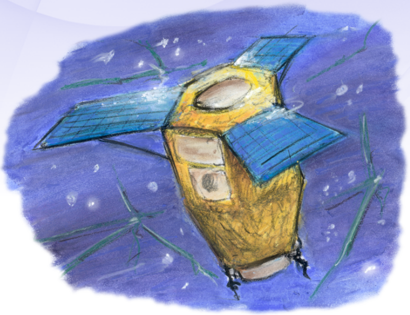
Fair Allocation of Indivisible Goods: Modelling, Compact Representation using Logic, and Complexity



MARA-revival Workshop. 6th June 2008.

Sylvain Bouveret

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The resource allocation problem. . .

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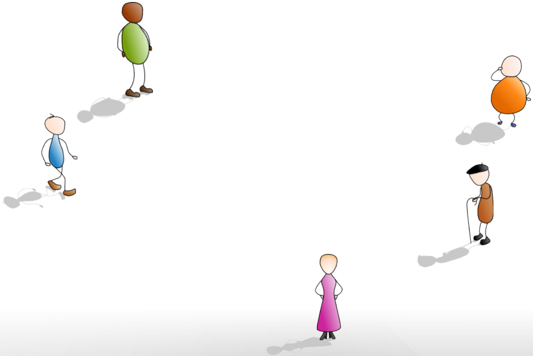
Inputs

- A finite set $\mathscr{N}$ of **agents** .



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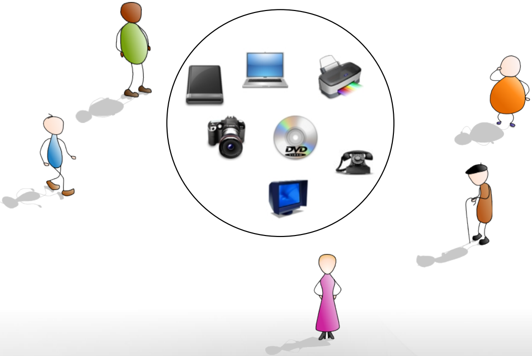

The resource allocation problem. . .

Inputs

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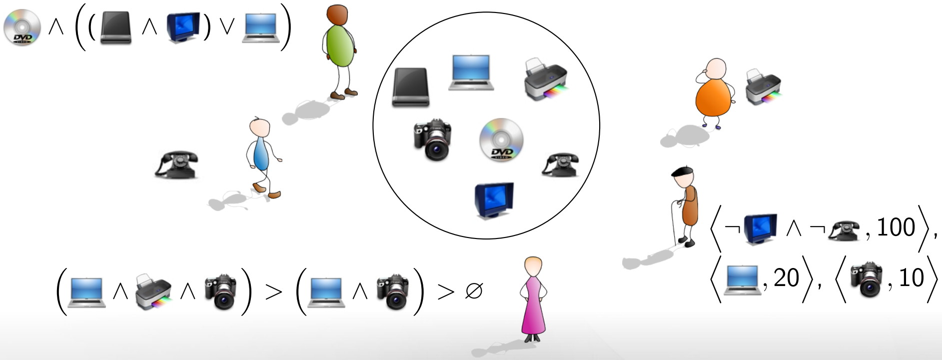

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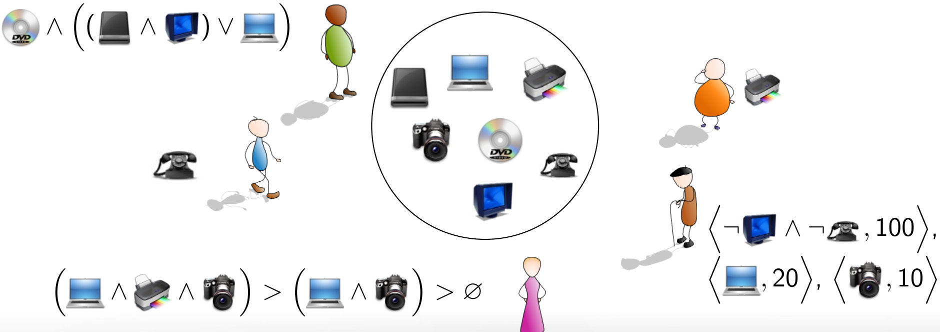

A bundle cannot exceed the transport capacity of an agent.

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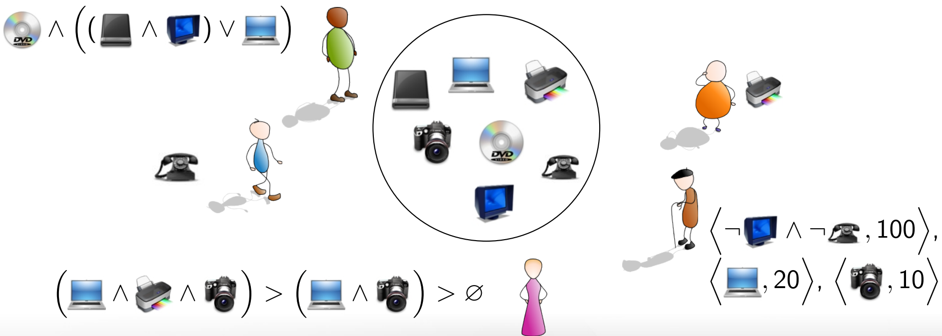

