College Quantumgroepen, Koornwinder, 17-9-96

Tensor fields on manifolds (*tutorial*)

C^∞-manifolds. Let M be a C^∞-manifold of dimension m. Thus M is a topological space which is Hausdorff (i.e., distinct points have disjoint open neighbourhoods), and each point has an open neighbourhood which is homeomorphic to some open part of $\mathbb{R}^m$. We can write this homeomorphism as $p \mapsto (x_1(p), \ldots, x_m(p))$ and we can consider the $x_1, \ldots, x_m$ locally as coordinates on M. It is also required that, in case of overlap of two sets on which coordinates, say $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$, are given, the transition from the x to the y coordinates is a C^∞-diffeomorphism. There is one more technical condition: M as a topological space is *second countable*, i.e., there is a countable collection of open subsets U_j of M such that any open subset of M is a union of U_j's. (This implies that M can have only countably many connected components.)

Vector fields. Let $C^\infty(M)$ be the space of all real-valued C^∞-functions on M. A *vector field* (always assumed to be C^∞) on M is a linear map $X\colon f \mapsto X(f)\colon C^\infty(M) \to C^\infty(M)$ which can locally, in terms of coordinates $(x_1, \ldots, x_m)$, be written as

$$(X(f))(x) = \sum_{j=1}^m a_j(x)\, \frac{\partial f(x)}{\partial x_j}$$

with the functions a_j being C^∞. The commutator $[X,Y] := XY - YX$ of two vector fields X, Y is again a vector field. Write

$$X_x(f) := (X(f))(x).$$

Fix $x \in M$. Then, the linear functionals $X_x\colon C^\infty(M) \to \mathbb{R}$ (X vector field) form an m-dimensional real vector space, the so-called *tangent space* $\mathcal{T}_x M$. Locally, in terms of coordinates $(x_1, \ldots, x_m)$, the $\partial/\partial x_j$ evaluated at x form a basis of $\mathcal{T}_x M$.

Proposition X is a vector field on M iff X is a derivation of the algebra $C^\infty(M)$, i.e., iff $X\colon C^\infty(M) \to C^\infty(M)$ is a linear map satisfying $X(fg) = X(f)\, g + f\, X(g)$.

1-forms. The *cotangent space* $\mathcal{T}_x^* M$ is the linear dual of $\mathcal{T}_x M$. Write the pairing as $\langle\, .\,,\, .\,\rangle\colon \mathcal{T}_x M \times \mathcal{T}_x^* M \to \mathbb{R}$. Any $f \in C^\infty(M)$ determines an element $(df)_x \in \mathcal{T}_x^* M$ by

$$\langle X_x, (df)_x\rangle := X_x(f) \qquad (X_x \in \mathcal{T}_x M).$$

A *1-form* (always assumed to be C^∞) on M is an element $\omega = (\omega_x)_{x\in M}$ with $\omega_x \in \mathcal{T}_x^* M$ such that the function $x \mapsto \langle X_x, \omega_x\rangle\colon M \to \mathbb{R}$ is C^∞ for all vector fields X. We will write this function as $\langle X, \omega\rangle$, so

$$\langle X, \omega\rangle(x) = \langle X_x, \omega_x\rangle.$$

If $f \in C^\infty(M)$ then $df = ((df)_x)_{x\in M}$ is a 1-form and

$$\langle X, df\rangle = X(f).$$

Locally, in terms of coordinates $(x_1, \ldots, x_m)$, we can write

$$X = \sum_j a_j\, \frac{\partial}{\partial x_j}, \quad df = \sum_i \frac{\partial f}{x_i}\, dx_i,$$

and

$$X(f) = \langle X, df\rangle = \sum_{i,j} a_j\, \frac{\partial f}{\partial x_i}\, \langle \partial/\partial x_j, dx_i\rangle = \sum_j a_j\, \frac{\partial f}{\partial x_j}\,.$$

Contravariant tensor fields. A *contravariant tensor field of degree* r (always assumed to be C^∞) on M is an element $T = (T_x)_{x\in M}$ with $T_x \in (\mathcal{T}_x)^{\otimes r}$ (r-fold tensor product) such that the function $x \mapsto \langle T_x, (df_1)_x \otimes \cdots \otimes (df_r)_x\rangle$ is C^∞ for all $f_1, \ldots, f_r \in C^\infty(M)$. Locally, in terms of coordinates $(x_1, \ldots, x_m)$, we can write

$$\langle T_x, (df_1)_x \otimes \cdots \otimes (df_r)_x\rangle = \sum_{j_1,\ldots,j_r} T_{j_1,\ldots,j_r}(x)\, \frac{\partial f_1(x)}{\partial x_{j_1}} \cdots \frac{\partial f_r(x)}{\partial x_{j_r}}\,,$$

where the functions $T_{j_1,\ldots,j_r}$ are C^∞. By an *r-vector field* we mean an antisymmetric contravariant tensor field of degree r. For $r = 2$ we speak about a *bivector field.*

Covariant tensor fields. A *covariant tensor field of degree* r (always assumed to be C^∞) on M is an element $\omega = (\omega_x)_{x\in M}$ with $\omega_x \in (\mathcal{T}_x^*)^{\otimes r}$ such that the function $x \mapsto \langle T_x, \omega_x\rangle$ is C^∞ for all contravariant tensor fields T of degree r. Locally, in terms of coordinates $(x_1, \ldots, x_m)$, we can write

$$\omega_x = \sum_{j_1,\ldots,j_r} \omega_{j_1,\ldots,j_r}(x)\, dx_{j_1} \otimes \cdots \otimes dx_{j_r}\,,$$

$$\langle T_x, \omega_x\rangle = \sum_{j_1,\ldots,j_r} T_{j_1,\ldots,j_r}(x)\, \omega_{j_1,\ldots,j_r}(x).$$

r-forms. An *r-form* is a covariant tensor field of degree r which is anti-symmetric. If $\omega_1, \ldots, \omega_r$ are 1-forms then

$$\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_r := \frac{1}{r!} \sum_{\sigma\in S_r} \varepsilon(\sigma)\omega_{\sigma(1)} \otimes \cdots \otimes \omega_{\sigma(r)}$$

is an r-form. Here $\varepsilon(\sigma)$ denotes the signum of the permutation σ. Note that, for $\sigma \in S_r$,

$$\omega_{\sigma(1)} \wedge \omega_{\sigma(2)} \wedge \ldots \wedge \omega_{\sigma(r)} = \varepsilon(\sigma)\, \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_r.$$

Locally, in terms of coordinates $(x_1, \ldots, x_m)$, each r-form can be written in the form

$$\omega = r! \sum_{j_1<j_2<\ldots<j_r} \omega_{j_1,\ldots,j_r}\, dx_{j_1} \wedge \ldots \wedge dx_{j_r}\,,$$

where the functions $\omega_{j_1,\ldots,j_r}$ are C^∞. Note that nonzero r-forms exist iff $r \le m$. The *differential* $d\omega$ of the above r-form ω is an $(r+1)$-form given by

$$d\omega := r! \sum_{j_1<j_2<\ldots<j_r} d\omega_{j_1,\ldots,j_r} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_r}.$$

An r-form ω is called *closed* if $d\omega = 0$. We always have that $d(d\omega) = 0$.

Differential of a C^∞-map. Let M and N be C^∞-manifolds and let $F: N \to M$ be a C^∞-map. Let $x \in N$. Then the *differential of F at x* is the linear map $dF_x: \mathcal{T}_x N \to \mathcal{T}_{F(x)} M$ defined by

$$(dF_x(X_x))(f) := X_x(f \circ F) \qquad (X_x \in \mathcal{T}_x,\ f \in C^\infty(M)).$$

Hence

$$\langle dF_x(X_x), (df)_{F(x)}\rangle = \langle X_x, (d(f \circ F))_x\rangle.$$

Submanifold. Let M be a C^∞-manifold. Let S be a subset of M and denote by $i: S \to M$ the natural injection. The subset S is called a *submanifold* of M if S is a C^∞-manifold itself such that the map $i: S \to M$ is C^∞ and, for all $x \in S$, the linear map $di_x: \mathcal{T}_x S \to \mathcal{T}_x M$ is injective. Thus the tangent space $\mathcal{T}_x S$ can be identified with a linear subspace of $\mathcal{T}_x M$. We will mostly deal with submanifolds which are *regularly embedded*, i.e., with differentiable structure compatible with the topology inherited from M.

Let M have dimension m and let its regularly embedded submanifold S have dimension s. It can be shown that each point of S has an open neighbourhood on which there are coordinates $x_1, \ldots, x_m$ for M such that $(x_1, \ldots, x_m) \in S$ iff $x_{s+1} = \ldots = x_m = 0$, and that then $x_1, \ldots, x_s$ are coordinates locally on S.

Integral curves. Let X be a vector field on a C^∞-manifold M. Let α be a C^∞-curve in M, i.e., a C^∞-map $\alpha: t \mapsto \alpha(t): (a,b) \to M$. Then the curve α is called an *integral curve* of the vector field X if $d\alpha_s(\frac{d}{dt}\big|_{t=s}) = X_{\alpha(s)}$ for all $s \in (a,b)$, or equivalently, if

$$X_{\alpha(t)}(f) = \frac{d}{dt}\, f(\alpha(t))$$

for all $f \in C^\infty(M)$ and all $t \in (a,b)$. Locally, in terms of coordinates $(x_1, \ldots, x_m)$ we can write

$$X = \sum_j a_j \, \frac{\partial}{\partial x_j} \quad \text{and} \quad \alpha(t) = (\alpha_1(t), \ldots, \alpha_m(t)).$$

Then α is an integral curve of X iff $\alpha_j'(t) = a_j(\alpha(t))$ for $j = 1, \ldots, m$.

Additions to C&P, §1.1

Proposition Let M be a C^∞-manifold. Then the identity

$$\langle w, df \otimes dg \rangle = \{f, g\} \quad (f, g \in C^\infty(M))$$

establishes a 1-1 correspondence between bivector fields w on M and antisymmetric bilinear forms $\{\,.\,,\,.\,\}$ on $C^\infty(M)$ satisfying the Leibniz identity (3).

Locally, in terms of coordinates $(x_1, \ldots, x_m)$, write $w = (w_{ij})$. Then $\{\,.\,,\,.\,\}$ satisfies moreover the Jacobi identity (2) iff

$$\sum_r \left(w_{ri} \, \frac{\partial w_{jk}}{\partial x_r} + w_{rj} \frac{\partial w_{ki}}{\partial x_r} + w_{rk} \frac{\partial w_{ij}}{\partial x_r} \right) = 0.$$

Let these equivalent conditons hold. Then M is called a *Poisson manifold*, $\{\,.\,,\,.\,\}$ is called a *Poisson bracket*, w is called a *Poisson bivector*, and for each $f \in C^\infty(M)$ the corresponding *hamiltonian vector field* X_f is given by

$$X_f(g) := \{g, f\} \quad (g \in C^\infty(M)).$$

Re: Definition-Proposition 1.1.2 The collection $(B_x(\mathcal{T}_x^* M))_{x \in M}$ is an example of an *involutive distribution* $\mathcal{D}$ on M: a (smooth) choice of linear subspace $\mathcal{D}_x$ of $\mathcal{T}_x M$ for each $x \in M$ such that, for vector fields X, Y on M with the property that $X_x, Y_x \in \mathcal{D}_x$ for all $x \in M$, we also have that $[X, Y]_x \in \mathcal{D}_x$ for all $x \in M$. A submanifold S is called an *integral manifold* of the distribution $\mathcal{D}$ if $\mathcal{T}_x S = \mathcal{D}_x$ for all $x \in S$. In case of constant rank ($\dim \mathcal{D}_x$ constant), the existence of a unique maximal integral manifold through each point of M is stated by a theorem of Frobenius. The extension of this result to the case of non-constant rank for the distribution $(B_x(\mathcal{T}_x^* M))_{x\in M}$ is possible because the rank does not change along integral curves of hamiltonian vector fields. See Kirillov, *l.c.*, and R. Hermann, *Cartan connections and the equivalence problem for geometric structures*, in *Contributions to Differential Equations, Vol. 3*, Wiley, 1964, pp. 199–248.

Exercises (Submit an answer to Exercise 7)

Exercise 1. Prove the second part of the above Proposition by showing that

$$\begin{aligned}&\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} \\ &\quad = \sum_{i,j,l} \sum_k \left(w_{kl} \frac{\partial w_{ij}}{\partial x_k} + w_{ki} \frac{\partial w_{jl}}{\partial x_k} + w_{kj} \frac{\partial w_{li}}{\partial x_k} \right) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \frac{\partial h}{\partial x_l}\,.\end{aligned}$$

Exercise 2. Let M be a Poisson manifold. Show that $[X_f, X_g] = X_{\{g,f\}}$.

Exercise 3. Let M be a Poisson manifold. Work locally, in terms of coordinates $(x_1, \ldots, x_m)$, so x_j can also be considered as a smooth function $p \mapsto x_j(p)$. Let $X_f.X_{x_j}$ denote the hamiltonian vector fields corresponding to $f, x_j \in C^\infty(M)$. Show that

$$X_f = \sum_i \left(\sum_j w_{ij} \frac{\partial f}{\partial x_j} \right) \frac{\partial}{\partial x_i} = \sum_j \frac{\partial f}{\partial x_j} X_{x_j}\,.$$

Exercise 4. Let M, N be C^∞-manifolds and let $F\colon N \to M$ be a C^∞-map. Prove that F is a Poisson map iff $(dF_x \otimes dF_x)(w_N)_x = (w_M)_{F(x)}$ for all $x \in N$.

Exercise 5. Let M be a Poisson manifold and let S be a submanifold of M such that $(w_M)_x \in \mathcal{T}_x S \otimes \mathcal{T}_x S$ for all $x \in S$. Prove that S then can be made into a Poisson manifold with Poisson bivector $(w_S)_x := (w_M)_x$ $(x \in S)$.

Exercise 6. Let M be a Poisson manifold. Let $f \in C^\infty(M)$ and let α be an integral curve of the hamiltonian vector field X_f. Show that locally, in terms of coordinates $(x_1, \ldots, x_m)$,

$$\alpha_i'(t) = \sum_j w_{ij}(\alpha(t)) \left.\frac{\partial f(x)}{\partial x_j}\right|_{x=\alpha(t)}.$$

Conclude that, if S is a Poisson submanifold of M and the curve α has a point in common with S, then the curve α lies completely in S. Conclude from this that, if S is a Poisson submanifold of M and $x \in S$ then any symplectic leaf containg x is a subset of S.

Exercise 7. Show that $\mathbb{R}^{2n+s}$ with

$$\{f,g\} := \sum_{j=1}^n \left(\frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_{j+n}} - \frac{\partial f}{\partial x_{j+n}} \frac{\partial g}{\partial x_j} \right)$$

is a Poisson manifold. Show that the symplectic leaves are the submanifolds $\{(x,y) \mid x \in \mathbb{R}^{2n}\}$ $(y \in \mathbb{R}^s)$. For any $x \in \mathbb{R}^{2n}$, $y \in \mathbb{R}^s$ also give $f \in C^\infty(\mathbb{R}^{2n+s})$ such that a suitable integral curve of X_f connects $(0,y)$ with (x,y).

