathrm{dom}(f)$ and all $x, y \in A$,

$$C(\underline{A}, x, y) \in \mathcal{W}^f_{(x,y)} \Leftrightarrow k \in C(\underline{A}, x, y).$$

This yields $\mathcal{W}^f_{(x,y)} = \{S \subseteq I : k \in S\}$.

The requirement of the preservation of the two axioms characterizing weak orders has some immediate implications for the induced families of winning coalitions. In particular, completeness implies the strongness[2] of any of these families of winning coalitons.

Lemma 5 (Strongness) *Let $f : \Omega^I \to \Omega$ be a social welfare function with universal domain which satisfies independence of irrelevant alternatives (and suppose $\#A \geq 2$). Then for any pair of distinct alternatives $x, y \in A$ and any coalition $U \in 2^I$*

$$U \notin \mathcal{W}^f_{(x,y)} \Rightarrow I \backslash U \in \mathcal{W}^f_{(y,x)}.$$

Proof. Let $x, y \in A$ with $x \neq y$ and $U \notin \mathcal{W}^f_{(x,y)}$. Since f is a social welfare function with universal domain, we can construct a profile $\underline{\mathfrak{A}} \in \mathrm{dom}(f)$ such that
   (a) for all $i \in I$, $\mathfrak{A}_i \models \neg R(x,y) \vee \neg R(y,x)$
   (completeness of the negated order), and
   (b) $\{i \in I : \mathfrak{A}_i \models R(x,y)\} = U$.
   Then, on the one hand $I \setminus U = \{i \in I : \mathfrak{A}_i \not\models R(x,y)\} = \{i \in I : \mathfrak{A}_i \models \neg R(x,y)\} = \{i \in I : \mathfrak{A}_i \models R(y,x)\}$, because our choice of $\underline{\mathfrak{A}}$ and completeness imply $\mathfrak{A}_i \models (\neg R(x,y) \leftrightarrow R(y,x))$ for all $i \in I$ ("$\rightarrow$" by completeness, "$\leftarrow$" by (a)).
   On the other hand, by the assumption $U \notin \mathcal{W}^f_{(x,y)}$, we may deduce $f(\underline{\mathfrak{A}}) \not\models R(x,y)$, which by completeness (of the social preference ordering) yields $f(\underline{\mathfrak{A}}) \models R(y,x)$.
   Combining this, we conclude $I \setminus U \in \mathcal{W}^f_{(y,x)}$. ∎
   Similarly, preservation of transitivity has strong implications on the relation between families of winning coalitions for different pairs of alternatives which amount to a property of intersection and superset closure across pairs of alternatives.

Lemma 6 (Monotonicity Lemma) *Let $f : \Omega^I \to \Omega$ be a social welfare function with universal domain which satisfies independence of irrelevant alternatives (and suppose $\#A \geq 3$). Then for any triple of distinct alternatives $x, y, z \in A$, any winning coalitions $U \in \mathcal{W}^f_{(x,y)}$ and $V \in \mathcal{W}^f_{(y,z)}$, $W \in \mathcal{W}^f_{(x,z)}$ for all $W \supseteq U \cap V$.*

Proof. Since f is a social welfare function with universal domain, we can construct a profile $\underline{\mathfrak{A}} \in \Omega^I = \mathrm{dom}(f)$ such that
   (a) $\{i \in I : \mathfrak{A}_i \models R(x,y)\} = U$,
   (b) $\{i \in I : \mathfrak{A}_i \models R(y,z)\} = V$, and
   (c) $\{i \in I : \mathfrak{A}_i \models R(x,z)\} = W$.
   (This is possible due to the assumption of $W \supseteq U \cap V$ and x, y, z being distinct.)
   By (a), (b) and the decisiveness of U, V, $f(\underline{\mathfrak{A}}) \models R(x,y) \wedge R(y,z)$ and hence, by transitivity, $f(\underline{\mathfrak{A}}) \models R(x,z)$. Thus, by independence, $\{i \in I : \mathfrak{A}_i \models R(x,z)\} \in \mathcal{W}^f_{(x,z)}$, whence by (c), $W \in \mathcal{W}^f_{(x,z)}$. ∎
   Social welfare functions that satisfy the above normative conditions except nondictatorship will be called *Arrovian*.