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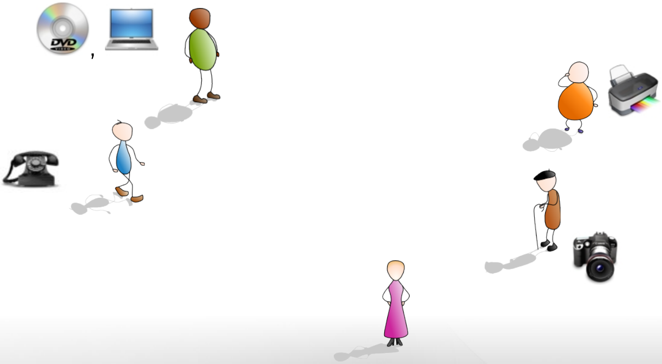

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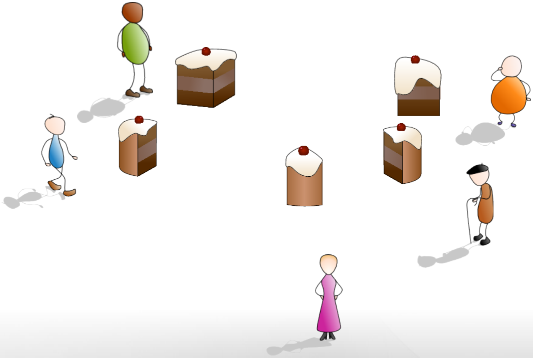

Real-world applications

An ubiquitous problem. . .

- Fair share of Earth Observation Satellites.
- Tasks or subjects allocation.
- Combinatorial auctions problems [Cramton et al., 2006].
- Computer network sharing, rostering problems, allocation of take-off and landing slots in airports [Faltings, 2005],. . ..

Cramton, P., Shoham, Y., and Steinberg, R., editors (2006).
Combinatorial Auctions.
MIT Press.

Faltings, B. (2005).
A budget-balanced, incentive-compatible scheme for social choice.
In Faratin, P. and Rodriguez-Aguilar, J. A., editors, *Agent-Mediated Electronic Commerce VI*, volume 3435 of *LNAI*, pages 30–43. Springer.

Outline of the talk

We focus on **fair** and **constrained** resource allocation problems, on **combinatorial domains** :

- Basic concepts and modelling.
- Compact representation and complexity.

Outline

1. **The elements of the fair resource allocation problem**
   - The resource
   - Admissibility constraints
   - The agents' preferences
   - Welfarism

2. **Compact representation and complexity**
   - About compact representation. . .
   - Collective utility maximization problem: representation and complexity
   - Efficiency and envy-freeness: representation and complexity

The resource allocation problem

Inputs
- A set $\mathscr{N}$ of **agents** expressing **preferences** on the resource.
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- A set of **constraints** (physical, legal, moral,...).
- A decision or optimisation **criterion**

Sortie
- The allocation of a part of or the whole resource to each agent / no violated constraint / criterion optimized or verified.

The resource allocation problem

Inputs

- A set $\mathcal{N}$ of **agents** expressing **preferences** on the resource.
- **A limited common resource.**

$\rightsquigarrow$ Continuous resource, discrete, indivisible, mixed ;

$\rightsquigarrow$ Possibility of monetary compensations.

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Indivisible resource, share, allocation

- Indivisible resource : set of objects $\mathcal{O}$.
- Share of an agent : $\pi \subseteq \mathcal{O}$.
- Allocation : $\overrightarrow{\pi} \in 2^{\mathcal{O}^n}$.

The resource allocation problem

Inputs
- A set $\mathscr{N}$ of **agents** expressing **preferences** on the resource.
- **The resource** $\rightsquigarrow$ a finite set $\mathscr{O}$ of indivisible objects.
- **A set of constraints (physical, legal, moral,. . .).**
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Constraints on the resource

Admissibility constraint, admissible allocation

- Constraint : subset $C \subseteq 2^{\mathscr{O}^n}$.
- Admissible allocation : allocation $\overrightarrow{\pi} \in \bigcap_{C \in \mathscr{C}} C$.

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Preemption constraint

An object cannot be allocated to more than one agent :

$$C_{preempt} = \{\overrightarrow{\pi} \mid \forall i \neq j,\ \pi_i \cap \pi_j \neq \varnothing\}$$

Constraints on the resource

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- Preemption constraint.
- Exclusion constraint.
- Volume constraint.

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Preference structure

Usual model in decision theory :

Preference structure

Binary reflexive relation $\Re_S$ on the set of alternatives $\mathscr{E}$.
$x \Re_S y \Leftrightarrow x$ is at least as good as y.

Main kinds of preference structures

- Ordinal preference structure.
    - Dichotomous preference structure.
- Cardinal preference structure.
- Semi-orders (threshold models), interval orders (variable threshold models), fuzzy preference structure,. . .