College Quantumgroepen, Koornwinder, 24-9-96

Lie groups and corresponding Lie algebras (*tutorial*)

Left invariant vector fields. A *Lie group* is a group G which has also the structure of a C^∞-manifold such that the multiplication $\mu\colon (x,y) \mapsto xy\colon G \times G \to G$ is a C^∞-map. Then it can be shown that the inversion $x \mapsto x^{-1}$ is also a C^∞-map. Let $L_x\colon y \mapsto xy\colon G \to G$ and $R_x\colon y \mapsto yx\colon G \to G$ denote left and right multiplication, respectively. For each $x \in G$ the maps L_x and R_x are diffeomorphisms of G.

A vector field X on G is called *left invariant* if $(dL_x)_y(X_y) = X_{xy}$ for all $x,y \in G$. Then:

(a) The space of all left invariant vector fields is a Lie algebra over $\mathbb{R}$ with Lie bracket $[X,Y] := XY - YX$ (using the product of linear operators on $C^\infty(G)$). This Lie algebra is denoted by $\mathrm{Lie}(G)$.

(b) Each left invariant vector field X is completely determined by X_e since $X_x = (dL_x)_e(X_e)$. Also, for any $v \in \mathcal{T}_e(G)$ the formula $X_x := (dL_x)_e(v)$ defines a left invariant vector field X. Thus there is a linear bijection $X \mapsto X_e\colon \mathrm{Lie}(G) \to \mathcal{T}_e G$.

When we speak about $\mathrm{Lie}(G)$ or about the *Lie algebra of the Lie group G*, we mean by definition the Lie algebra of left invariant vector fields, but often, silently using the identification of linear spaces in (b), one means instead the tangent space $\mathcal{T}_e G$. A priori, this tangent space is just a vector space. We make $\mathcal{T}_e G$ into a Lie algebra by inducing on it the Lie algebra structure of the space of left invariant vector fields: $[X_e, Y_e] := [X,Y]_e$, where X,Y are left invariant vector fields.

The Lie algebra of $GL(n,\mathbb{R})$. For so-called linear Lie groups G, the induced Lie algebra structure of $\mathcal{T}_e G$ coincides with another natural Lie algebra structure. As a prototype of this situation consider $G := GL(n,\mathbb{R})$, the group of all real invertible $(n \times n)$ matrices. By writing $T \in GL(n,\mathbb{R})$ as $T = (T_{ij})_{i,j=1,\ldots,n}$ we can consider $GL(n,\mathbb{R})$ as an open subset of $\mathbb{R}^{n^2}$, by which it becomes a C^∞-manifold and a Lie group. Let $gl(n,\mathbb{R})$ denote the real vector space of all real $(n \times n)$ matrices, which becomes a Lie algebra with Lie bracket $[A,B] := AB - BA$ (using the matrix product). With any $A \in gl(n,\mathbb{R})$ we can associate a vector field X_A on $GL(n,\mathbb{R})$ given by

$$(X_A)_T := \sum_{i,j=1,\ldots,n} (TA)_{ij} \, \frac{\partial}{\partial T_{ij}}.$$

Then:

(a) The vector field X_A is left invariant.

(b) The mapping $A \mapsto X_A\colon gl(n,\mathbb{R}) \to \mathrm{Lie}(GL(n,\mathbb{R}))$ is an isomorphism of Lie algebras, so it is a linear bijection and it preserves the Lie bracket: $X_{[A,B]} = [X_A, X_B]$.
Note that

$$(X_A)_I = \sum_{i,j=1,\ldots,n} A_{ij} \, \frac{\partial}{\partial T_{ij}}\Big|_{T=I}.$$

In general, if S is a submanifold of $\mathbb{R}^n$ and $x \in S$, there is a natural identification (linear bijection) between the tangent space $\mathcal{T}_x S$ and some linear subspace of $\mathbb{R}^n$. The linear bijection $A \mapsto (X_A)_I\colon gl(n,\mathbb{R}) \to \mathcal{T}_I GL(n,\mathbb{R})$ in the formula above gives this natural identification for the case of the submanifold $GL(n,\mathbb{R})$ of $gl(n,\mathbb{R}) \simeq \mathbb{R}^{n^2}$, where the tangent space is taken at I.

Lie group homomorphisms and Lie subgroups. Let G and H be Lie groups and let $F: G \to H$ be a *Lie group homomorphism*, i.e., a group homomorphism which is also a C^∞-mapping. Then there is a unique Lie algebra homomorphism $f: \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ such that $(f(X))_e = (dF_e)(X_e)$ for all $X \in \mathrm{Lie}(G)$. Thus, if $X \in \mathrm{Lie}(G)$ and $Y := f(X)$ then $Y_{F(x)} = dF_x(X_x)$. Instead of f we also write dF.

Let G be a Lie group, let $H \subset G$ and write $i: H \to G$ for the natural injection. We call H a *Lie subgroup of* G if H is both a subgroup and a submanifold of G. Then $di: \mathrm{Lie}(H) \to \mathrm{Lie}(G)$ is an injective Lie algebra homomorphism, so it identifies $\mathrm{Lie}(H)$ with a Lie subalgebra of $\mathrm{Lie}(G)$.

Linear Lie groups. Let G be a Lie subgroup of $GL(n,\mathbb{R})$, so G is a submanifold of $gl(n,\mathbb{R}) \simeq \mathbb{R}^{n^2}$. Thus $T_I G$ can be naturally identified with a linear subspace $\mathfrak{g}$ of $gl(n,\mathbb{R})$. It turns out that $\mathfrak{g}$ is a Lie subalgebra of $gl(n,\mathbb{R})$ and that the map $A \mapsto X_A: \mathfrak{g} \to \mathrm{Lie}(G)$ (the restriction of the map $A \mapsto X_A: gl(n,\mathbb{R}) \to \mathrm{Lie}(GL(n,\mathbb{R}))$ given above) is a Lie algebra isomorphism.

As a generalisation of the description of $\mathrm{Lie}(GL(n,\mathbb{R}))$ we can consider the group $GL(n,\mathbb{C})$ of all complex invertible $(n \times n)$ matrices. By considering $\mathrm{Re}\, T_{ij}$ and $\mathrm{Im}\, T_{ij}$ $(i,j = 1,\ldots,n)$ as real coordinates of $T \in GL(n,\mathbb{C})$, we can consider $GL(n,\mathbb{C})$ as an open subset of $\mathbb{R}^{2n^2}$. Thus $GL(n,\mathbb{C})$ becomes a C^∞-manifold and a Lie group. (In fact it is even a complex analytic manifold and a complex Lie group.) The complex vector space $gl(n,\mathbb{C})$ of all complex $(n \times n)$ matrices becomes a Lie algebra over $\mathbb{C}$ with Lie bracket $[A,B] := AB - BA$, so, by restricting scalar multiplication to real scalars, it becomes also a Lie algebra over $\mathbb{R}$. With any $A \in gl(n,\mathbb{C})$ we can associate a vector field X_A on $GL(n,\mathbb{C})$ given by

$$(X_A)_T := \sum_{i,j=1,\ldots n} \Big(\mathrm{Re}\,(TA)_{ij} \frac{\partial}{\partial(\mathrm{Re}\, T_{ij})} + \mathrm{Im}\,(TA)_{ij} \frac{\partial}{\partial(\mathrm{Im}\, T_{ij})} \Big).$$

Then the vector field X_A is left invariant and the mapping $A \mapsto X_A: gl(n,\mathbb{C}) \to \mathrm{Lie}(GL(n,\mathbb{C}))$ is an isomorphism of Lie algebras.

A Lie group G is called a *linear Lie group* if it is a Lie subgroup of $GL(n,\mathbb{C})$ for certain n. We can essentially repeat here what we wrote above for Lie subgroups of $GL(n,\mathbb{R})$. If G is a Lie subgroup of $GL(n,\mathbb{C})$ then G is a submanifold of $gl(n,\mathbb{C}) \simeq \mathbb{R}^{2n^2}$, so $T_I G$ can naturally be identified with a real linear subspace $\mathfrak{g}$ of $gl(n,\mathbb{C})$. The map $A \mapsto X_A: \mathfrak{g} \to \mathrm{Lie}(G)$ is then a Lie algebra isomorphism. Finally observe that, since $GL(n,\mathbb{R})$ is a Lie subgroup of $GL(n,\mathbb{C})$, any Lie subgroup of $GL(n,\mathbb{R})$ is a Lie subgroup of $GL(n,\mathbb{C})$.

The exponential map. Let G be a Lie group. There exists a unique map $\exp: \mathrm{Lie}(G) \to G$ such that $\exp(0) = e$ and, for each $X \in \mathrm{Lie}(G)$, the map $t \mapsto \exp(tX): \mathbb{R} \to G$ is C^∞ and gives an integral curve of the (left invariant) vector field X. Then also $\exp((s+t)X) = \exp(sX)\,\exp(tX)$. The map exp is called the *exponential map* associated with the Lie group G. The definition of exp implies that

$$(Xf)(y) = \frac{d}{dt} f(y\,\exp(tX))\Big|_{t=0} \qquad (X \in \mathrm{Lie}(G),\ f \in C^\infty(G),\ y \in G).$$

In particular,

$$X_e(f) = \frac{d}{dt} f(\exp(tX))\Big|_{t=0} \qquad (X \in \mathrm{Lie}(G),\ f \in C^\infty(G)).$$

If G is a linear Lie group and if $\mathfrak{g}$ is the real Lie subalgebra of $gl(n,\mathbb{C})$ associated with $\mathcal{T}_I G$ then

$$\exp(X_A) = e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!} \qquad (A \in \mathfrak{g}).$$

For an arbitrary Lie group G there are open subsets U of 0 in $\mathrm{Lie}(G)$ and V of e in G such that exp is a diffeomorphism of U onto V.

If $F: G \to H$ is a homomorphism of Lie groups and if $f: \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is the corresponding Lie algebra homomorphism then $\exp_H(f(X)) = F(\exp_G(X))$ for all $X \in \mathrm{Lie}(G)$.

If H is a Lie subgroup of G then $\exp_H(X) = \exp_G(X)$ for all $X \in \mathrm{Lie}(H)$.

The adjoint representations Ad *and* ad. Let G be a Lie group and let $y \in G$. The map $a_y: x \mapsto yxy^{-1}: G \to G$ is a Lie group automorphism. Denote the differential of a_y by Ad_y. Then $\mathrm{Ad}_y: \mathrm{Lie}(G) \to \mathrm{Lie}(G)$ is a Lie algebra automorphism. Furthermore, the map $\mathrm{Ad}: y \mapsto \mathrm{Ad}_y: G \to GL(\mathrm{Lie}(G))$ is a Lie group homomorphism. Hence Ad is a representation of G on the vector space $\mathfrak{g}$. The representation Ad is called the *adjoint representation* of the Lie group G. If $G \subset GL(n,\mathbb{C})$ is a linear Lie group with corresponding Lie subalgebra $\mathfrak{g} \subset gl(n,\mathbb{C})$ then $\mathrm{Ad}_T(X_A) = X_{TAT^{-1}}$.

The *adjoint representation* of any Lie algebra $\mathfrak{g}$ is defined by $\mathrm{ad}_Y(X) := [Y,X]$ $(X,Y \in \mathfrak{g})$. Then $\mathrm{ad}: Y \mapsto \mathrm{ad}_Y: \mathfrak{g} \to gl(\mathfrak{g})$ is a Lie algebra homomorphism and the map $\mathrm{ad}_Y: X \mapsto [Y,X]: \mathfrak{g} \to \mathfrak{g}$ is a derivation of the Lie algebra $\mathfrak{g}$.

Now consider the adjoint representation of the Lie algebra $\mathrm{Lie}(G)$ (G a Lie group). Since the Lie algebra $\mathrm{Lie}(GL(\mathrm{Lie}(G)))$ can be identified with $gl(\mathrm{Lie}(G))$, we can consider the differential of the Lie group homomorphism $\mathrm{Ad}: G \to GL(\mathrm{Lie}(G))$ as a Lie algebra homomorphism of $\mathfrak{g}$ to $gl(\mathrm{Lie}(G))$. It is a theorem that this differential is equal to ad. Thus

$$\mathrm{ad}_Y(X) = \frac{d}{dt}\,\mathrm{Ad}_{\exp(tY)}(X)\Big|_{t=0} \qquad (X,Y \in \mathrm{Lie}(G)),$$

and for a linear Lie group $G \subset GL(n,\mathbb{C})$ with Lie algebra $\mathfrak{g} \subset gl(n,\mathbb{C})$:

$$\mathrm{ad}_B(A) = \frac{d}{dt}\,(e^{tB}\,A\,e^{-tB})\big|_{t=0} \qquad (A,B \in \mathfrak{g}).$$

While we can go downwards from a_y to the adjoint representations Ad and ad by taking suitable differentials, we can go upwards from ad to Ad and a_y by using the exponential map. For a Lie group G we have the formulas

$$\mathrm{Ad}_{\exp X}(Y) = e^{\mathrm{ad}_X}(Y) = \sum_{k=0}^{\infty} \frac{1}{k!}\,(\mathrm{ad}_X)^k(Y) \qquad (X,Y \in \mathrm{Lie}(G)),$$
$$a_y(\exp(X)) = \exp(\mathrm{Ad}_y(X)) \qquad (y \in G,\ X \in \mathrm{Lie}(G)).$$

For a linear Lie group $G \subset GL(n,\mathbb{C})$ with Lie algebra $\mathfrak{g} \subset gl(n,\mathbb{C})$ these formulas become:

$$e^A B e^{-A} = e^{\mathrm{ad}_B}(A) \quad \text{and} \quad T e^A T^{-1} = e^{TAT^{-1}} \qquad (A,B \in \mathfrak{g},\ T \in G).$$

The co-adjoint representations Ad^* *and* ad^*. Let G be a Lie group and write $\mathfrak{g} := \mathrm{Lie}(G)$. Let $\mathfrak{g}^*$ be the linear dual of $\mathfrak{g}$. Let $\langle X,\xi\rangle$ denote the pairing between $X \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$. For $x \in G$ and $X \in \mathfrak{g}$ define $\mathrm{Ad}^*_x \in GL(\mathfrak{g}^*)$ and $\mathrm{ad}^*_X \in gl(\mathfrak{g}^*)$ by

$$< Y, \mathrm{Ad}^*_x(\xi)\rangle := \langle \mathrm{Ad}_x(Y),\xi\rangle \quad \text{and} \quad \langle Y, \mathrm{ad}^*_X(\xi)\rangle := \langle \mathrm{ad}_X(Y),\xi\rangle \qquad (Y \in \mathfrak{g},\ \xi \in \mathfrak{g}^*).$$

Then the map $x \mapsto \mathrm{Ad}^*_{x^{-1}}: G \to GL(\mathfrak{g}^*)$ is a Lie group homomorphism and the map $X \mapsto -\mathrm{ad}^*_X: \mathfrak{g} \to gl(\mathfrak{g}^*)$ is a Lie algebra homomorphism. We have:

$$\frac{d}{dt}\,\mathrm{Ad}^*_{\exp(tX)}\big|_{t=0} = \mathrm{ad}^*_X.$$

This follows by duality from a similar formula for Ad and ad.

Additions to C&P, §1.1 (continued)

In the situation of Exercise 1 take for f, g, h functions which are locally equal to the coordinate functions x_i, x_j, x_l, respectively. Then it follows that locally

$$\{\{x_i, x_j\}, x_l\} + \{\{x_j, x_l\}, x_i\} + \{\{x_l, x_i\}, x_j\} = \sum_k \left(w_{kl}\,\frac{\partial w_{ij}}{\partial x_k} + w_{ki}\,\frac{\partial w_{jl}}{\partial x_k} + w_{kj}\,\frac{\partial w_{li}}{\partial x_k} \right).$$

Hence, we have locally that

$$\begin{aligned}&\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\}\\ &\quad = \sum_{i,j,l} \big(\{\{x_i, x_j\}, x_l\} + \{\{x_j, x_l\}, x_i\} + \{\{x_l, x_i\}, x_j\}\big)\,\frac{\partial f}{\partial x_i}\,\frac{\partial g}{\partial x_j}\,\frac{\partial h}{\partial x_l}\,.\end{aligned}$$

This implies that the bracket $\{\,.\,,\,.\,\}$ satisfies the Jacobi identity for general f, g, h if it satisfies the Jacobi identity on the coordinate functions.