Definition 7 *A social welfare function $f : \mathrm{dom}(f) \to \Omega$ is called **Arrovian** if and only if it has universal domain ($\mathrm{dom}(f) = \Omega^I$), is weakly Paretian and satisfies independence of irrelevant alternatives.*

In the following we give a simple proof of a generalization of Arrow's theorem which establishes its relation to the ultraproduct construction in model theory by showing that

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[2] A familiy $\mathcal{W} \subseteq 2^I$ of subsets of some set I is said to be *strong* if for any $U \in 2^I$ $I \backslash U \in \mathcal{W}$ whenever $U \notin \mathcal{W}$.

an Arrovian social welfare function is equivalent to the reduction of a direct product of preference relations over an ultrafilter on the set of individuals.

Recall that a *filter* on the set I is a family $\mathcal{W} \subset 2^I$ such that
(F1) $\mathcal{W} \neq \varnothing$ and $\varnothing \notin \mathcal{W}$ (non-triviality)
(F2) $U \cap V \in \mathcal{W}$ for all $U, V \in \mathcal{W}$ (finite intersection closure)
(F3) $V \in \mathcal{W}$ whenever $V \supseteq U$ for some $U \in \mathcal{W}$ (superset closure).

A filter is an *ultrafilter* on I if for any $U \subseteq I$ either $U \in \mathcal{W}$ or $I \backslash U \in \mathcal{W}$.
An ultrafilter $\mathcal{W}$ on I is *principal* if and only if there exists some $k \in I$ such that $\mathcal{W} = \{U \subseteq I : k \in I\}$.
The reduction of a direct product $\underline{\mathfrak{A}}$ over an ultrafilter $\mathcal{W}$ is known as an *ultraproduct* and is denoted by $\underline{\mathfrak{A}}/\mathcal{W}$ (for an introduction to ultraproducts see (2)).

Theorem 8 *Let $f : \Omega^I \to \Omega$ be an Arrovian social welfare function. Then there exists an ultrafilter $\mathcal{W} \subset 2^I$ such that*
(i) for any profile $\underline{\mathfrak{A}} \in \Omega^I$ and for all pairs of alternatives $x, y \in A$, $f(\underline{\mathfrak{A}}) \models R(x, y)$ if and only if $\{i \in I : \mathfrak{A}_i \models R(x, y)\} \in \mathcal{W}$, and
(ii) for any profile $\underline{\mathfrak{A}} \in \Omega^I$ and for all pairs of alternatives $x, y \in A$
$f(\underline{\mathfrak{A}}) \models R(x, y)$ if and only if $\underline{\mathfrak{A}}/\mathcal{W} \models R(x, y)$.
In particular, if I is finite, then there is no non-dictatorial Arrovian social welfare function.

For the proof of this theorem we use several lemmas.

The first of these lemmas is in the spirit of the contagion lemmas known from classical Arrovian social choice theory.

Lemma 9 (Contagion Lemma) *Let $f : \Omega^I \to \Omega$ be an Arrovian social welfare function. Then for any two pairs of (possibly nondistinct) alternatives $a, b \in A$ and $x, y \in A$*

$$\mathcal{W}^f_{(x,y)} = \mathcal{W}^f_{(a,b)}$$

Proof. Let $a, b, x, y \in A$. It is sufficient to prove the inclusion $\mathcal{W}^f_{(x,y)} \subseteq \mathcal{W}^f_{(a,b)}$ as the converse inclusion will then follow from interchanging a, b, x, y.

Let hence $U \in \mathcal{W}^f_{(x,y)}$. Since f is a social welfare function with universal domain, we can construct a profile $\underline{\mathfrak{A}} \in \text{dom}(f)$ such that
(a) for all $i \in I$, $\mathfrak{A}_i \models R(a, x) \wedge R(y, b) \wedge R(x, a) \wedge R(b, y)$ and
(b) $\{i \in I : \mathfrak{A}_i \models R(x, y)\} = U$.