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Ordinal preference structure

A complete preorder $\succeq$ on the alternatives ($\Re_S$ + transitivity + completeness).

Dichotomous preference structure

Degenerated kind of ordinal preferences, with two equivalence classes :

- a set of "good" alternatives,
- a set of "bad" alternatives.

Main kinds of preference structures

- Ordinal preference structure.
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- **Cardinal preference structure.**
- Semi-orders (threshold models), interval orders (variable threshold models), fuzzy preference structure,. . .

Cardinal preference structure

Refinement of the ordinal model by a **utility function** $u : \mathscr{E} \to \mathscr{V}$.
$\mathscr{V}$ totally ordered valuation space (*e.g.* $\mathbb{R}$, $\mathbb{N}$).

Main kinds of preference structures

- Ordinal preference structure.
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- **Semi-orders (threshold models), interval orders (variable threshold models), fuzzy preference structure,. . .**

Target space of the preferences

On which set of alternatives do the agents express their preferences ?

Assumption (non exogenous preferences) : Each agent can only express preferences on the set of possible allocations (in particular, s/he cannot take into account what the others receive).

set of alternatives = set of possible shares. For an agent i, $2^{\mathscr{O}}$.

The resource allocation problem

Inputs

- A set $\mathscr{N}$ of **agents** expressing **preferences** on the resource using preorders $\succeq_i$ or utility functions u_i.
- **The resource** $\leadsto$ a finite set $\mathscr{O}$ of indivisible objects.
- **Some constraints** $\leadsto$ a finite set $\mathscr{C} \subset 2^{2^{\mathscr{O}^n}}$.
- **A decision or optimisation criterion**

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Preference aggregation...

The problem : *How to distribute the resource among the agents, in a way such that it takes into account in an equitable way their antagonistic preferences ?*

Preference aggregation. . .

The problem : *How to distribute the resource among the agents, in a way such that it takes into account in an equitable way their antagonistic preferences ?*

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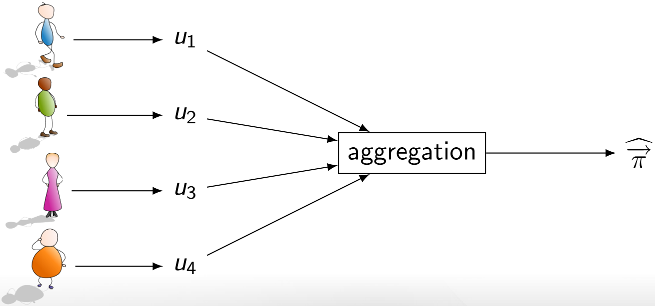

The cardinal welfarism

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Social Welfare Ordering

A **social welfare ordering** is a preorder $\preceq$ on $\mathscr{V}^n$.

A social welfare ordering reflects the **collective preference ordering** regarding the set of possible allocations.

Collective utility function

A **collective utility function** is a function from $\mathscr{V}^n$ to $\mathscr{V}$.

A collective utility function represents a particular social welfare ordering.

Fairness ?

Fairness [Young, 1994]

"[...] appropriate to the need, status and contribution of [the society's] various members."

Four principles of distributive justice from Aristotle (*Nicomachean Ethics, Book V*) – see [Moulin, 2003] :

- compensation ;
- merits ;
- exogenous rights ;
- fitness.

Moulin, H. (2003).
Fair Division and Collective Welfare.
MIT Press.

Young, H. P. (1994).
Equity in Theory and Practice.
Princeton University Press.

Basic properties of Social Welfare Orderings

Unanimity

A utility vector $\overrightarrow{u}$ **Pareto-dominates** another utility vector $\overrightarrow{v}$ iff for all i, $u_i \geq v_i$ and there is an i s.t. $u_i > v_i$.
A non Pareto-dominated vector is said **Pareto-efficient**.
A Social Welfare Ordering $\preceq$ satisfies **unanimity** iff :

$$\overrightarrow{u} \text{ Pareto-dominates } \overrightarrow{v} \Rightarrow \overrightarrow{u} \succ \overrightarrow{v}.$$

Anonymity

$$(u_1, \ldots, u_n) \sim (u_{\sigma(1)}, \ldots, u_{\sigma(n)}),$$

for all permutation σ of $[\![1, n]\!]$.

Fairness. . .

- Properties of Social Welfare Orderings :
  - Anonymity (property of fairness *ex-ante*).
  - Pareto-compatible.
  - Fair share guaranteed.
  - Reduction of inequalities.
- Properties of allocations :
  - Pareto-efficiency.
  - Fair share test.
  - Inequality measurement. Atkinson and Gini indices, Lorenz curve. . .
  - Envy-freeness test.