Re: Example 1.1.3 Let $\xi \in \mathfrak{g}^*$. We can identify the tangent space $\mathcal{T}_\xi \mathfrak{g}^*$ with $\mathfrak{g}^*$ by the linear bijection $\eta \mapsto \eta_\xi: \mathfrak{g}^* \to \mathcal{T}_\xi \mathfrak{g}^*$, where

$$\eta_\xi(f) := \frac{d}{dt}\, f(\xi + t\eta)\big|_{t=0} \qquad (f \in C^\infty(\mathfrak{g}^*)).$$

So $\mathcal{T}^*_\xi \mathfrak{g}^*$ can be identified with $\mathfrak{g}$. Write $\langle x, \eta\rangle$ for the pairing between $x \in \mathfrak{g}$ and $\eta \in \mathfrak{g}^*$. For $\xi \in \mathfrak{g}^*$ and $f \in C^\infty(\mathfrak{g}^*)$ the element $(df)_\xi \in \mathcal{T}^*_\xi \mathfrak{g}^*$ can be considered as an element of $\mathfrak{g}$ which is determined by the rule

$$\langle (df)_\xi, \eta\rangle = \frac{d}{dt}\, f(\xi + t\eta)\big|_{t=0} \qquad (\eta \in \mathfrak{g}^*).$$

Let $x \in \mathfrak{g}$. Then x can be considered as an element of $C^\infty(\mathfrak{g}^*)$ by the rule $x(\xi) := \langle x, \xi\rangle$ $(\xi \in \mathfrak{g}^*)$. Thus

$$\langle (dx)_\xi, \eta\rangle = \frac{d}{dt}\,\langle x, \xi + t\eta\rangle\big|_{t=0} = \langle x, \eta\rangle \qquad (\eta \in \mathfrak{g}^*).$$

Hence $(dx)_\xi = x$.

For $f, g \in C^\infty(\mathfrak{g}^*)$ define the function $\{f, g\}$ on $\mathfrak{g}^*$ by

$$\{f,g\}(\xi) := \langle [(df)_\xi, (dg)_\xi], \xi\rangle.$$

This is again a C^∞-function. In particular

$$\{x,y\}(\xi) = \langle [x,y], \xi\rangle \quad \text{for all } \xi \in \mathfrak{g}^*, \text{ so} \quad \{x,y\} = [x,y].$$

It is evident that the bracket $\{\,.\,,\,.\,\}$ is bilinear and anti-symmetric and that it satisfies the Leibniz identity. For $x,y,z\in\mathfrak{g}$ it satisfies the Jacobi identity:

$$\{\{x,y\},z\}+\{\{y,z\},x\}+\{\{z,x\},y\}=[[x,y],z]+[[y,z],x]+[[z,x],y]=0.$$

If $x_1,\ldots,x_m$ is a basis of $\mathfrak{g}$ then the functions $x_1,\ldots,x_m$ can be considered as global coordinate functions on $\mathfrak{g}^*$: identify $\xi\in\mathfrak{g}^*$ with $(\langle x_1,\xi\rangle,\ldots\langle x_m,\xi\rangle)$. Since the bracket $\{\,.\,,\,.\,\}$ satisfies the Jacobi identity on the coordinate functions, it satisfies the Jacobi identity on any C^∞-functions. So $\mathfrak{g}^*$ becomes a Poisson manifiold with this bracket.

For $\xi\in\mathfrak{g}^*$, we have the Poisson bivector $w_\xi\in\mathcal{T}_\xi\mathfrak{g}^*\otimes\mathcal{T}_\xi\mathfrak{g}^*\simeq\mathfrak{g}^*\otimes\mathfrak{g}^*$ such that

$$\langle(df)_\xi\otimes(dg)_\xi,w_\xi\rangle=\{f,g\}(\xi).$$

In particular, for $x,y\in\mathfrak{g}$:

$$\langle x\otimes y,w_\xi\rangle=\langle[x,y],\xi\rangle,$$

This shows that w_ξ depends linearly on ξ. Also note that the linear map $\xi\mapsto w_\xi\colon\mathfrak{g}^*\to\mathfrak{g}^*\otimes\mathfrak{g}^*$ is the linear adjoint of the linear map $x\otimes y\mapsto[x,y]\colon\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$.

Let G be a connected Lie group with $\mathfrak{g}=\mathrm{Lie}(G)$. Another way to see that the symplectic leaves of $\mathfrak{g}^*$ are the coadjoint orbits $\{\mathrm{Ad}_g^*(\xi)\}_{g\in G}$ is as follows. The symplectic leaves are the maximal connected integral manifolds of the involutive distribution $\mathcal{D}$ on $\mathfrak{g}^*$ defined by

$$\mathcal{D}_\xi:=\{(X_f)_\xi\mid f\in C^\infty(\mathfrak{g}^*)\}=\mathrm{Span}\{(X_{x_i})_\xi\}_{i=1,\ldots,m}$$

For $x\in\mathfrak{g}$ considered as function on $\mathfrak{g}^*$, we compute the corresponding Hamiltonian vector field X_x. Let $f\in C^\infty(\mathfrak{g}^*)$, $\xi\in\mathfrak{g}^*$. Then

$$\langle(df)_\xi,(X_x)_\xi\rangle=(X_x)_\xi(f)=\{f,x\}(\xi).$$

In particular, for $f:=y$ with $y\in\mathfrak{g}$, we obtain:

$$\langle y,(X_x)_\xi\rangle=\{y,x\}(\xi)=\langle[y,x],\xi\rangle=-\langle\mathrm{ad}_x(y),\xi\rangle=-\langle y,\mathrm{ad}_x^*(\xi)\rangle.$$

Hence $(X_x)_\xi=-\mathrm{ad}_x^*(\xi)$. So $\mathcal{D}_\xi=\{\mathrm{ad}_x^*(\xi)\}_{x\in\mathfrak{g}}$.

On the other hand, for $\xi\in\{\mathrm{Ad}_g^*(\eta)\}_{g\in G}$ we determine the tangent space at ξ to the submanifold $\{\mathrm{Ad}_g^*(\eta)\}_{g\in G}$ of $\mathfrak{g}^*$. We identify this tangent space with a linear subspace of $\mathfrak{g}^*$. The elements of this linear subspace are the elements

$$\frac{d}{dt}\mathrm{Ad}_{\exp(tx)}^*(\xi)=\mathrm{ad}_x^*(\xi)\qquad(x\in\mathfrak{g}).$$

Thus this tangent space coincides with $\mathcal{D}_\xi$.

Additions to C&P, §1.2 and §1.3

Another expression for Ad_x Let G be a Lie group. For $x\in G$ we saw that the linear map $\mathrm{Ad}_x\colon\mathrm{Lie}(G)\to\mathrm{Lie}(G)$ is the differential of $a_x\colon G\to G$. Since $\mathrm{Lie}(G)$ can be identified with $\mathcal{T}_eG$ as a linear space, we can consider Ad_x also as a linear mapping of $\mathcal{T}_eG$ to itself by the rule $\mathrm{Ad}_x(Y_e) := (\mathrm{Ad}_x(Y))_e$ ($Y \in \mathrm{Lie}(G)$). Then $\mathrm{Ad}_x = (da_x)_e$. Since $a_x(y) = xyx^{-1} = R_{x^{-1}}(L_x(y))$, we have

$$\mathrm{Ad}_x = (da_x)_e = (dR_{x^{-1}})_x \circ (dL_x)_e.$$

Lemma K1 Let G be a Lie group with bracket $\{\,.\,,\,.\,\}$ on $C^\infty(G)$ which is bilinear, antisymmetric and satisfies the Leibniz identity. Let w be the corresponding bivector. Put $\mathfrak{g} := \mathcal{T}_e G$ with the Lie algebra structure induced by $\mathrm{Lie}(G)$. Define a C^∞-mapping $w^R\colon G \to \mathfrak{g} \otimes \mathfrak{g}$ (taking anti-symmetric values) by

$$w^R(g) := \big((dR_{g^{-1}})_g \otimes (dR_{g^{-1}})_g\big)(w_g).$$

Then the following three statements are equivalent:

(a) $\{f_1, f_2\}_G(gh) = \{f_1 \circ \mu, f_2 \circ \mu\}_{G\times G}(g,h) \qquad (f_1, f_2 \in C^\infty(G),\ g, h \in G);$

(b) $w_{gh} = \big((dL_g)_h \otimes (dL_g)_h\big)(w_h) + \big((dR_h)_g \otimes (dR_h)_g\big)(w_g) \qquad (g, h \in G);$

(c) $w^R(gh) = (\mathrm{Ad}_g \otimes \mathrm{Ad}_g)(w^R(h)) + w^R(g) \qquad (g, h \in G).$

Proof The left-hand side of (a) equals $\langle w_{gh}, (df_1)_{gh} \otimes (df_2)_{gh}\rangle$. The right-hand side of (a) equals

$$\begin{aligned}
&\{(f_1 \circ \mu)(.,h), (f_2 \circ \mu)(.,h)\}_G(g) + \{(f_1 \circ \mu)(g,.), (f_2 \circ \mu)(g,.)\}_G(h)\\
&= \{f_1 \circ R_h, f_2 \circ R_h\}_G(g) + \{f_1 \circ L_g, f_2 \circ L_g\}_G(h)\\
&= \langle w_g, (d(f_1 \circ R_h))_g \otimes (d(f_2 \circ R_h))_g\rangle + \langle w_h, (d(f_1 \circ L_g))_h \otimes (d(f_2 \circ L_g))_h\rangle\\
&= \langle \big((dR_h)_g \otimes (dR_h)_g\big)(w_g), (df_1)_{gh} \otimes (df_2)_{gh}\rangle + \langle \big((dL_g)_h \otimes (dL_g)_h\big)(w_h), (df_1)_{gh} \otimes (df_2)_{gh}\rangle.
\end{aligned}$$

This establishes the equivalence of (a) and (b).

Identity (c) can be obtained from (b) by applying $\big((dR_{g^{-1}})_g \otimes (dR_{g^{-1}})_g\big) \circ \big((dR_{h^{-1}})_{gh} \otimes (dR_{h^{-1}})_{gh}\big)$ to both sides of (b). (Use the alternative expression for Ad_g given above, and use that the operations of left multiplication and right multiplication commute.) We can go back from (c) to (b) by applying the inverse of the above operator. $\square$

Note that (b) implies that $w_e = 0$ and that (c) implies that $w^R(e) = 0$.

Cocycles and coboundaries of a group Let G be a group and let π be a representation of G on a finite dimensional vector space V. A V-valued function $f\colon G \to V$ is called a *1-cocycle* if

$$\pi(g)\, f(h) + f(g) - f(gh) = 0 \qquad (g, h \in G).$$

In particular, the reader can verify immediately that all functions $f\colon G \to V$ of the form

$$f(g) := \pi(g)\, v - v \qquad (g \in G)$$

for some $v \in V$ (the so-called *coboundaries*) are 1-cocycles.

Thus, condition (c) in Lemma K1 can be rephrased as stating that the function $w^R\colon G \to \mathfrak{g} \otimes \mathfrak{g}$ is a 1-cocycle for the representation $\mathrm{Ad} \otimes \mathrm{Ad}$ of G on $\mathfrak{g} \otimes \mathfrak{g}$.

Let $\mathfrak{g}\wedge\mathfrak{g}$ be the antisymmetric part of $\mathfrak{g}\otimes\mathfrak{g}$, i.e., the linear span of all elements $A\wedge B := \frac{1}{2}(A\otimes B - B\otimes A)$ $(A,B\in\mathfrak{g})$. Take in (c) for w^R a coboundary

$$w^R(g) := (\mathrm{Ad}_g\otimes\mathrm{Ad}_g - \mathrm{id})\,(r) \qquad (g\in G)$$

for some $r\in\mathfrak{g}\wedge\mathfrak{g}$. So (c) is valid and, equivalently, (b) is valid with

$$w_g = ((dL_g)_e\otimes(dL_g)_e - (dR_g)_e\otimes(dR_g)_e)\,(r) \qquad (g\in G).$$

Re: C-P, §1.3A, formulas (11) and (12) Let G be a Lie group, put $\mathfrak{g}:=\mathcal{T}_eG$, and let $\{\,.\,,\,.\,\}$, w and w^R be related as in the assumptions of Lemma K1. Put $\delta := (dw^R)_e$, a linear map. Since $w^R\colon G\to\mathfrak{g}\otimes\mathfrak{g}$, we have $\delta\colon\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$ and δ, like w^R, maps into $\mathfrak{g}\wedge\mathfrak{g}$. There is also the formula

$$\delta(A) = \frac{d}{dt}\,w^R(\exp(tA))\big|_{t=0} \qquad (A\in\mathfrak{g}).$$

The linear antisymmetric map $\delta^*\colon\mathfrak{g}^*\otimes\mathfrak{g}^*\to\mathfrak{g}^*$ is defined by duality:

$$\langle A,\delta^*(\xi\otimes\eta)\rangle := \langle\delta(A),\xi\otimes\eta\rangle \qquad (\xi,\eta\in\mathfrak{g}^*,\ A\in\mathfrak{g}).$$

If $f\in C^\infty(G)$ then $(df)_e\in\mathcal{T}_e^*G=\mathfrak{g}^*$.

Proposition K2 Let $\{\,.\,,\,.\,\}$, w and w^R be related as in the assumptions of Lemma K1. Assume that $w^R(e)=0$ (certainly true if condition (c) of Lemma K1 holds). Then:

(i) $(d\{f_1,f_2\})_e = \delta^*\big((df_1)_e\otimes(df_2)_e\big)$ $\qquad (f_1,f_2\in C^\infty(G))$. Thus we then have an antisymmetric bilinear map $(\xi,\eta)\mapsto[\xi,\eta]\colon\mathfrak{g}^*\times\mathfrak{g}^*\to\mathfrak{g}^*$ given by:
$[\xi,\eta] := (d\{f_1,f_2\})_e$ if $\xi=(df_1)_e$, $\eta=(df_2)_e$.

(ii) If the bracket $\{\,.\,,\,.\,\}$ satisfies the Jacobi identity (i.e., if it is a Poisson bracket) then the bracket $[\,.\,,\,.\,]$ on $\mathfrak{g}^*$ satisfies the Jacobi identity, so it makes $\mathfrak{g}^*$ into a Lie algebra.

(iii) If condition (c) of Lemma K1 is satisfied then
$\delta([A,B]) = (\mathrm{ad}_A\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{ad}_A)\,\delta(B) - (\mathrm{ad}_B\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{ad}_B)\,\delta(A) \quad (A,B\in\mathfrak{g}).$

(iv) If G is a Poisson-Lie group then δ induces a Lie algebra structure on $\mathfrak{g}^*$ according to (i) and (ii), and δ satisfies the identity given in (iii).