Then, transitivity implies for all $i \in I$, $\mathfrak{A}_i \models (R(a, b) \leftrightarrow R(x, y))$. This yields on the one hand an alternative description of U: $\{i \in I : \mathfrak{A}_i \models R(a, b)\} = U$. On the other hand, (a) entails, via the Pareto principle, first $f(\underline{\mathfrak{A}}) \models R(a, x) \wedge R(y, b) \wedge R(x, a) \wedge R(b, y)$ and then by transitivity also $f(\underline{\mathfrak{A}}) \models (R(a, b) \leftrightarrow R(x, y))$. However, we already know that $f(\underline{\mathfrak{A}}) \models R(x, y)$ due to (b) and our assumption $U \in \mathcal{W}^f_{(x,y)}$. Hence, actually we already have $f(\underline{\mathfrak{A}}) \models R(a, b)$ and thus by the previously established alternative description of U, we arrive at $U \in \mathcal{W}^f_{(a,b)}$. ∎

This neutrality property immediately strenghtens independence to a property known as systematicity in the literature on judgment aggregation:

Proposition 10 *Let $f : \Omega^I \to \Omega$ be an Arrovian social welfare function. Then f is **systematic**, i.e. for all $x, y \in A$*

$$\mathcal{W}^f_{(x,y)} = \bigcup_{a,b \in A} \mathcal{W}^f_{(a,b)} = \bigcap_{a,b \in A} \mathcal{W}^f_{(a,b)}$$

In view of this equality, we may henceforth suppress the subscript of $\mathcal{W}^f$; the family of winning coalitions inherits the strongness property of any of the $\mathcal{W}^f_{(x,y)}$.

With these results, the proof of the theorem follows almost immediately.

Proof. Let $\mathcal{W}$ be the family $\mathcal{W}^f$ of winning coalitions. We verify (i) and (ii) in the Theorem, as follows:

(i) Non-triviality (F1) follows directly from the weak Pareto property combined with the strongness property (which ensures $I \in \mathcal{W}$), while intersection (F2) and superset closure (F3) follow from the Monotonicity Lemma. Moreover, given that $\mathcal{W}$ is a filter, the strongness property implies that it is an ultrafilter.

(ii) Follows directly from part (i) and the (elementary) atomic case of Łoś's theorem.[3]

Finally, let I be finite, and suppose, for a contradiction, f were a non-dictatorial Arrovian social welfare function. The finiteness of I implies, by a well-known lemma from Boolean algebra (cf. e.g. Ch. 6, §1, Lemma 1.3 in (2)), that $\mathcal{W}$ is principal. Hence in light of (i), there is some individual $k \in I$ such that for all $\underline{\mathfrak{A}} \in \Omega^I$ and all $x, y \in A$, $f(\underline{\mathfrak{A}}) \models R(x,y)$ if and only if $\mathfrak{A}_k \models R(x,y)$. Such an individual k is a dictator, contradiction.

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4 Conclusion

According to Arrow's theorem, it is the requirement of the preservation of the first-order properties of the individual preference relations by an Arrovian social welfare function which establishes the equivalence of the latter with the model-theoretic construction later known as ultraproduct, i.e. the reduction of the direct product over an ultrafilter on the index set of the individuals. A typical preservation problem thus lies at the origin of the development of Arrovian social theory. As dictatorship is just a consequence of the ultrafilter structure of the family of winning coalitions on a finite set of individuals, preservation problems can be seen to lie at the heart of impossibility results in aggregation theory.

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[3]Łoś's theorem is the central theorem on ultraproducts. It asserts in particular (cf. e.g. Ch. 5, §2, Corollary 2.2 in (2)), that for any profile $\underline{\mathsf{A}} \in \Omega^I$ and any sentence φ, $\underline{\mathsf{A}}/\mathcal{W} \models \varphi$ if and only if $\{i \in I : \mathsf{A}_i \models \varphi\} \in \mathcal{W}$. In our proof, we only need this result for atomic φ, viz. for every $\underline{\mathsf{A}} \in \Omega^I$ and all $x, y \in A$,

$$\underline{\mathsf{A}}/\mathcal{W} \models R(x,y) \Leftrightarrow \{i \in I : \mathsf{A}_i \models R(x,y)\} \in \mathcal{W},$$

which is an immediate consequence of the definition of an ultraproduct and Tarski's definition of truth.

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