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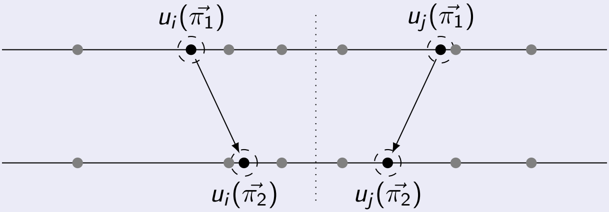
Reduction of inequalities (Pigou-Dalton principle)



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Envy-freeness

$\overrightarrow{\pi}$ is envy-free iff for each $i \neq j$, $\pi_i \succ_i \pi_j$.

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Usual Social Welfare Orderings

- Classical utilitarian order.
- Egalitarian order.
- Leximin egalitarian order.
- Compromises between classical utilitarianism and egalitarianism : Nash ($\times$), families OWA and sum of powers,. . .

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Classical utilitarianism [Harsanyi]

$\overrightarrow{u} \preceq \overrightarrow{v} \Leftrightarrow \sum_{i=1}^{n} u_i \leq \sum_{i=1}^{n} v_i$.

Features

Conveys the sum-fitness principle (resource goes to who makes the best use of it).
Indifferent to inequalities (Pigou-Dalton) $\rightsquigarrow$ can lead to huge inequalities between the agents.

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Conveys the compensation principle : the least well-off must be made as well-off as possible (justice according to needs) $\rightsquigarrow$ tends to equalize the utility profile.

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Conveys the compensation principle : the least well-off must be made as well-off as possible (justice according to needs) $\rightsquigarrow$ tends to equalize the utility profile.

However, it can lead to non Pareto-efficient decisions (drowning effect).

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Egalitarian SWO and Pareto-efficiency

$\langle 1,1,1,1\rangle \sim \langle 1000,1,1000,1000\rangle$, whereas $\langle 1,1,1,1\rangle$ and $\langle 1000,1,1000,1000\rangle$ are very different !

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Leximin egalitarianism [Sen, 1970 ; Kolm, 1972]

Let $\overrightarrow{x}$ be a vector. We write $\overrightarrow{x^{\uparrow}}$ the sorted version of $\overrightarrow{x}$.
$\overrightarrow{u} \succ_{leximin} \overrightarrow{v} \Leftrightarrow \exists k$ such that $\forall i \leq k,\ u_i^{\uparrow} = v_i^{\uparrow}$ and $u_{k+1}^{\uparrow} > v_{k+1}^{\uparrow}$.
This is a lexicographical comparison over sorted vectors.

Perform a leximin comparison...

Two vectors to compare : $\overrightarrow{u} = \langle 4, 10, 3, 5 \rangle$ and $\overrightarrow{v} = \langle 4, 3, 6, 6 \rangle$.

- We sort the two vectors : $\begin{cases} \overrightarrow{u}^{\uparrow} = \langle 3, 4, 5, 10 \rangle \\ \overrightarrow{v}^{\uparrow} = \langle 3, 4, 6, 6 \rangle \end{cases}$
- We lexicographically sort the ordered vectors : $\overrightarrow{u}^{\uparrow} \prec_{lexico} \overrightarrow{v}^{\uparrow}$

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Features

This SWO both refines the egalitarian SWO and the Pareto relation $\rightsquigarrow$ it inherits of the fairness features of egalitarism, while overcoming drowning effect.

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Leximin SWO leximin and Pareto-efficiency

$\langle 1, 1, 1, 1 \rangle \prec \langle 1000, 1, 1000, 1000 \rangle$ (the second value of the two vectors is discriminating).

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(Ex-post) Fairness and efficiency in resource allocation

Two different points of view :

- **Reduction of inequalities :**
  - Aggregation of utilities using a SWO or CUF compatible with the Pigou-Dalton principle (and with the Pareto relation).
  - Example : leximin.
  - Needs the **interpersonnal comparison** of utilities.
- **Envy-freeness :**
  - One looks for an envy-free (and Pareto-efficient) allocation.
  - Only based on the agents' personnal point of view.
  - Purely **ordinal** property.
  - However, not always relevant (for ethical or technical reasons).

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The resource allocation problem

Inputs

- A set $\mathscr{N}$ of **agents** expressing **preferences on the resource**.
- **The resource** $\rightsquigarrow$ a finite set $\mathscr{O}$ of indivisible objects.
- **Some constraints** $\rightsquigarrow$ a finite set $\mathscr{C} \subset 2^{2^{\mathscr{O}\,n}}$.
- A **criterion** $\rightsquigarrow$ maximization of a SWO or of a CUF, or efficiency and envy-freeness.

Sortie

- The allocation of a part of or the whole resource to each agent / no violated constraint / criterion optimized or verified.

Some other issues

- Unequal exogenous rights :
  - One weight (hierarchy, age, ...) per agent.
  - Duplication of agents principle.
- Repeated resource allocation :
  - Possibility of compensation over time.
  - Using exogenous rights to bias future resource allocations ?
- Partial knowledge.
  - The resource allocator has a partial knowledge of the agents' preferences.
  - The agents have partial knowledge of the other agents, and of their preferences.

Outline

1. The elements of the fair resource allocation problem
   - The resource
   - Admissibility constraints
   - The agents' preferences
   - Welfarism

2. **Compact representation and complexity**
   - About compact representation...
   - Collective utility maximization problem: representation and complexity
   - Efficiency and envy-freeness: representation and complexity

A representation language

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No idea on how the instances are **formally represented**, and how they should be implemented.