Proof First we prove (i). We have

$$\{f_1,f_2\}(g) = \langle w_g,(df_1)_g\otimes(df_2)_g\rangle = \langle w^R(g),(d(f_1\circ R_g))_e\otimes(d(f_2\circ R_g))_e\rangle.$$

Hence, for $A\in\mathfrak{g}$,

$$\{f_1,f_2\}(\exp(tA)) = \langle w^R(\exp(tA),(d(f_1\circ R_{\exp(tA)}))_e\otimes(d(f_2\circ R_{\exp(tA)}))_e\rangle.$$

Differentiate both sides with respect to t at 0. Since $w^R(e)=0$, we only need to differentiate the right-hand side with respect to the occurrence of t in $w^R(\exp(tA))$. Thus

$$\langle A,(d\{f_1,f_2\})_e\rangle = \langle\delta(A),(df_1)_e\otimes(df_2)_e\rangle = \langle A,\delta^*((df_1)_e\otimes(df_2)_e)\rangle. \qquad \square$$

The proof of (ii) is immediate. For the proof of (iii) note that iteration of condition (c) of Lemma K1 implies that

$$w^R(ghg^{-1}) = (\mathrm{Ad}_g\otimes\mathrm{Ad}_g)(\mathrm{Ad}_h\otimes\mathrm{Ad}_h)(w^R(g^{-1}))+(\mathrm{Ad}_g\otimes\mathrm{Ad}_g)(w^R(h))+w^R(g) \qquad (g,h\in G).$$

Put $h:=\exp(tB)$ and differentiate both sides with respect to t at 0. This yields:

$$\delta(\mathrm{Ad}_g(B)) = (\mathrm{Ad}_g\otimes\mathrm{Ad}_g)(\mathrm{ad}_B\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{ad}_B)(w^R(g^{-1}))+(\mathrm{Ad}_g\otimes\mathrm{Ad}_g)(\delta(B)).$$

Next put $g:=\exp(sA)$ and differentiate both sides with respect to s at 0. Also use that $w^R(e)=0$. Then we obtain (iii). $\square$

Cocycles and coboundaries of a Lie algebra Let $\mathfrak{g}$ be a Lie algebra and let ρ be a representation of $\mathfrak{g}$ on a vector space V. A linear map $\theta\colon \mathfrak{g} \to V$ is called a *1-cocycle* if

$$\rho(A)\,\theta(B) - \rho(B)\,\theta(A) - \theta([A,B]) = 0 \qquad (A, B \in \mathfrak{g}).$$

In particular, the reader can verify immediately that all linear mappings $\theta\colon \mathfrak{g} \to V$ of the form

$$\theta(A) := \rho(A)\,v \qquad (A \in \mathfrak{g})$$

for some $v \in V$ (the so-called *coboundaries*) are 1-cocycles.

Thus the identity in (iii) of Proposition K2 can be rephrased as stating that the linear map $\delta\colon \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is a 1-cocycle for the representation of $\mathfrak{g}$ on $\mathfrak{g} \otimes \mathfrak{g}$ which is the tensor product of the adjoint representation ad of $\mathfrak{g}$ on $\mathfrak{g}$.

Observe that, corresponding to the coboundary $w^R(g) := (\mathrm{Ad}_g \otimes \mathrm{Ad}_g - \mathrm{id})\,(r)$ (for some $r \in \mathfrak{g} \wedge \mathfrak{g}$), we have the coboundary $\delta\colon \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ given by

$$\delta(A) := (\mathrm{ad}_A \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_A)\,r \qquad (A \in \mathfrak{g}).$$

Let $\mathfrak{g}$ be a finite dimensional Lie algebra which is *semisimple*, i.e., for which the *Killing form*

$$\kappa(A, B) := \mathrm{tr}\,(\mathrm{ad}_A \circ \mathrm{ad}_B) \qquad (A, B \in \mathfrak{g})$$

is a nondegenerate bilinear form. A Lemma of Whitehead (see for instance V. S. Varadarajan, *Lie groups, Lie algebras, and their representations*, Prentice-Hall, 1974, Theorem 3.12.1) states that, for a finite dimensional representation of $\mathfrak{g}$, any cocycle is a coboundary.

Exercises

Exercise 8. Let M and N be Poisson manifolds with Poisson bivectors w_M and w_N, respectively. Observe that, for $(x, y) \in M \times N$, the tangent space $\mathcal{T}_{x,y}(M \times N)$ can be identified with the direct sum $\mathcal{T}_x M \oplus \mathcal{T}_y N$ such that, if $u \in \mathcal{T}_x M$ and $v \in \mathcal{T}_y N$, then $u + v \in \mathcal{T}_{(x,y)}(M \times N)$ is given by

$$(u + v)(f) := u(f(.,y)) + v(f(x,.)) \qquad (f \in C^\infty(M \times N)).$$

Let $\{\,.\,,\,.\,\}_{M\times N}$ be defined as in C-P, §1.1.B, formula (6). Show that this bracket is antisymmetric and that it satisfies the Leibniz identity. Show that the corresponding bivector $w_{M\times N}$ satisfying

$$\langle (w_{M\times N})_{(x,y)}, (df)_{(x,y)} \otimes (dg)_{(x,y)} \rangle = \{f, g\}(x, y) \qquad (f, g \in C^\infty(M \times N))$$

equals

$$(w_{M\times N})_{(x,y)} = (w_M)_x + (w_N)_y.$$

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The Schouten bracket

The Schouten bracket was mentioned in C&P, p.17. In addition to Schouten, 1940, *l.c*, see A. Nijenhuis, *Jacobi-type identities for bilinear differential concomitants of certain tensor fields. I, II*, Indag. Math. 42 (1955), 391–397, 398–403.

Definition-Proposition K3 Let M be a C^∞-manifold of dimension m. For positive integers a, b let A be a a-vector field on M and B a b-vector field. Then there is a unique $(a+b-1)$-vector field $[A,B]$, called the *Schouten bracket* of A and B, such that for all $(a+b-1)$-forms β on M the following holds:

$$\begin{aligned}\langle [A,B],\beta\rangle =&(-1)^a\binom{a+b-1}{a}\langle B, d(\beta(A\otimes .))\rangle + (-1)^{(a-1)b}\binom{a+b-1}{b}\langle A, d(\beta(B\otimes .))\rangle\\ &-\binom{a+b}{a}\langle A\otimes B, d\beta\rangle.\end{aligned}\tag{K1}$$

Here the $(b-1)$-form $\beta(A\otimes .)$ is defined by

$$\langle X_1\otimes\cdots\otimes X_{b-1},\beta(A\otimes .)\rangle := \langle A\otimes X_1\otimes\cdots\otimes X_{b-1},\beta\rangle \qquad (X_1,\ldots,X_{b-1}\text{ vector fields.})$$

Sketch of Proof Evidently, the right-hand side of (K1) evaluated at some $x\in M$ defines a linear functional on the space of $(a+b-1)$-forms β. In order to see that this linear functional only depends on β_x it is sufficient to show that the right-hand side is linear in β over the ring $C^\infty(M)$. For the proof of this let $f\in C^\infty(M)$, take the right-hand side with β replaced by $f\beta$ and subtract from this $f\times$ the right-hand side. This difference is equal to

$$\begin{aligned}(-1)^a\binom{a+b-1}{a}\langle B, df\wedge\beta(A\otimes .)\rangle+(-1)^{(a-1)b}\binom{a+b-1}{b}\langle A, df\wedge\beta(B\otimes .)\rangle\\ -\binom{a+b}{a}\langle A\otimes B, df\wedge\beta\rangle.\end{aligned}$$

This last expression can be shown to be zero by replacing A by $X_1\otimes\cdots\otimes X_a$ and B by $X_{a+1}\otimes\cdots\otimes X_{a+b}$ ($X_1,\ldots,X_{a+b}$ vecor fields) and by showing that the expression thus obtained can be reduced to zero. □

See Exercise 9 for two simple properties of the Schouten bracket. Let us next work locally on M in coordinates $x_1,\ldots,x_m$ and write an a-vector field A as $(A_{i_1,\ldots,i_a})$ with respect to these coordinates. Then:

Proposition K4

$$\begin{aligned}[A,B]_{i_1,\ldots,i_{a+b-1}} &= \frac{1}{(a-1)!\,b!}\sum_{\sigma\in S_{a+b-1}}\sum_{j=1}^m \operatorname{sgn}(\sigma)\,A_{j,i_{\sigma(1)},\ldots,i_{\sigma(a-1)}}\frac{\partial}{\partial x_j}B_{i_{\sigma(a)},\ldots,i_{\sigma(a+b-1)}}\\ &+\frac{(-1)^a}{a!\,(b-1)!}\sum_{\sigma\in S_{a+b-1}}\sum_{j=1}^m \operatorname{sgn}(\sigma)\,B_{j,i_{\sigma(a+1)},\ldots,i_{\sigma(a+b-1)}}\frac{\partial}{\partial x_j}A_{i_{\sigma(1)},\ldots,i_{\sigma(a)}}.\end{aligned}$$

Sketch of Proof Take $\beta := dx_{i_1} \wedge \ldots \wedge dx_{i_{a+b-1}}$ in formula (K1), so $d\beta = 0$. Since $[A,B]$ is anti-symmetric, we have:

$$[A,B]_{i_1,\ldots,i_{a+b-1}} := \langle [A,B], dx_{i_1} \otimes \cdots \otimes dx_{i_{a+b-1}} \rangle = \langle [A,B], dx_{i_1} \wedge \cdots \wedge dx_{i_{a+b-1}} \rangle.$$

Now use that

$$(dx_{i_1} \wedge \ldots \wedge dx_{i_{a+b-1}})(A \otimes .) = \frac{1}{(a+b-1)!} \sum_{\sigma \in S_{a+b-1}} \operatorname{sgn}(\sigma)\, A_{i_{\sigma(1)},\ldots,i_{\sigma(a)}}\, dx_{i_{\sigma(a+1)}} \wedge \ldots \wedge dx_{i_{\sigma(a+b-1)}}.$$

Hence the differential of this $(b-1)$-form equals

$$\frac{1}{(a+b-1)!} \sum_{\sigma \in S_{a+b-1}} \sum_{j=1}^{m} \operatorname{sgn}(\sigma) \frac{\partial A_{i_{\sigma(1)},\ldots,i_{\sigma(a)}}}{\partial x_j}\, dx_j \wedge dx_{i_{\sigma(a+1)}} \wedge \ldots \wedge dx_{i_{\sigma(a+b-1)}}.$$

By substitution in formula (K1) of this last expression and a similar one, the result follows readily. □

See Exercise 10. Thus the bracket $\{\,.\,,\,.\,\}$ defined by a bivector field w on a C^∞-manifold M satisfies the Jacobi identity iff $[w,w] = 0$.

Let $\mathfrak{g}$ be a Lie algebra. For $r \in \mathfrak{g} \otimes \mathfrak{g}$ C&P (p.51) define elements $[r_{12}, r_{13}]$, etc. of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$. In order to see that this is well-defined, define (just for the moment) the linear map $\lambda: a \otimes b \to [a,b]: \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$. Let $\sigma: a \otimes b \to b \otimes a: \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ be the flip operator. Let $v, w \in \mathfrak{g}$. Then $[v_{12}, w_{13}]$ is defined as an element of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ by:

$$[v_{12}, w_{13}] := \lambda_{12}(v_{13} \otimes w_{24}) = (\lambda \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(v \otimes w).$$

If $v = \sum_i a_i \otimes b_i$ and $w = \sum_j c_j \otimes d_j$ then

$$[v_{12}, w_{13}] = \sum_{i,j} [a_i, c_j] \otimes b_i \otimes d_j.$$

Other brackets in C&P, p.51 can be dealt with in a similar way. Now let, for $r \in \mathfrak{g} \otimes \mathfrak{g}$,

$$[[r,r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}].$$

Note that, if $r \in \mathfrak{g} \wedge \mathfrak{g}$, this can be rewritten as:

$$[[r,r]] = [r_{12}, r_{13}] + [r_{23}, r_{21}] + [r_{31}, r_{32}],$$

and then $[[r,r]] \in \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$. We say that $r \in \mathfrak{g} \otimes \mathfrak{g}$ satisfies the *classical Yang-Baxter equation* if $[[r,r]] = 0$ (see C&P, p.54).

Proposition K5 Let M be a C^∞-manifold and let $\mathfrak{g}$ be the Lie algebra of vector fields on M. Let w be a bivector field on M, so $w \in \mathfrak{g} \wedge \mathfrak{g}$ and $[[w,w]]$ is a well-defined 3-vector field on M. Let $[w,w]$ be the Schouten bracket. Then

$$[w,w] = -2[[w,w]].$$

Sketch of Proof Work locally in terms of coordinates $x_1,\ldots,x_m$ and write $w=(w_{ij})$. Then, by linearity, we may suppose that $w_{ij} = u_1 v_j - u_j v_i$ with u,v vector fields on M, since a general w will be a sum of such elements. Recall that $[u,v]_j = \sum_l (u_l(\partial v_j/\partial x_l) - v_l(\partial u_j/\partial x_l)$. Now observe that

$$\begin{aligned}&\sum_{l=1}^m \left(w_{li}\,\frac{\partial w_{jk}}{\partial x_l} + w_{lj}\,\frac{\partial w_{ki}}{\partial x_l} + w_{lk}\,\frac{\partial w_{ij}}{\partial x_l}\right)\\ &= \sum_{l=1}^m \left(u_l\,\frac{\partial v_i}{\partial x_l} - v_l\,\frac{\partial u_i}{\partial x_l}\right)(v_j u_k - v_k u_j) + \text{cyclic permutation in } i,j,k. \qquad \square\end{aligned}$$

We conclude from Proposition K5 that the bracket $\{\,.\,,\,.\,\}$ defined by a bivector field w on a C^∞-manifold M satisfies the Jacobi identity iff w satisfies the classical Yang-Baxter equation $[[w,w]]=0$, where $[[\,.\,]]$ is taken with respect to the Lie algebra structure of the space of vector fields on M.

Let us apply this to the Poisson manifold structure of $\mathfrak{g}^*$, where $\mathfrak{g}$ is a Lie algebra. We have seen that the corresponding Poisson bivector field w on $\mathfrak{g}^*$ can be characterized by

$$\langle x \otimes y, w_\xi \rangle = \langle [x,y], \xi \rangle \quad (x,y \in \mathfrak{g},\ \xi \in \mathfrak{g}^*).$$

Let $x_1,\ldots,x_m$ be a basis for $\mathfrak{g}$ and let $\xi_1,\ldots,\xi_m$ be the dual basis for $\mathfrak{g}^*$. Write

$$[x_i,x_j] = \sum_k c_{ij}^k x_k$$

(see C&P, p.20). Here the c_{ij}^k are the structure constants for the Lie algebra $\mathfrak{g}$. Then

$$w_\xi = \sum_{i,j,k} c_{ij}^k\, x_k(\xi)\, \xi_i \otimes \xi_j, \quad \text{hence} \quad (w_\xi)_{ij} = \sum_k c_{ij}^k\, x_k(\xi).$$

It follows from the preceding remark that w will satisfy the classical Yang-Baxter equation with respect to the Lie algebra structure of the vector fields on $\mathfrak{g}^*$. Still otherwise said, for given structure constants of a Lie algebra, the identities for these structure constants which follow from the Jacobi identity can also be viewed as the classical Yang Baxter equation for suitable linear bivector fields expressed in terms of these structure constants.

Coboundary Poisson-Lie groups A Poisson-Lie group G with Lie algebra $\mathfrak{g}$ identified with $\mathcal{T}_e G$ is called a *coboundary Poisson-Lie group* if the Poisson bivector field w has the form

$$w_g = ((dL_g)_e \otimes (dL_g)_e - (dR_g)_e \otimes (dR_g)_e)\,(r) \qquad (g \in G)$$

for some $r \in \mathfrak{g} \wedge \mathfrak{g}$. This condition is equivalent to the fact that $w^R\colon G \to G \otimes G$ is a coboundary for the representation $\mathrm{Ad} \otimes \mathrm{Ad}$ of G on $\mathfrak{g} \otimes \mathfrak{g}$ (see pp.10,11 of these notes). For a given $r \in \mathfrak{g} \wedge \mathfrak{g}$ we see that w satisfies property (b) of Lemma K1. However, the bracket $\{\,.\,,\,.\,\}$ corresponding to w will not necessarily satisfy the Jacobi identity. A necessary and sufficient condition for the Jacobi identity is that the Schouten bracket $[w,w]$ equals 0.

Define a left invariant bivector field $w^{(1)}$ and a right invariant bivector field $w^{(2)}$ on G by:

$$w_g^{(1)} := ((dL_g)_e \otimes (dL_g)_e)(r),$$
$$w_g^{(2)} := ((dR_g)_e \otimes (dR_g)_e)(r).$$

Then $w = w^{(1)} - w^{(2)}$ and

$$[w,w] = [w^{(1)}, w^{(1)}] + [w^{(2)}, w^{(2)}] - 2[w^{(1)}, w^{(2)}],$$

Hence, by Exercise 12,

$$[w,w] = [w^{(1)}, w^{(1)}] + [w^{(2)}, w^{(2)}].$$

Also, since $w_e = 0$, we have $[w,w]_e = 0$ (write $[w,w]$ locally in terms of coordinates). Hence

$$-[w^{(2)}, w^{(2)}]_e = [w^{(1)}, w^{(1)}]_e = [[w^{(1)}, w^{(1)}]]_e = [[r,r]],$$

where the last identity follows since $\mathfrak{g} \simeq \mathcal{T}_e G$ has the Lie algebra structure induced from the Lie algebra structure of the space of left invariant vector fields. Now we have:

$[w,w] = 0$

iff

$[w^{(2)}, w^{(2)}]_g = -[w^{(1)}, w^{(1)}]_g$ for all $g \in G$

iff

$((dR_g)_e \otimes (dR_g)_e \otimes (dR_g)_e)([w^{(2)}, w^{(2)}]_e) = -((dL_g)_e \otimes (dL_g)_e \otimes (dL_g)_e)([w^{(1)}, w^{(1)}]_e)$ for all $g \in G$

iff

$(\mathrm{Ad}_g \otimes \mathrm{Ad}_g \otimes \mathrm{Ad}_g)([w^{(1)}, w^{(1)}]_e) = [w^{(1)}, w^{(1)}]_e$ for all $g \in G$

iff

$[[r,r]]$ is Ad_G-invariant.

So we have proved:

Proposition K6 Let G be a Lie group with Lie algebra $\mathfrak{g} \simeq \mathcal{T}_e G$. Let $r \in \mathfrak{g} \wedge \mathfrak{g}$. Then G is a coboundary Poisson-Lie group with Poisson bivector field

$$w_g = ((dL_g)_e \otimes (dL_g)_e - (dR_g)_e \otimes (dR_g)_e)\,(r) \qquad (g \in G)$$

iff $[[r,r]]$ is invariant under the representation $\mathrm{Ad} \otimes \mathrm{Ad} \otimes \mathrm{Ad}$ of G on $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$.

Example K7 Let $G = SU(2)$ with Lie algebra $su(2)$ (of dimension 3). Hence $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ has dimension 1. Hence the representation $\mathrm{Ad} \otimes \mathrm{Ad} \otimes \mathrm{Ad}$ on $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ can be equivalently considered as a representation of $SU(2)$ on $\mathbb{R}$. Since $SU(2)$ is compact and connected, this will be the trivial representation. Hence, for any $r \in \mathfrak{g} \wedge \mathfrak{g}$, we see that $[[r,r]]$ is Ad-invariant and that r will yield a coboundary Poisson-Lie group by Proposition K6. In particular, the bivector field w_G in C&P, Example 1.2.5 indeed defines a Poisson-Lie group structure on $SU(2)$.

Triangular and quasi-triangular Poisson-Lie groups Keep the assumptions of Proposition K6. Clearly, $[[r,r]]$ is Ad_G-invariant if it equals zero. A coboundary Poisson-Lie group corresponding to an element $r \in \mathfrak{g} \wedge \mathfrak{g}$ satisfying $[[r,r]] = 0$ is called a *triangular* Poisson-Lie group.

Next assume that $u \in \mathfrak{g} \otimes \mathfrak{g}$, $[[u,u]] = 0$ and $s := \frac{1}{2}(u_{12} + u_{21})$ (the symmetric part of u) is Ad_G-invariant, hence also $\mathrm{ad}_{\mathfrak{g}}$-invariant. Let $r := \frac{1}{2}(u_{12} - u_{21})$ (the antisymmetric part of u). Then, by Exercise 11, $0 = [[u,u]] = [[r,r]] + [[s,s]]$. Hence $[[r,r]] = -[[s,s]]$, and it is Ad_G-invariant (see Exercise 14). So, by Proposition K6, there is a coboundary Poisson-Lie group corresponding to r. Such a Poisson-Lie group is called *quasi-triangular*. Note that, for such r, the Poisson bivector field w on G also satisfies

$$w_g = ((dL_g)_e \otimes (dL_g)_e - (dR_g)_e \otimes (dR_g)_e)\,(u) \qquad (g \in G).$$

Exercises (submit 11 or 12)

Exercise 9. Let A be a a-vector field and B a b-vector field on a C^∞-manifold M. Show that

$$[A,B] = (-1)^{ab}\,[B,A].$$

Show also that, for $a = b = 1$, the Schouten bracket $[A,B]$ coincides with the commutator $AB - BA$ of vector fields.

Exercise 10. Let A be a bivector field on a C^∞-manifold M. Show that locally, in terms of coordinates $x_1, \ldots, x_m$ we have:

$$[A,A]_{ijk} = 2\sum_{l=1}^{m} \left(A_{li}\,\frac{\partial A_{jk}}{\partial x_l} + A_{lj}\,\frac{\partial A_{ki}}{\partial x_l} + A_{lk}\,\frac{\partial A_{ij}}{\partial x_l} \right).$$

Exercise 11. Let $\mathfrak{g}$ be a Lie algebra, let $a \in \mathfrak{g} \wedge \mathfrak{g}$ and $b \in \mathfrak{g} \otimes \mathfrak{g}$. Show that

$$[[a+b, a+b]] = [[a,a]] + [[b,b]] + [a_{12} + a_{32}, b_{13}] + [a_{12} + a_{13}, b_{23}] - [a_{13} + a_{23}, b_{12}].$$

Conclude that

$$[[a+b, a+b]] = [[a,a]] + [[b,b]]$$

if $a \in \mathfrak{g} \wedge \mathfrak{g}$, $b \in \mathfrak{g} \otimes \mathfrak{g}$ and b is ad-invariant, i.e., if $(\mathrm{ad}_x \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_x)(b) = 0$ for all $x \in \mathfrak{g}$.

Exercise 12. Let G be a Lie group. Show that $[X,Y] = 0$ if X is a left invariant vector field and Y is a right invariant vector field on G. Use this in order to show that the Schouten bracket $[A,B]$ of a left invariant bivector field A and a right invariant bivector field B equals zero.
Hint Write $2[A,B] = [A+B, A+B] - [A,A] - [B,B] = [[A+B, A+B]] - [[A,A]] - [[B,B]]$ and use the first identity in Exercise 11.

Exercise 13. Let G be a Lie group. Let A be a bivector field on G. Show that that the 3-vector field $[A,A]$ is left (right) invariant if A is left (right) invariant.
Hint Use that $[A,A] = [[A,A]]$ taken with respect to the Lie algebra structure of the vector fields on M.

Exercise 14. Let G be a Lie group with Lie algebra $\mathfrak{g} \simeq \mathcal{T}_e G$. Let $b \in \mathfrak{g} \otimes \mathfrak{g}$ be Ad_G-invariant. Show that $[[b,b]]$ is Ad_G-invariant.

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The modified classical Yang-Baxter equation (see C&P, pp.54,55)

Let $\mathfrak{g}$ be a finite dimensional Lie algebra equipped with a non-degenerate bilinear symmetric form $\langle\,.\,,\,.\,\rangle$ which is $\mathrm{ad}_{\mathfrak{g}}$-invariant, i.e., for which

$$\langle \mathrm{ad}_x(y), z\rangle + \langle y, \mathrm{ad}_x(z)\rangle = 0 \qquad (x,y,z\in\mathfrak{g}),$$

or equivalently,

$$\langle [x,y], z\rangle = \langle x, [y,z]\rangle \qquad (x,y,z\in\mathfrak{g}).$$

If G is connected Lie group with $\mathrm{Lie}(G)=\mathfrak{g}$, then $\mathrm{ad}_{\mathfrak{g}}$-invariance of the form $\langle\,.\,,\,.\,\rangle$ is equivalent to Ad_G-invariance:

$$\langle \mathrm{Ad}_g(y), \mathrm{Ad}_g(z)\rangle = \langle y,z\rangle \qquad (y,z\in\mathfrak{g},\ g\in G).$$

Such a form exists uniquely, up to a constant factor, if $\mathfrak{g}$ is simple (i.e., having no nontrivial ideals). If $\mathfrak{g}$ is semisimple (i.e., direct sum of simple Lie algebras) or reductive (i.e., direct sum of a semisimple and an abelian Lie algebra) then such a form exists, but not necessarily uniquely.

Now define $t\in\mathfrak{g}\otimes\mathfrak{g}$ and $\omega\in\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$ by

$$\langle x\otimes y, t\rangle := \langle x,y\rangle \qquad (x,y\in\mathfrak{g}), \tag{K2}$$

$$\langle x\otimes y\otimes z, \omega\rangle := \langle [x,y], z\rangle \qquad (x,y,z\in\mathfrak{g}). \tag{K3}$$

Note that t is symmetric and $\mathrm{ad}_{\mathfrak{g}}$-invariant (and Ad_G-invariant), and that ω is antisymmetric and $\mathrm{ad}_{\mathfrak{g}}$-invariant (and Ad_G-invariant). It can be shown (see Exercise 15) that

$$[[t,t]] = \omega. \tag{K4}$$

Let $r\in\mathfrak{g}\wedge\mathfrak{g}$. Then:

$$[[r+t, r+t]] = 0 \quad\Longleftrightarrow\quad [[r,r]] = -\omega.$$

We call the equation $[[r,r]]=-\omega$ the *modified clasical Yang-Baxter equation*. See the example for $sl(2)$ in Exercise 16. We also have:

$$[[r+it, r+it]] = 0 \quad\Longleftrightarrow\quad [[r,r]] = \omega.$$

See an example of this last case for $su(2)$ in Exercise 17.

Linear coboundary Poisson-Lie groups First observe that, on a Poisson manifold M, the Poisson bracket can be extended to a complex bilinear mapping for complex-valued C^∞-functions on M. Now let $G\subset GL(n,\mathbb{C})$ be a linear Lie group. Then each $g\in G$ can be written as an element $g=(g_{ij})$ of $GL(n,\mathbb{C})$ and the functions $\tau_{ij}\colon g\to g_{ij}$ are (complex-valued) C^∞-functions on G. Suppose that G is also a coboundary Poisson-Lie group. Let $\mathcal{A}(G)$ be the algebra (under pointwise multiplication) generated by the functions τ_{ij}. It will turn out that for any pair τ_{ij},τ_{kl} their Poisson bracket is in $\mathcal{A}(G)$ with an explicit (quadratic) expression. Thus, because of the Leibniz rule, $\mathcal{A}(G)$ is closed under taking the Poisson bracket.

Proposition K8 Let $G \subset GL(n,\mathbb{C})$ be a linear Lie group with Lie algebra $\mathfrak{g} \subset gl(n,\mathbb{C})$. Let $r \in \mathfrak{g} \wedge \mathfrak{g}$ such that G becomes a coboundary Poisson-Lie group with Poisson bivector field as in Proposition K6. Then

$$\begin{aligned}\{\tau_{ij}, \tau_{kl}\}(g) &= ((g \otimes g)r)_{ik,jl} - (r(g \otimes g))_{ik,jl} \\ &= \sum_{p,q} g_{ip} g_{kq} r_{pq,jl} - \sum_{p,q} r_{ik,pq} g_{pj} g_{ql}.\end{aligned} \tag{K5}$$

Proof We give the proof for $G \subset GL(n,\mathbb{R})$ with Lie algebra $\mathfrak{g} \subset gl(n,\mathbb{R})$. (The proof for $G \subset GL(n,\mathbb{C})$ is similar.) Recall that

$$\{\tau_{ij}, \tau_{kl}\}(g) = \langle w_g, (d\tau_{ij})_g \otimes (d\tau_{kl})_g \rangle.$$

Substitute the expression for w_g as given in Proposition K5. It follows that

$$\{\tau_{ij}, \tau_{kl}\}(g) = \langle r, (d(\tau_{ij} \circ L_g))_e \otimes (d(\tau_{kl} \circ L_g))_e \rangle - \langle r, (d(\tau_{ij} \circ R_g))_e \otimes (d(\tau_{kl} \circ R_g))_e \rangle.$$

Now observe that, for $v = (v_{ij}) \in \mathfrak{g}$ and $f \in C^\infty(G)$ we have:

$$\begin{aligned}\langle v, (d(f \circ L_g))_e \rangle &= \sum_{p,q} (gv)_{pq} \frac{\partial f(g)}{\partial g_{pq}}, \\ \langle v, (d(f \circ R_g))_e \rangle &= \sum_{p,q} (vg)_{pq} \frac{\partial f(g)}{\partial g_{pq}}.\end{aligned}$$

Hence

$$\langle v, (d(\tau_{ij} \circ L_g))_e \rangle = (gv)_{ij}, \quad \langle v, (d(\tau_{ij} \circ R_g))_e \rangle = (vg)_{ij}.$$

A moment's thought now yields (K5). $\square$

See Exercise 18 for an application of this Proposition to the case $G = SL(2,\mathbb{R})$.

Cohomology of groups and of Lie algebras First we discuss cohomology of groups, see for instance A. A. Kirillov, *Elements of the theory of representations*, Springer, 1976, §2.5. Let G be a group and let π be a representation of G on a finite dimensional vector space V. By an *n-dimensional cochain* we mean a V-valued function $f\colon G^{n+1} \to V$ such that

$$f(gg_0, gg_1, \ldots, gg_n) = \pi(g)\, f(g_0, g_1, \ldots, g_n).$$

Denote the vector space of all n-dimensional cochains by $C^n(G,\pi)$. Define the linear operator $d\colon C^n(G,\pi) \to C^{n+1}(G,\pi)$ by

$$(df)(g_0, \ldots, g_{n+1}) := \sum_{i=0}^{n+1} (-1)^i\, f(g_0, \ldots, \widehat{g_i}, \ldots, g_{n+1}).$$

Then it follows easily that $d \circ d = 0$. A cochain f is called a *coboundary* if $f = dg$ for some cochain g. The space of all n-dimensional coboundaries is denoted by $B^n(G,\pi)$. A cochain f is called a *cocycle* if $df = 0$. The space of all n-dimensional cocycles is denoted by $Z^n(G,\pi)$. Clearly, every coboundary is a cocycle, so $B^n(G,\pi)$ is a linear subspace of

$Z^n(G,\pi)$. The quotient space $Z^n(G,\pi)/B^n(G,\pi)$ is called the *n-dimensional cohomology space* of G for the representation π.