These precisions are crucial, particularly for the representation of **constraints** and **preferences**.

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Compact preference representation

Example

Resource allocation problem with 2 objects o_1 and o_2.
Expression of the utility function :
$u(\varnothing) = 0,\ u(o_1) = 5,\ u(o_2) = 7,\ u(\{o_1, o_2\}) = 3.$

Compact preference representation

Example

Resource allocation problem with 4 objects o_1, o_2, o_3 and o_4.
Expression of the utility function :
$u(\varnothing) = 0$, $u(o_1) = 5$, $u(o_2) = 7$, $u(o_3) = 2$, $u(o_4) = 8$, $u(\{o_1, o_2\}) = 3$, $u(\{o_1, o_3\}) = 5$, $u(\{o_1, o_4\}) = 3$, $u(\{o_2, o_3\}) = 0$, $u(\{o_2, o_4\}) = 6$, $u(\{o_3, o_4\}) = 2$, $u(\{o_1, o_2, o_3\}) = 8$, $u(\{o_1, o_2, o_4\}) = 9$, $u(\{o_1, o_3, o_4\}) = 10$, $u(\{o_2, o_3, o_4\}) = 3$, $u(\{o_1, o_2, o_3, o_4\}) = 10$.

Compact preference representation

Example

Resource allocation problem with 20 objects $o_1, \ldots, o_{20}$

Expression of the utility function :

$u(\varnothing) = 0,\ u(o_1) = 5,\ u(o_2) = 7,\ u(o_3) = 2,\ u(o_4) = 8\ u(o_5) = 5,\ u(o_6) = 0,\ u(o_7) = 1,$
$u(o_8) = 15\ u(o_9) = 4,\ u(o_{10}) = 6,\ u(o_{11}) = 6,\ u(o_{12}) = 8,\ u(o_{13}) = 5,\ u(o_{14}) = 7,$
$u(o_{15}) = 2,\ u(o_{16}) = 8,\ u(o_{17}) = 7,\ u(o_{18}) = 2,\ u(o_{19}) = 8,\ u(o_{20}) = 7,\ u(\{o_1, o_2\}) = 15,$
$u(\{o_1, o_3\}) = 12,\ u(\{o_1, o_4\}) = 5,\ u(\{o_1, o_5\}) = 1,\ u(\{o_1, o_6\}) = 4,\ u(\{o_1, o_7\}) = 2,$
$u(\{o_1, o_8\}) = 8,\ u(\{o_1, o_9\}) = 10,\ u(\{o_1, o_{10}\}) = 3,\ u(\{o_1, o_{11}\}) = 11,\ u(\{o_1, o_{12}\}) = 12,$
$u(\{o_1, o_{13}\}) = 5,\ u(\{o_1, o_{14}\}) = 13,\ u(\{o_1, o_{15}\}) = 3,\ u(\{o_1, o_{16}\}) = 15,\ u(\{o_1, o_{17}\}) = 1,$
$u(\{o_1, o_{18}\}) = 3,\ u(\{o_1, o_{19}\}) = 11,\ u(\{o_2, o_3\}) = 12,\ u(\{o_2, o_4\}) = 5,\ u(\{o_2, o_5\}) = 1,$
$u(\{o_2, o_6\}) = 4,\ u(\{o_2, o_7\}) = 2,\ u(\{o_2, o_8\}) = 8,\ u(\{o_2, o_9\}) = 10,\ u(\{o_2, o_{10}\}) = 3,$
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$u(\{o_3, o_{15}\}) = 3,\ u(\{o_3, o_{16}\}) = 15,\ u(\{o_3, o_{17}\}) = 1,\ u(\{o_3, o_{18}\}) = 3,\ u(\{o_3, o_{19}\}) = 11,$

Compact preference representation

Example

Resource allocation problem with 20 objects $o_1, \ldots, o_{20}$

Expression of the utility function :

$u(\emptyset) = 0,\ u(o_1) = 5,\ u(o_2) = 7,\ u(o_3) = 2,\ u(o_4) = 8\ u(o_5) = 5,\ u(o_6) = 0,\ u(o_7) = 1,$
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1048576 values $\rightsquigarrow$ the expression needs more than 12 days (supposing the agent expresses 1 value per second).

Compact preference representation

Three possible answers to combinatorial explosion :

1. Ignore it and suppose that the number of objects is low [Herreiner and Puppe, 2002].
2. Add some restrictive assumptions on the preferences (for example : additivity) that make the expression possible [Brams et al., 2003] and [Demko and Hill, 1998].
3. Use a **compact representation language**.

Brams, S. J., Edelman, P. H., and Fishburn, P. C. (2003).
Fair division of indivisible items.
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Compact preference representation languages

- Dichotomous preferences :
  - propositional logics.
- Ordinal preferences :
  - prioritized goals (best-out, discrimin, leximin...),
  - CP-nets, TCP-nets.
- Cardinal Preferences :
  - k-additive languages, GAI-nets,
  - weighted-goals based languages,
  - bidding languages for combinatorial auctions (OR, XOR, ...),
  - UCP-nets,
  - valued CSP.