There is the following 1-1 linear bijection $f \leftrightarrow \widetilde{f}$ between the space of n-dimensional cochains and the space of V-valued functions on G^n:

$$\widetilde{f}(h_1,\dots,h_n) = f(e, h_1, h_1h_2, \dots, h_1h_2\dots h_n).$$

Then

$$\begin{aligned}\widetilde{df}(h_1,\dots,h_{n+1}) &= \pi(h_1)\,\widetilde{f}(h_2,\dots,h_{n+1}) \\ &+ \sum_{i=1}^{n}(-1)^i\,\widetilde{f}(h_1,\dots,h_{i-1},h_ih_{i+1},h_{i+2},\dots,h_{n+1}) + (-1)^{n+1}\,\widetilde{f}(h_1,\dots,h_n).\end{aligned}$$

In particular

$$\widetilde{df}(h) = \pi(h)\,\widetilde{f} - \widetilde{f}, \quad \widetilde{df}(h_1,h_2) = \pi(h_1)\,\widetilde{f}(h_2) - \widetilde{f}(h_1h_2) + \widetilde{f}(h_1).$$

Next we discuss cohomology of Lie algebras. Let $\mathfrak{g}$ be a Lie algebra and let ρ be a representation of $\mathfrak{g}$ on a finite dimensional vector space V. By an *n-dimensional cochain* we mean a linear anti-symmetric mapping $\phi\colon \mathfrak{g}^n \to V$. Denote the vector space of all n-dimensional cochains by $C^n(\mathfrak{g},\rho)$. Define the linear operator $d\colon C^n(\mathfrak{g},\rho) \to C^{n+1}(\mathfrak{g},\rho)$ by

$$\begin{aligned}(d\phi)(x_1\wedge\ldots\wedge x_{n+1}) &:= \sum_{i=1}^{n+1}(-1)^{i+1}\,\rho(x_i)\,\phi(x_1\wedge\ldots\wedge\widehat{x_i}\wedge\ldots\wedge x_{n+1}) \\ &+ \sum_{i<j}(-1)^{i+j}\,\phi([x_i,x_j]\wedge x_1\wedge\ldots\wedge\widehat{x_i}\wedge\ldots\wedge\widehat{x_j}\wedge\ldots\wedge x_{n+1}).\end{aligned}$$

Then $d\circ d = 0$. The special cases $n = 0, 1$ of the operator d are:

$$(d\phi)(x) = \rho(x)\,\phi, \quad (d\phi)(x,y) = \rho(x)\,\phi(y) - \rho(y)\,\phi(x) - \phi([x,y]).$$

A cochain ϕ is called a *coboundary* if $\phi = d\psi$ for some cochain ψ. The space of all n-dimensional coboundaries is denoted by $B^n(\mathfrak{g},\rho)$. A cochain ϕ is called a *cocycle* if $d\phi = 0$. The space of all n-dimensional cocycles is denoted by $Z^n(\mathfrak{g},\rho)$. Clearly, every coboundary is a cocycle, so $B^n(\mathfrak{g},\rho)$ is a linear subspace of $Z^n(\mathfrak{g},\rho)$. The quotient space $Z^n(\mathfrak{g},\rho)/B^n(\mathfrak{g},\rho)$ is called the *n-dimensional cohomology space* of $\mathfrak{g}$ for the representation ρ.

Proposition K9 Let G be a connected Lie group with Lie algebra $\mathfrak{g} \simeq \mathcal{T}_eG$. Let π be a continuous (hence smooth) representation of G on a finite dimensional vector space V. Let $\rho := d\pi$ be the corresponding representation of $\mathfrak{g}$ on V. Then

(i) If $f\colon G \to V$ is a group 1-cocycle then $\phi := (df)_e\colon \mathfrak{g} \to V$ is a Lie algebra 1-cocycle. Also, f is uniquely determined by ϕ. Furthermore, f is coboundary iff ϕ is coboundary.

(ii) If G is simply connected then to any Lie algebra 1-cocycle ϕ there corresponds a Lie group 1-cocycle f such that $\phi = (df)_e$.

(iii) If $\mathfrak{g}$ is semisimple then every 1-cocycle $\phi\colon \mathfrak{g} \to V$ on $\mathfrak{g}$ is a coboundary, i.e., $\phi(x) = \rho(x)\,v$ for some $v \in V$.

(iv) If G is compact then every 1-cocycle $f\colon G \to V$ on G is a coboundary, i.e., $f(g) = \pi(g)\,v - v$ for some $v \in V$.

Parts (i) and (iv) are left as exercises to the reader. For the proof of (ii) see J.-H. Lu, *Multiplicative and affine Poisson structures on Lie groups*, Dissertation, University of California, Berkeley, 1990, Lemma 2.13. Part (iii) is Whitehead's Lemma, see p.12 of these notes.

Lie bialgebras Read at p.11 of these notes Proposition K2 and the paragraph preceding it. Lu (*l.c.*, Theorem 2.18) shows that, under certain conditions, the converse of part (ii) of Proposition K2 also holds:

Proposition K10 Let G be a connected Lie group with bracket $\{\,.\,,\,.\,\}$ on $C^\infty(G)$ which is bilinear, antisymmetric, satisfies the Leibniz identity and satisfies condition (a) of Lemma K1. Let $\mathfrak{g} \simeq \mathcal{T}_e G$ be the Lie algebra of G. Define a bracket $[\,.\,,\,.\,]$ on $\mathfrak{g}^*$ by the rule
$[\xi,\eta] := (d\{f_1,f_2\})_e \quad \text{if } \xi = (df_1)_e,\ \eta = (df_2)_e.$
Then $\{\,.\,,\,.\,\}$ satisfies the Jacobi identity iff $[\,.\,,\,.\,]$ satisfies the Jacobi identity.

See now C&P, Definition 1.3.1 (definition of a *Lie bialgebra*). As a corollary of Propositions K2, K9 and K10 we have a result stated in C&P, Theorem 1.3.2:

Theorem K11 Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Every Poisson-Lie group structure on G determines, as in Proposition K2, a Lie bialgebra structure on $\mathfrak{g}$, and to every Lie bialgebra structure on $\mathfrak{g}$ there corresponds at most one Poisson-Lie group structure on G. If G is moreover simply connected then, for every Lie bialgebra structure on $\mathfrak{g}$, there exists a corresponding Poisson-Lie group structure on G.

Manin triples See in C&P the definition of a *Manin triple* (Definition 1.3.3) and the 1-1 correspondence between Lie bialgebras and Manin triples stated in Proposition 1.3.4. Proposition 1.3.4 follows from Lemma 1.3.5. We restate this Lemma.

Let $\mathfrak{g}$ and $\mathfrak{g}^*$ be finite dimensional vector spaces, dual to each other. Extend the bilinear pairing $\langle\,.\,,\,.\,\rangle$ on $\mathfrak{g}\times\mathfrak{g}^*$ to a symmetric bilinear form $\langle\,.\,,\,.\,\rangle$ on $\mathfrak{g}\oplus\mathfrak{g}^*$ such that $\langle x,y\rangle = 0$ $(x,y\in\mathfrak{g})$ and $\langle\xi,\eta\rangle = 0$ $(\xi,\eta\in\mathfrak{g}^*)$. Assume that both $\mathfrak{g}$ and $\mathfrak{g}^*$ have a Lie algebra structure. Extend the Lie brackets on $\mathfrak{g}$ and on $\mathfrak{g}^*$ to an antisymmetric bilinear bracket $[\,.\,,\,.\,]$ on $\mathfrak{g}\oplus\mathfrak{g}^*$ such that the form $\langle\,.\,,\,.\,\rangle$ on $\mathfrak{g}\times\mathfrak{g}^*$ is "ad-invariant", i.e.,

$$\langle [x+\xi, y+\eta], z+\zeta\rangle = \langle x+\xi, [y+\eta, z+\zeta]\rangle \quad (x,y,z\in\mathfrak{g},\ \xi,\eta,\zeta\in\mathfrak{g}^*).$$

Hence, for $x\in\mathfrak{g}$ and $\eta\in\mathfrak{g}^*$, the $\mathfrak{g}$-part $[x,\eta]_{\mathfrak{g}}$ and the $\mathfrak{g}^*$-part $[x,\eta]_{\mathfrak{g}^*}$ of $[x,\eta]$ are determined by

$$\begin{aligned} \langle [x,\eta]_{\mathfrak{g}}, \zeta\rangle &= \langle x, [\eta,\zeta]_{\mathfrak{g}^*}\rangle \quad (\zeta\in\mathfrak{g}^*),\\ \langle [x,\eta]_{\mathfrak{g}^*}, z\rangle &= -\langle \eta, [x,z]_{\mathfrak{g}}\rangle \quad (z\in\mathfrak{g}). \end{aligned}$$

Define $\delta\colon \mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$ by

$$\langle \delta(x), \eta\otimes\zeta\rangle := \langle x, [\eta,\zeta]\rangle \quad (\eta,\zeta\in\mathfrak{g}^*).$$

Lemma K12 The bracket $[\,.\,,\,.\,]$ satisfies the Jacobi identity iff δ is a 1-cocycle for the representation ad of $\mathfrak{g}$ on $\mathfrak{g}\otimes\mathfrak{g}$.

For the proof see the proof of C&P, Lemma 1.3.5.

Exercises (submit 16 or 18, preferably both)

Exercise 15. Let $\mathfrak{g}$ be a finite dimensional Lie algebra equipped with a non-degenerate bilinear symmetric form. Let $t \in \mathfrak{g} \otimes \mathfrak{g}$ and $\omega \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ be defined by (K2), (K3). Prove (K4).
Hint For instance, if $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$ then there will be a basis $x_1, \ldots, x_m$ of $\mathfrak{g}$ such that $\langle x_i, x_j \rangle = \lambda_j^{-1}\, \delta_{ij}$ for certain real nonzero λ_j. Now express t and ω in terms of the x_j, λ_j and (for ω) the structure coefficients of $\mathfrak{g}$.

Exercise 16 Let $\mathfrak{g} = sl(2, \mathbb{F})$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). Take h, b, c as a basis for $\mathfrak{g}$ with

$$h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad c := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

An $\mathrm{ad}_{\mathfrak{g}}$-invariant bilinear symmetric form (unique up to a constant factor) on $\mathfrak{g}$ is given by $\langle h, h \rangle := 2$, $\langle b, c \rangle = \langle c, b \rangle := 1$, while the form on other pairs of basis elements equals zero. Show that t and ω, defined by (K2) and (K3), are equal to

$$t = \tfrac{1}{2}\, h \otimes h + b \otimes c + c \otimes b, \quad \omega = 6\, h \wedge c \wedge b.$$

Let

$$r := 2\, c \wedge b.$$

Show that $[[r, r]] = -\omega$.

Exercise 17 Let $\mathfrak{g} = su(2)$. It has a basis consisting of

$$ih = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad b - c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad i(b + c) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Here h, b, c are as in Exercise 17. The restriction to $su(2)$ of the ad-invariant bilinear symmetric form on $sl(2, \mathbb{C})$ given in Exercise 16 yields: $\langle ih, ih \rangle = -2$, $\langle b - c, b - c \rangle = -2$, $\langle i(b + c), i(b + c) \rangle = -2$, while the form on other pairs of basis elements equals zero. Show that t and ω, defined by (K2) and (K3), are equal to

$$t = -\tfrac{1}{2}\, (ih) \otimes (ih) - \tfrac{1}{2}\, (b - c) \otimes (b - c) - \tfrac{1}{2}\, (ib + ic) \otimes (ib + ic), \quad \omega = -3\, h \wedge (ib + ic) \wedge (b - c).$$

Let

$$r := (ib + ic) \wedge (b - c).$$

Show that $[[r, r]] = \omega = [[t, t]]$ and that $[[r + it, r + it]] = 0$.

Exercise 18 Let $\alpha, \beta, \gamma, \delta$ be the functions on $SL(2, \mathbb{C})$ which send $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to a, b, c, d, respectively. Apply Proposition K8 to the case $G = SL(2, \mathbb{R})$ with r as in Exercise 16. Show that

$$\{\alpha, \beta\} = \alpha\beta, \quad \{\alpha, \gamma\} = \alpha\gamma, \quad \{\alpha, \delta\} = 2\beta\gamma, \quad \{\beta, \gamma\} = 0, \quad \{\beta, \delta\} = \beta\delta, \quad \{\gamma, \delta\} = \gamma\delta.$$

Hint Rewrite (K5) as an identitity for (4×4) matrices with the left-hand side equal to

$$\begin{pmatrix} \{\alpha,\alpha\}(g) & \{\alpha,\beta\}(g) & \{\beta,\alpha\}(g) & \{\beta,\beta\}(g) \\ \{\alpha,\gamma\}(g) & \{\alpha,\delta\}(g) & \{\beta,\gamma\}(g) & \{\beta,\delta\}(g) \\ \{\gamma,\alpha\}(g) & \{\gamma,\beta\}(g) & \{\delta,\alpha\}(g) & \{\delta,\beta\}(g) \\ \{\gamma,\gamma\}(g) & \{\gamma,\delta\}(g) & \{\delta,\gamma\}(g) & \{\delta,\delta\}(g) \end{pmatrix}$$

and

$$g \otimes g = \begin{pmatrix} a^2 & ab & ba & b^2 \\ ac & ad & bc & bd \\ ca & cb & da & db \\ c^2 & cd & dc & d^2 \end{pmatrix}, \quad r = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Exercise 19 Prove Proposition K9 (i).
Hint For the first part imitate the proof of Proposition K2 (iii). For the second part assume $\phi = 0$ and show first that $\rho(x)f(g) = 0$ for all $x \in \mathfrak{g}$ and all $g \in G$.

Exercise 20 Prove Proposition K9 (iv).
Hint Integrate the cocycle condition with respect to the Haar measure on G.