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Resource allocation and compact representation

We will introduce two compact representation languages, based on **propositional logic**, for the two following problems :

- Maximizing collective utility.
- Existence of a Pareto-efficient and envy-free allocation.

Agents, objects and allocation

Allocation of indivisible goods among agents

- Set of **agents** $\mathcal{N} = \{1, \ldots, n\}$.
- Set of **items** $\mathcal{O}$.
- Allocation $\overrightarrow{\pi} = \langle \pi_1, \ldots, \pi_n \rangle$ ($\pi_i \subseteq \mathcal{O}$ is agent i's **share**).

Constraints

A propositional language $L_{\mathcal{O}}^{alloc}$:

- a set of propositional symbols $\{alloc(o,i) \mid o \in \mathcal{O}, i \in \mathcal{N}\}$.
- the usual connectives $\neg, \wedge, \vee$

Constraint

A constraint is a formula of $L_{\mathcal{O}}^{alloc}$.

Example

The preemption constraint can be expressed by the set of formulae :

$$\{\neg(alloc(o,i) \wedge alloc(o,j)) \mid i,j \in \mathcal{N}, i \neq j\}.$$

A language based on weighted logic

Preference representation :

- A propositional language $L_{\mathscr{O}}$...
  - a set of propositional symbols $\mathscr{O}$,
  - the usual connectives $\neg, \wedge, \vee$
- ... and some weights $w \in \mathscr{V}$.

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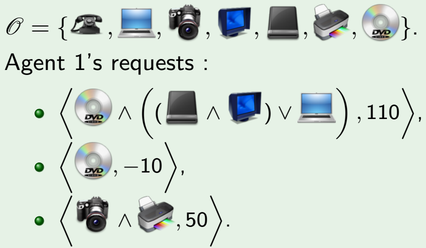
Example

- $\mathscr{O} = \{$ ... $\}$.
- Agent 1's requests :



Individual utility

Expresses the satisfaction of an agent regarding an allocation. Depends on :

- her share (assumption of non exogenity),
- her weighted requests,

and is obtained by **aggregating** the weights of the satisfied formulas, using an operator $\oplus$.

Individual utility

Given an agent i, her requests Δ_i, an allocation $\overrightarrow{\pi}$, her individual utility is :

$$u_i(\pi_i) = \bigoplus \{w \mid \langle \varphi, w \rangle \in \Delta_i \text{ et } x_i \models \varphi\}.$$

Two reasonable choices for $\oplus$: $+$ or max.

Individual utility

Example

- $\mathcal{O} = \{$  ,  ,  ,  ,  ,  ,  $\}$.
- Agent 1's requests :
  - $\langle$  $\wedge$ $(($  $\wedge$  $) \vee$  $), 110 \rangle$,
  - $\langle$  $, -10 \rangle$,
  - $\langle$  $\wedge$  $, 50 \rangle$.

Computation of individual utility ($\oplus = +$) :

$\pi_1 = \{$  ,  ,  ,  $\}$

Individual utility

Example

- $\mathscr{O} = \{$ , , , , , ,  $\}$.
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Individual utility

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Computation of individual utility ($\oplus = +$) :

$\pi_1 = \{$ , , ,  $\} \Rightarrow u_1(\pi_1) = 110 - 10 + 0 =$ **100**

Collective utility

Expressed as an aggregation of individual utilities.

Collective utility

Given : an allocation $\overrightarrow{\pi}$, a set of agents $\mathcal{N}$ and their individual utilities,

$$uc(\overrightarrow{\pi}) = g(u_1(\pi_1), \ldots, u_n(\pi_n)),$$

with g a commutative and non-decreasing function from $\mathcal{V}^n$ to $\mathcal{V}$.

Two levels of aggregation :

$$\left.\begin{array}{ccc} w_1^1, \ldots, w_{p_1}^1 & \overset{\oplus}{\longmapsto} & u_1 \\ & \vdots & \\ w_1^n, \ldots, w_{p_n}^n & \overset{\oplus}{\longmapsto} & u_n \end{array}\right\} \overset{g}{\longmapsto} uc.$$

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The resource allocation problem

To sum-up :

Instance of the resource allocation problem

Inputs

- A finite set $\mathcal{N}$ of **agents** expressing **requests** $\{\Delta_1, \ldots, \Delta_n\}$ under weighted propositional form $L_{\mathcal{O}} \times \mathcal{V}$
- A finite set $\mathcal{O}$ of indivisible items.
- A finitie set $\mathcal{C}$ of constraints expressed in a propositional language $L_{\mathcal{O}}^{alloc}$.
- A pair of aggregation operators $(\oplus, g)$.

Output

- An allocation $\overrightarrow{\pi} \in 2^{\mathcal{O}^n}$ such that $\{alloc(o, i) \mid o \in \pi_i\} \models \bigwedge_{C \in \mathcal{C}} C$ and that maximizes the collective utility function defined as :

$$uc(\overrightarrow{\pi}) = g(u_1, \ldots, u_n), \text{ with}$$

$$u_i = \bigoplus \{w \mid \langle \varphi, w \rangle \in \Delta_i \text{ et } x_i \models \varphi\}.$$

The collective utility maximization problem

What is the complexity of the problem of maximizing collective utility ?

Problem [MAX-CUF]

Given an instance of the resource allocation problem, and an integer K ($\mathscr{V} = \mathbb{N}$), does an admissible allocation $\overrightarrow{\pi}$ exists, such that $uc(\overrightarrow{\pi}) \geq K$?

This problem is **NP-complete**.

Does it remain **NP**-complete in the following cases :

- restrictions on the operators ($\oplus \in \{+, \max\}$, $g \in \{+, \min, \text{leximin}\}$),
- restrictions on the constraints (preemption, volume, exclusion),
- restriction on the preferences (atomic) ?

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The complexity results

[MAX-CUF]

Any kind of constraints : **NPC**

No constraint : **P**

Exclusion constraints only

⊕ \ g	+	(lexi)min
+	**NPC**	**NPC**
max	**NPC**	**NPC**

Volume constraints only

⊕ \ g	+	(lexi)min
+	**NPC**	**NPC**
max	**NPC**	**NPC**

Preemption constraints

Atomic requests

⊕ \ g	+	min	leximin
+	**P**	**NPC**, **P** if eq. wgts	**NPC**
max	**P**	**P**	?

Any kind of requests

⊕ \ g	+	(lexi)min
+	**NPC**	**NPC**
max	**NPC**	**NPC**

Envy-freeness

Another way to consider the notion of equity : **envy-freeness**.

Envy-freeness alone is not enough : we need an **efficiency** criterion (**Pareto-efficiency**, completeness, CUF maximization, ...).

But... There does not always exist an envy-free and efficient allocation does not always exist, and it could be **complex** to determine if there is one.

How complex it is to determine if there is an efficient and envy-free allocation, when the agents' preferences are expressed compactly, with preemption constraint only ?

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Of dichotomous preferences. . .

We will study the particular case where preferences are **dichotomous**.

Dichotomous preference relation

$\succeq$ is dichotomous $\Leftrightarrow$ there exists a set of "good" bundles *Good* such that $\pi \succeq \pi' \Leftrightarrow \pi \in \mathit{Good}$ ou $\pi' \notin \mathit{Good}$.

Example :

$$\mathscr{O} = \{o_1, o_2, o_3\}$$

$$\Rightarrow 2^{\mathscr{O}} = \{\varnothing, \{o_1\}, \{o_2\}, \{o_3\}, \{o_1, o_2\}, \{o_1, o_3\}, \{o_2, o_3\}, \{o_1, o_2, o_3\}\}$$

$$\mathit{Good} \longrightarrow \{\{o_1, o_2\}, \{o_2, o_3\}\}$$

$$\overline{\mathit{Good}} \longrightarrow \{\varnothing, \{o_1\}, \{o_2\}, \{o_3\}, \{o_1, o_3\}, \{o_1, o_2, o_3\}\}$$

Once again, propositional logic. . .

A dichotomous preference relation is represented by its set *Good*. A direct way to represent this set is to use propositional logic.

Example :

		
$Good_i$	$\{\{o_1, o_2\}, \{o_2, o_3\}\}$	$\{\{o_2\}\{o_2, o_3\}\}$
φ_i	$(o_1 \land o_2 \land \neg o_3) \lor (\neg o_1 \land o_2 \land o_3)$	$o_2 \land \neg o_1$

Preemption, envy-freeness and Pareto-efficiency

- The **preemption** constraint : a logical formula of $L_{\mathscr{O}}^{alloc} \rightsquigarrow \Gamma_{\mathcal{P}}$.
- The **envy-freeness** property can be expressed as a formula of $L_{\mathscr{O}}^{alloc} \rightsquigarrow \Lambda_{\mathcal{P}}$.
- The **Pareto-efficiency** property is equivalent to :
  - satisfying a maximal number (in the inclusion sense) of agents,
  - the consistency of $F(\overrightarrow{\pi})$ with a maximal-consistent subset of formulae from $\{\varphi_1^*, \ldots, \varphi_n^*\}$.

Existence of a Pareto-efficient and envy-free allocation

$\exists \mathscr{S}$ maximal $\Gamma_{\mathcal{P}}$-consistent subset of $\{\varphi_1^*, \ldots, \varphi_n^*\}$ such that $\bigwedge_{\varphi \in \mathscr{S}} \varphi \wedge \Gamma_{\mathcal{P}} \wedge \Lambda_{\mathcal{P}}$ is consistent.

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A skeptical inference problem

It is actually a well-known problem in the field of non-monotonic reasoning : *skeptical inference with normal defaults without prerequisites* [Reiter, 1980].
The [EEF-EXISTENCE] problem can be reduced to :

$$\langle \Gamma_{\mathcal{P}}, \{\varphi_1^*, \ldots, \varphi_n^*\}\rangle \not\mid\!\sim^{\forall} \neg\Lambda_{\mathcal{P}}$$

Reiter, R. (1980).
A logic for default reasoning.
Artificial Intelligence, 13 :81–132.