College Quantumgroepen, Koornwinder, 22-10-96

Structure of semisimple Lie algebras (tutorial) See for instance:
J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer, 1972;
T. H. Koornwinder, *Real semisimple Lie algebras*, in *The structure of real semisimple Lie groups*, MC Syllabus 49, Mathematisch Centrum, Amsterdam, 1982.
Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Equivalently, let $\mathfrak{g}$ be a finite dimensional complex Lie algebra for which the Killing form $\kappa(X,Y) := \mathrm{tr}\,(\mathrm{ad}_X \circ \mathrm{ad}_Y)$ is a nondegenerate bilinear symmetric form on $\mathfrak{g}$. A *Cartan subalgebra* (CSA) of $\mathfrak{g}$ is a maximal abelian Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that, for each $H \in \mathfrak{h}$, the linear map $\mathrm{ad}_H \colon \mathfrak{g} \to \mathfrak{g}$ is semisimple (i.e., diagonalizable). There exists a CSA in $\mathfrak{g}$, and all CSA's are conjugate under the (complex) adjoint group of $\mathfrak{g}$. Let $\mathfrak{h}^*$ be the linear dual of $\mathfrak{h}$. For $\alpha \in \mathfrak{h}^*$ let the linear subspace $\mathfrak{g}_\alpha$ of $\mathfrak{g}$ be defined by

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid [H,X] = \alpha(H)\,X \quad \forall H \in \mathfrak{h}\}.$$

Let the set Δ consist of all $\alpha \in \mathfrak{h}^* \backslash \{0\}$ such that $\dim \mathfrak{g}_\alpha > 0$. Then $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Delta$ and

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha \qquad \text{(direct sum of vector spaces).}$$

The restriction of the Killing form κ to $\mathfrak{h}$ is a nondegenerate bilinear symmetric form on $\mathfrak{h}$. For $\lambda \in \mathfrak{h}^*$ let $T_\lambda \in \mathfrak{h}$ be defined by

$$\kappa(T_\lambda, H) := \lambda(H) \qquad (H \in \mathfrak{h}).$$

Define a nondegenerate bilinear symmetric form on $\mathfrak{h}^*$ by

$$\langle \lambda, \mu \rangle := \kappa(T_\lambda, T_\mu) \qquad (\lambda, \mu \in \mathfrak{h}^*).$$

Let $\mathfrak{h}_0$ be the $\mathbb{R}$-span of the T_α ($\alpha \in \Delta$). Then $\mathfrak{h}_0$ is a real form of $\mathfrak{h}$ and the restriction of the Killing form to $\mathfrak{h}_0$ is a positive definite bilinear symmetric form. The same holds for the restriction of $\langle\,.\,,\,.\,\rangle$ to $\mathfrak{h}_0^*$ (i.e., to the real linear dual of $\mathfrak{h}_0$). Thus $\mathfrak{h}_0$ and $\mathfrak{h}_0^*$ become inner product spaces. We will also use the notation

$$H_\alpha := \frac{2}{\langle \alpha, \alpha \rangle}\, T_\alpha \qquad (\alpha \in \Delta).$$

For $\alpha \in \mathfrak{h}_0^* \backslash \{0\}$ define the reflection map $s_\alpha \colon \mathfrak{h}_0^* \to \mathfrak{h}_0^*$ by

$$s_\alpha(\lambda) := \lambda - \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}\, \alpha \qquad (\lambda \in \mathfrak{h}_0^*).$$

The finite subset Δ of $\mathfrak{h}_0^* \backslash \{0\}$ is a *root system* in the inner product space $\mathfrak{h}_0^*$, i.e.,
(a) Δ spans $\mathfrak{h}_0^*$;
(b) If $\alpha, \beta \in \Delta$ then $s_\beta(\alpha) \in \Delta$;
(c) If $\alpha, \beta \in \Delta$ then $\frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$.

In particular, note that $-\alpha \in \Delta$ iff $\alpha \in \Delta$. It can also be shown that the root system Δ is *reduced*, i.e., if $\alpha \in \Delta$ then the only multiples of α in Δ are $\pm\alpha$. The elements of Δ are called *roots*, the nonzero elements of $\mathfrak{g}_\alpha$ $(\alpha \in \Delta)$ are called *root vectors*.

The *Weyl group* W corresponding to the root system Δ is the subgroup of the group of orthogonal transformations of $\mathfrak{h}_0^*$ which is generated by the reflections s_α $(\alpha \in \Delta)$.

If $\alpha, \beta \in \Delta$ and X, Y are nonzero elements of $\mathfrak{g}_\alpha, \mathfrak{g}_\beta$, respectively, then $[X, Y] \in \mathfrak{g}_{\alpha+\beta}$, we have $[X, Y] \neq 0$ iff $\alpha + \beta \in \Delta \cup \{0\}$, and we have $\kappa(X, Y) \neq 0$ iff $\alpha + \beta = 0$.

Choose some nonzero $\lambda \in \mathfrak{h}_0^*$ such that $\langle \lambda, \alpha \rangle \neq 0$ for all roots α. Let $\Delta_+ := \{\alpha \in \Delta \mid \langle \lambda, \alpha \rangle > 0\}$. Then Δ is the disjoint union of Δ_+ and $-\Delta_+$. The elements of Δ_+ are called *positive roots*. Define

$$\mathfrak{n}_+ := \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha, \quad \mathfrak{n}_- := \sum_{\alpha \in \Delta_+} \mathfrak{g}_{-\alpha}.$$

Then $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are Lie subalgebras of $\mathfrak{g}$. They are *nilpotent* Lie algebras, i.e., for each element X sufficiently high powers of ad_X acting on this Lie algebra vanish. Also define the so-called *Borel subalgebras*

$$\mathfrak{b}_+ := \mathfrak{h} + \mathfrak{n}_+, \quad \mathfrak{b}_- := \mathfrak{h} + \mathfrak{n}_-.$$

These are also Lie subalgebras of $\mathfrak{g}$.

For each $\alpha \in \Delta$ choose $X_\alpha \in \mathfrak{g}_\alpha \backslash \{0\}$. If $\alpha, \beta, \alpha + \beta \in \Delta$ then let $c_{\alpha,\beta} \in \mathbb{C}\backslash\{0\}$ be defined by

$$[X_\alpha, X_\beta] = c_{\alpha,\beta}\, X_{\alpha+\beta}.$$

Theorem K13 With notation as above the X_α's can be chosen such that:
(i) $[X_\alpha, X_{-\alpha}] = H_\alpha \qquad (\alpha \in \Delta)$;
(ii) if $\alpha, \beta, \alpha + \beta \in \Delta$ then $c_{\alpha,\beta} = -c_{-\alpha,-\beta}$ and $c_{\alpha,\beta} \in \mathbb{R}$;
(iii) $\kappa(X_\alpha, X_{-\alpha}) = 2/\langle \alpha, \alpha \rangle \qquad (\alpha \in \Delta)$.

Let $\mathfrak{g}$ be a complex Lie algebra and write $\mathfrak{g}_\mathrm{R}$ for $\mathfrak{g}$ considered as a real Lie algebra. By a *real form* of $\mathfrak{g}$ we mean a (real) Lie subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}_\mathrm{R}$ such that $\mathfrak{g}_\mathrm{R}$ is the direct sum of $\mathfrak{g}_0$ and $i\mathfrak{g}_0$.

Corollary K14 Let the X_α's be as in Theorem K13. Then $i\mathfrak{h}_0$ together with the elements $X_\alpha - X_{-\alpha}$ and $iX_\alpha + iX_{-\alpha}$ $(\alpha \in \Delta)$ span a real form $\mathfrak{u}$ of $\mathfrak{g}$ on which κ is a negative definite symmetric bilinear form. (We call $\mathfrak{u}$ a *compact real form* of $\mathfrak{g}$.)

Example K15 (Manin triple) (cf. C&P, Example 1.4.3)
Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Let $\mathfrak{h}$ be a CSA in $\mathfrak{g}$, let Δ_+ be a choice of positive roots and let $\mathfrak{n}_\pm$ and $\mathfrak{b}_\pm$ be as above. Let $P\colon \mathfrak{g} \to \mathfrak{h}$ be the projection onto $\mathfrak{h}$ which vanishes on $\mathfrak{n}_+ + \mathfrak{n}_-$. Then the restrictions $P\colon \mathfrak{b}_+ \to \mathfrak{h}$ and $P\colon \mathfrak{b}_- \to \mathfrak{h}$ are Lie algebra homomorphisms. Define a new Lie algebra $\mathfrak{q}$ as a direct sum by

$$\mathfrak{q} := \mathfrak{g} \oplus \mathfrak{h} := \{(X, H) \mid X \in \mathfrak{g},\ H \in \mathfrak{h}\}.$$

Then $\mathfrak{q}$ can also be written as a direct sum $\mathfrak{q} = \mathfrak{q}_+ + \mathfrak{q}_-$ of Lie subalgebras

$$\mathfrak{q}_\pm := \{(x, Px) \mid x \in \mathfrak{b}_\pm\}.$$

Indeed, for $X = X_+ + X_0 + X_-$ ($X_\pm \in \mathfrak{n}_\pm$, $X_0 \in \mathfrak{h}$) being an arbitrary element of $\mathfrak{g}$ and for $H \in \mathfrak{h}$ we have

$$(X,H) = (X_+ + \tfrac{1}{2}X_0 + \tfrac{1}{2}H, \tfrac{1}{2}X_0 + \tfrac{1}{2}H) + (X_- + \tfrac{1}{2}X_0 - \tfrac{1}{2}H, -\tfrac{1}{2}X_0 + \tfrac{1}{2}H),$$

where the first term is in $\mathfrak{q}_+$ and the second term in $\mathfrak{q}_-$. Now define a symmetric bilinear form $\langle\,.\,,\,.\,\rangle$ on $\mathfrak{q}$ by

$$\langle (X,H),(X',H')\rangle := \kappa(X,X') - \kappa(H,H') \qquad (X,X' \in \mathfrak{g},\ H,H' \in \mathfrak{h}). \tag{K5}$$

Then this form is nondegenerate and $\mathrm{ad}_\mathfrak{q}$-invariant, and the form restricted to $\mathfrak{q}_+$ or $\mathfrak{q}_-$ vanishes (see Exercise 21). We conclude that $(\mathfrak{q},\mathfrak{q}_+,\mathfrak{q}_-)$ is a Manin triple.

Example K16 (Manin triple) (cf. C&P, §1.4C; J,-H. Lu, Dissertation, Example 2.30). Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{u}$ a compact real form of $\mathfrak{g}$. Let $\mathfrak{n}_+$ be as defined on p.25. Put $\mathfrak{b}_0 := \mathfrak{h}_0 + \mathfrak{n}_+$, a real Lie subalgebra of $\mathfrak{g}$. Then $\mathfrak{g}_\mathrm{R}$ is the direct sum of $\mathfrak{u}$ and $\mathfrak{b}_0$. Define a real symmetric bilinear form on $\mathfrak{g}_\mathrm{R}$ by

$$\langle X,Y\rangle := \mathrm{Im}\,(\kappa(X,Y)) \qquad (X,Y \in \mathfrak{g}). \tag{K6}$$

Then this form is nondegenerate and $\mathrm{ad}_{\mathfrak{g}_\mathrm{R}}$-invariant, and the form restricted to $\mathfrak{u}$ or $\mathfrak{b}_0$ vanishes (see Exercise 22). We conclude that $(\mathfrak{g}_\mathrm{R},\mathfrak{u},\mathfrak{b}_0)$ is a Manin triple.

Coboundary Lie bialgebras Read in C&P Definition 2.1.1 of a *coboundary Lie bialgebra*. Read also Proposition 2.1.2, which characterizes those $r \in \mathfrak{g}\otimes\mathfrak{g}$ for which $\delta\colon X \mapsto X.r\colon \mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}$ gives rise to a Lie bialgebra structure. Note that the easy part of the proof is to show that $\delta\colon X \mapsto X.r$ is anti-symmetric iff $r_{12}+r_{21}$ is an $\mathrm{ad}_\mathfrak{g}$-invariant element of $\mathfrak{g}\otimes\mathfrak{g}$. If these two equivalent conditions hold then $\delta(X) = X.r = \frac{1}{2}X.(r_{12}-r_{21})$, so we may assume as well that $\delta(X) = X.r$ with r anti-symmetric. Note that an alternative way to the rest of the proof of the Proposition is by considering a simply connected Lie group G with Lie algebra $\mathfrak{g}$, and by taking $w^R(g) := \mathrm{Ad}_g(r) - r$ and by next defining the bivector field w on G as on the top of p.11 of these notes. Then use Proposition K6 and Theorem K11.

Read the definitions of a *triangular* Lie bialgebra and a *quasitriangular* Lie bialgebra in C&P, §2.1B, p.54. These are parallel to the definitions for the case of Poisson-Lie groups on top of p.17 of these notes.

If we say that $\mathfrak{g}$ is a Lie bialgebra then we imply that $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lie algebras. A Lie bialgebra $\mathfrak{g}$ is compeletely described by the pair $(\mathfrak{g},\mathfrak{g}^*)$ of Lie algebras. However, not every pair $(\mathfrak{g},\mathfrak{g}^*)$ of Lie algebras yields a Lie bialgebra. Note that, by the characterization of Lie bialgebras in terms of Manin triples, we see that, if $\mathfrak{g}$ (i.e. the pair of Lie algebras $(\mathfrak{g},\mathfrak{g}^*)$) is a Lie bialgebra, then so is $\mathfrak{g}^*$ (i.e. the pair of Lie algebras $(\mathfrak{g}^*,\mathfrak{g})$).

See C&P, Definition 1.3.1(b) for the definition of a *Lie bialgebra homomorphism* $\phi\colon \mathfrak{g} \to \mathfrak{h}$. So then $\mathfrak{g}$ and $\mathfrak{h}$ are Lie bialgebras and $\phi\colon \mathfrak{g} \to \mathfrak{h}$ and $\phi^*\colon \mathfrak{h}^* \to \mathfrak{g}^*$ are Lie algebra homomorphisms. Here ϕ^* is the dual of ϕ, i.e., $\langle X, \phi^*(\eta)\rangle = \langle \phi(X), \eta\rangle$ ($X \in \mathfrak{g}$, $\eta \in \mathfrak{h}^*$).

If $\mathfrak{g}$ is a Lie algebra then we mean by $\mathfrak{g}_\mathrm{op}$ the opposite Lie algebra, i.e., $[X,Y]_{\mathfrak{g}_\mathrm{op}} := -[X,Y]_\mathfrak{g}$. If $\mathfrak{g}$ is a Lie bialgebra, and thus a pair $(\mathfrak{g},\mathfrak{g}^*)$ of Lie algebras, then $\mathfrak{g}^\mathrm{op}$ is the Lie bialgebra given by the pair of Lie algebras $(\mathfrak{g},(\mathfrak{g}^*)_\mathrm{op})$, and $\mathfrak{g}_\mathrm{op}$ is the Lie bialgebra given by the pair of Lie algebras $(\mathfrak{g}_\mathrm{op},\mathfrak{g}^*)$. See also C&P, p.25, Remark [3].