The [EEF EXISTENCE] problem, dichotomous preferences

Proposition

The [EEF EXISTENCE] problem for agents having monotonic dichotomous preferences under logical form is Σ_2^p-complete ($\Sigma_2^p = \mathbf{NP}^{\mathbf{NP}}$).

This results holds even if preferences are not mononic.

- **Restrictions :**
  - identical preferences,
  - number of agents,
  - the propositional language.
- **Alternative efficiency criterion :**
  - completeness,
  - maximal number of satisfied agents.

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Non dichotomous preferences ?

Corollary

The [EEF EXISTENCE] problem for agents having monotonic preferences expressed in a compact language under logical form $\mathcal{L}$ is Σ_2^p-complete.

provided that :

- $\mathcal{L}$ is as compact as the previous language for dichotomous preferences ;
- Every pair of alternatives can be compared in polynomial time.

What about weighted logic and additive preferences ?

- **Weighted logic :** alternative efficiency based on collective utility maximization.
- **Additive preferences :**
  - **Completeness :** result already known [Lipton et al., 2004].
  - **Pareto-efficiency :** ? ? ?
    - identical preferences,
    - 0–1 preferences,
    - 0–1–...–k preferences (? ? ?),
    - number of objects lower than the number of agents.

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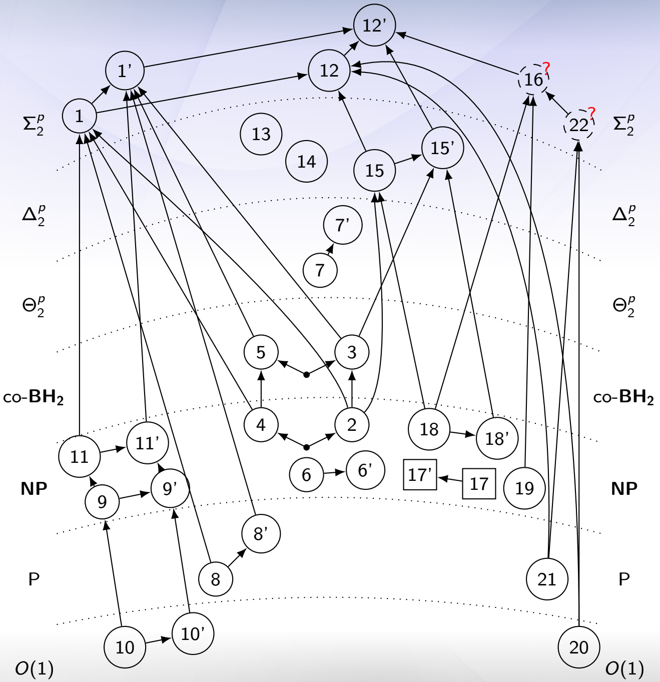
12'
1'
12
16?
1
22?
Σ_2^p
13
14
15
15'
Δ_2^p
7'
7
Θ_2^p
5
3
co-$\mathbf{BH_2}$
4
2
18
18'
11
11'
6
6'
17'
17
19
NP
9
9'
8'
P
8
21
10
10'
20
$O(1)$

Summary of the talk and contributions

1. *Modelling of resource allocation problems :* **A review of the basic concepts** and a formalism for taking exogenous rights into account in the welfarist framework.
2. *Compact representation :*
   - **Problem of maximizing the collective utility : weighted logic.**
   - **Existence of an envy-free and Pareto-efficient allocation : logic.**
3. *Computational complexity :* **[MAX-CUF] and [EEF EXISTENCE], and several of their restrictions.**
4. *Algorithmics :* Constraint programming for leximin optimization.
5. *Experiments :*
   - Generation of realistic instances of resource allocation problems.
   - Experimental comparison of leximin optimization algorithms.

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Perspectives and other issues

- Resource allocation and graphical languages for preference representation (CP-nets).
- Strategies and manipulation.
- A joint study of egalitarianism and envy-freeness (a few words about this in [Brams and King, 2005]).

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Perspectives and other issues (2)

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  - definition of this notion of approximation (measure of envy, approximated leximin),
  - approximation algorithms (PTAS, incomplete algorithms).
  - envy-freeness : limited knowledge of the agents EndrissAAAI07.
- Repeated allocation and temporal regulation.

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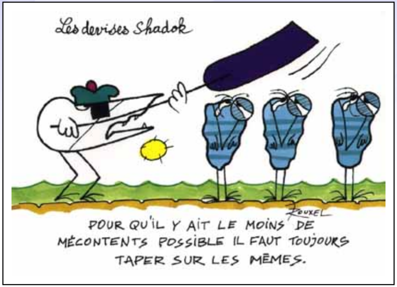
Les devises Shadok
ROUXEL
POUR QU'IL Y AIT LE MOINS DE
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