Definition-Theorem K17 (classical double) (see C&P, Proposition 1.4.2, Proposition 2.1.11) Let $\mathfrak{g}$ be a finite-dimensional Lie bialgebra. So we know that $\mathfrak{g}^*$ is a Lie algebra and that $\mathfrak{g}\oplus\mathfrak{g}^*$ is a Lie algebra having $\mathfrak{g}$ and $\mathfrak{g}^*$ as Lie subalgebras and with ad-invariant nondegenerate form obtained from the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. Write $\mathcal{D}(\mathfrak{g})=\mathcal{D}:=\mathfrak{g}\oplus\mathfrak{g}^*$ for this Lie algebra. Identify $\mathcal{D}^*$ as a linear space with $\mathfrak{g}^*\oplus\mathfrak{g}$. By means of the nondegenerate form on $\mathcal{D}$ we can identify $\mathcal{D}$ with $\mathcal{D}^*$. So the pairing of $x+\xi\in\mathfrak{g}\oplus\mathfrak{g}^*\simeq\mathcal{D}$ with $\eta+y\in\mathfrak{g}^*\oplus\mathfrak{g}\simeq\mathcal{D}^*$ is given by

$$\langle x+\xi,\eta+y\rangle=\langle x,\eta\rangle+\langle\xi,y\rangle.$$

identify $\mathcal{D}^*$ as a Lie algebra with $\mathfrak{g}^*\oplus\mathfrak{g}_{\rm op}$. Thus:

$$[\xi+X,\eta+Y]_{\mathcal{D}^*}:=[\xi,\eta]_{\mathfrak{g}^*}-[X,Y]_{\mathfrak{g}}.$$

Then $\mathcal{D}(\mathfrak{g})$ is a Lie bialgebra. It is called the *classical double* of the Lie bialgebra $\mathfrak{g}$. Moreover, $\mathcal{D}(\mathfrak{g})$ is a quasitriangular Lie bialgebra with $r\in\mathcal{D}(\mathfrak{g})\otimes\mathcal{D}(\mathfrak{g})$ given by

$$\langle r,(\xi+X)\otimes(\eta+Y)\rangle:=\langle\xi,Y\rangle\quad(X,Y\in\mathfrak{g},\ \xi,\eta\in\mathfrak{g}^*).$$

Proof Let r be as above. Define the linear map $\delta\colon\mathcal{D}\to\mathcal{D}\otimes\mathcal{D}$ by

$$\delta(Z+\zeta):=(Z+\zeta).r=(\mathrm{ad}_{Z+\zeta}\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{ad}_{Z+\zeta})(r)\quad(Z\in\mathfrak{g},\ \zeta\in\mathfrak{g}^*),$$

where ad is the adjoint representation for the Lie algebra $\mathcal{D}$. We will first show that

$$\langle\delta(Z+\zeta),(\xi\oplus X)\otimes(\eta\oplus Y)\rangle=\langle Z+\zeta,[\xi+X,\eta+Y]_{\mathcal{D}^*}\rangle.$$

This will imply that $\mathcal{D}$ is a coboundary Lie bialgebra. (Note that the antisymmetry of the Lie bracket on $\mathcal{D}^*$ implies by the above identity that $r_{12}+r_{21}$ is $\mathrm{ad}_{\mathcal{D}}$-invariant, and thus $\delta(Z+\zeta)=\frac12(Z+\zeta).(r_{12}-r_{21})$.) The identity for $\zeta=0$ is obtained by:

$$\begin{aligned}&\langle Z.r,(\xi+X)\otimes(\eta+Y)\rangle=\langle r,[\xi+X,Z]_{\mathcal{D}}\otimes(\eta+Y)\rangle+\langle r,(\xi+X)\otimes[\eta+Y,Z]_{\mathcal{D}}\rangle\\&=\langle[\xi,Z]_{\mathcal{D}},Y\rangle+\langle\xi,[\eta,Z]_{\mathcal{D}}\rangle+\langle\xi,[Y,Z]_{\mathcal{D}}\rangle=\langle Z,[\xi,\eta]_{\mathcal{D}}\rangle=\langle Z,[\xi,\eta]_{\mathfrak{g}^*}\rangle=\langle Z,[\xi,\eta]_{\mathcal{D}^*}\rangle.\end{aligned}$$

The identity for $z=0$ is obtained by:

$$\begin{aligned}&\langle\zeta.r,(\xi+X)\otimes(\eta+Y)\rangle=\langle r,[\xi+X,\zeta]_{\mathcal{D}}\otimes(\eta+Y)\rangle+\langle r,(\xi+X)\otimes[\eta+Y,\zeta]_{\mathcal{D}}\rangle\\&=\langle[\xi,\zeta]_{\mathcal{D}},Y\rangle+\langle[X,\zeta]_{\mathcal{D}},Y\rangle+\langle\xi,[Y,\zeta]_{\mathcal{D}}\rangle=-\langle\zeta,[X,Y]_{\mathcal{D}}\rangle=-\langle\zeta,[X,Y]_{\mathfrak{g}}\rangle=\langle\zeta,[X,Y]_{\mathcal{D}^*}\rangle.\end{aligned}$$

Finally we have to show that $[[r,r]]=0$. For this purpose choose a basis $\{X_r\}$ of $\mathfrak{g}$, let $\{X^s\}$ be the dual basis for $\mathfrak{g}^*$ and observe that $r=\sum_t X_t\otimes\xi^t$ (see Exercise 23). Now follow the proof of C&P, Proposition 2.1.11. □

Definition K18 Let $\mathfrak{g}$ be a finite dimensional Lie bialgebra and let $\mathfrak{k}$ be a linear subspace of $\mathfrak{g}$. We call $\mathfrak{k}$ a *Lie bialgebra ideal* of $\mathfrak{g}$ if $\mathfrak{k}$ is both an *ideal* of $\mathfrak{g}$ (i.e., $[\mathfrak{k},\mathfrak{g}]\subset\mathfrak{k}$) and a *co-ideal* of $\mathfrak{g}$ (i.e., $\delta(\mathfrak{k})\subset\mathfrak{g}\otimes\mathfrak{k}+\mathfrak{k}\otimes\mathfrak{g}$). (See C&P, p.25, Remark [5].) We call $\mathfrak{k}$ a *Lie subbialgebra* of $\mathfrak{g}$ if $\mathfrak{k}$ is both a *Lie subalgebra* of $\mathfrak{g}$ (i.e., $[\mathfrak{k},\mathfrak{k}]\subset\mathfrak{k}$) and a *Lie subcoalgebra* of $\mathfrak{g}$ (i.e., $\delta(\mathfrak{k})\subset\mathfrak{k}\otimes\mathfrak{k}$). Next consider the statements of Exercise 24.

Example K19 (classical double) Consider the situation of Example K15. We know from Definition-Theorem K17 that $\mathcal{D} = \mathcal{D}(\mathfrak{q}_+) \simeq \mathfrak{q} := \mathfrak{g} \oplus \mathfrak{h}$ is a Lie bialgebra. $\mathcal{D}^* \simeq \mathfrak{g} \oplus \mathfrak{h}$ as linear spaces, but $\mathcal{D}^* \simeq \mathfrak{q}_+ \oplus (\mathfrak{q}_-)_{\mathrm{op}}$ as a Lie algebra. Let $X = X_+ + X_0 + X_-$ and $X' = X'_- + X'_0 + X'_-$ ($X_\pm, X'_\pm \in \mathfrak{n}_\pm$, $X_0, X'_0 \in \mathfrak{h}$) be arbitrary elements of $\mathfrak{g}$. Let $H, H' \in \mathfrak{h}$. Then

$$\begin{aligned}
&[(X,H),(X',H')]_{\mathcal{D}} = ([X,X']_{\mathfrak{g}},0),\\
&[(X,H),(X',H')]_{\mathcal{D}^*} = [(X_+ + \tfrac12 X_0 + \tfrac12 H, \tfrac12 X_0 + \tfrac12 H),(X'_+ + \tfrac12 X'_0 + \tfrac12 H', \tfrac12 X'_0 + \tfrac12 H')]_{\mathfrak{q}_+}\\
&\quad - [(X_- + \tfrac12 X_0 - \tfrac12 H, -\tfrac12 X_0 + \tfrac12 H),(X'_- + \tfrac12 X'_0 - \tfrac12 H', -\tfrac12 X'_0 + \tfrac12 H')]_{\mathfrak{q}_-}\\
&\quad = ([X_+ + \tfrac12 X_0 + \tfrac12 H, X'_+ + \tfrac12 X'_0 + \tfrac12 H']_{\mathfrak{g}},0) - ([X_- + \tfrac12 X_0 - \tfrac12 H, X'_- + \tfrac12 X'_0 - \tfrac12 H']_{\mathfrak{g}},0),\\
&\langle (X,H),(X',H')\rangle = \kappa(X,X') - \kappa(H,H') \quad \text{(pairing between } (X,H)\in\mathcal{D} \text{ and } (X',H')\in\mathcal{D}^*\text{)},\\
&\langle r,(X,H)\otimes(X',H')\rangle = \langle (X_- + \tfrac12 X_0 - \tfrac12 H, -\tfrac12 X_0 + \tfrac12 H),(X'_+ + \tfrac12 X'_0 + \tfrac12 H', \tfrac12 X'_0 + \tfrac12 H')\rangle\\
&\quad = \kappa(X_- + \tfrac12 X_0 - \tfrac12 H, X'_+ + \tfrac12 X'_0 + \tfrac12 H') - \kappa(-\tfrac12 X_0 + \tfrac12 H, \tfrac12 X'_0 + \tfrac12 H')\\
&\quad = \kappa(X_-, X'_+) + \tfrac12 \kappa(X_0 - H, X'_0 + H').
\end{aligned}$$

Clearly the subspace $\mathfrak{k} := \{(0,H) \mid H \in \mathfrak{h}\}$ is a Lie algebra ideal of $\mathcal{D}$. We will show that it is also a coideal, hence a bialgebra ideal. Equivalently, we show that the subspace $\mathfrak{k}^\perp \simeq \{(X,0) \mid X \in \mathfrak{g}\}$ is a Lie subalgebra of $\mathcal{D}^*$. Indeed, $\mathcal{D}^* \simeq \mathfrak{g} \oplus \mathfrak{h}$ as linear spaces, but $\mathcal{D}^* \simeq \mathfrak{q}_+ \oplus (\mathfrak{q}_-)_{\mathrm{op}}$ as a Lie algebra. Then observe that $[\mathfrak{q}_+,\mathfrak{q}_+]$ and $[\mathfrak{q}_-,\mathfrak{q}_-]$ are both included in $\mathfrak{k}^\perp$.

Since $\mathcal{D}$ is a quasitriangular bialgebra, it follows from Exercise 24 that this induces the structure of a quasitriangular bialgebra on $\mathcal{D}/\mathfrak{k}$. As linear spaces we can identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by the Killing form and we can make the identifications:

$$X \simeq (X,H) \bmod \mathfrak{k}\colon \mathfrak{g} \simeq \mathcal{D}/\mathfrak{k}, \quad Y \simeq (Y,0)\colon \mathfrak{g}^* \simeq \mathfrak{k}^\perp.$$

Let $X = X_+ + X_0 + X_-$ and $X' = X'_- + X'_0 + X'_-$ ($X_\pm, X'_\pm \in \mathfrak{n}_\pm$, $X_0, X'_0 \in \mathfrak{h}$) be arbitrary elements of $\mathfrak{g}$. The induced quasitriangular bialgebra structure on $\mathfrak{g}$ is:

- the Lie algebra structure of $\mathfrak{g}$;
- the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ obtained from the Kiling form;
- the Lie bracket on $\mathfrak{g}^*$ given by

$$[X,X']_{\mathfrak{g}^*} = [X_+ + \tfrac12 X_0, X'_+ + \tfrac12 X'_0]_{\mathfrak{g}} - [X_- + \tfrac12 X_0, X'_- + \tfrac12 X'_0]_{\mathfrak{g}};$$

- the element $r \in \mathfrak{g}\otimes\mathfrak{g}$ given by

$$\langle r, X\otimes X'\rangle = \kappa(X_-, X'_+) + \tfrac12 \kappa(X_0, X'_0).$$

The antisymmetric part of r is given by

$$\langle \tfrac12 (r_{12} - r_{21}), X\otimes X'\rangle = \tfrac12 \kappa(X_-, X'_+) - \tfrac12 \kappa(X_+, X'_-).$$

Then it follows from Theorem K13 that

$$\tfrac12 (r_{12} - r_{21}) = \sum_{\alpha\in\Delta_+} \frac{\langle\alpha,\alpha\rangle}{2} X_\alpha \wedge X_{-\alpha}.$$

Exercises (submit 25)

Exercise 21 Let $\langle\,.\,,\,.\,\rangle$ be the bilinear symmetric form on $\mathfrak{q}$ defined by (K5). Show that this form is nondegenerate and $\mathrm{ad}_{\mathfrak{q}}$-invariant, and that the form restricted to $\mathfrak{q}_+$ or $\mathfrak{q}_-$ vanishes.

Exercise 22 Let $\langle\,.\,,\,.\,\rangle$ be the bilinear symmetric form on $\mathfrak{g}_{\mathrm{R}}$ defined by (K6). Show that this form is nondegenerate and $\mathrm{ad}_{\mathfrak{g}_{\mathrm{R}}}$-invariant, and that the form restricted to $\mathfrak{u}$ or $\mathfrak{b}_0$ vanishes.

Exercise 23 With the assumptions of Definition-Theorem K17, choose a basis $\{X_r\}$ of $\mathfrak{g}$ and let $\{\xi^s\}$ be the dual basis for $\mathfrak{g}^*$. Show that $r = \sum_t X_t \otimes \xi^t$.

Exercise 24 If $\mathfrak{k}$ is a linear subspace of a finite dimensional vector space $\mathfrak{g}$, put $\mathfrak{k}^\perp := \{\xi \in \mathfrak{g}^* \mid \langle \xi, K\rangle = 0 \quad \forall K \in \mathfrak{k}\}$. Now suppose that $\mathfrak{g}$ is a finite dimensional Lie bialgebra and $\mathfrak{k}$ is a linear subspace of $\mathfrak{g}$. Prove the following statements.

(a) $\mathfrak{k}$ is a Lie bialgebra ideal of $\mathfrak{g}$ iff $\mathfrak{k}^\perp$ is a Lie subbialgebra of $\mathfrak{g}^*$.

(b) If $\mathfrak{k}$ is a Lie bialgebra ideal of $\mathfrak{g}$ then the quotient space $\mathfrak{g}/\mathfrak{k}$ inherits a Lie bialgebra structure from $\mathfrak{g}$ and we can identify $(\mathfrak{g}/\mathfrak{k})^* \simeq \mathfrak{k}^\perp$ as Lie bialgebras.

(c) If $\mathfrak{g}$ is a quasitriangular Lie bialgebra with respect to $r \in \mathfrak{h} \otimes \mathfrak{g}$, (i.e., such that $\delta(X) = X.r$ and $[[r, r]] = 0$) and if $\mathfrak{k}$ is a Lie bialgebra ideal of $\mathfrak{g}$, then the Lie bialgebra $\mathfrak{g}/\mathfrak{k}$ is quasitriangular with respect to $r \bmod (\mathfrak{g} \otimes \mathfrak{k} + \mathfrak{k} \otimes \mathfrak{g})$.

Exercise 25 Show that the example of Exercise 16 can be considered as a special case of the Lie bialgebra $\mathfrak{g}$ obtained in Example K19. Also show that the example of Exercise 17 can be considered as a special case of the Lie bialgebra $\mathfrak{u}$ obtained in Example K16.

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Normal real form Let $\mathfrak{g}$ be a complex semisimple Lie algebra. In Corollary K14 we gave a compact real form $\mathfrak{u}$ of $\mathfrak{g}$. Theorem K13 immediately implies another real form of $\mathfrak{g}$: the *normal real form* of $\mathfrak{g}$ which is the real span of $\mathfrak{h}_0$ together with the elements X_α ($\alpha \in \Delta$). We will notate this normal real form by $\mathfrak{g}_0$. (So $\mathfrak{g}_0$ will no longer be used as a notation for $\mathfrak{h}$.) The Killing form restricted to $\mathfrak{g}_0$ is real and nondegenerate, but it is indefinite: κ is positive definite on the real span of $\mathfrak{h}_0$ together with the elements $X_\alpha + X_{-\alpha}$ ($\alpha \in \Delta_+$) and κ is negative definite on the real span of the elements $X_\alpha - X_{-\alpha}$ ($\alpha \in \Delta_+$).

Two remarks on Lie bialgebras Let $\mathfrak{g}$ be a complex Lie bialgebra, so, in particular, both $\mathfrak{g}$ and its complex linear dual $\mathfrak{g}^*$ have the structure of a complex Lie algebra. Let the real Lie algebra $\mathfrak{g}_0$ be a real form of the Lie algebra $\mathfrak{g}$, consider the real linear dual $\mathfrak{g}_0^*$ of $\mathfrak{g}_0$ as a real linear subspace of $\mathfrak{g}^*$ and suppose that $\mathfrak{g}_0^*$ is a real Lie subalgebra of $\mathfrak{g}^*$, so the real Lie algebra $\mathfrak{g}_0^*$ is a real form of the Lie algebra $\mathfrak{g}^*$. Then it follows immediately that $(\mathfrak{g}_0, \mathfrak{g}_0^*)$ is a real Lie bialgebra. We call it a *real form* of the complex Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$.

In particular, for the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ given by Example K19 ($\mathfrak{g}$ a complex semisimple Lie algebra) we obtain as a real form the Lie bialgebra $(\mathfrak{g}_0, \mathfrak{g}_0^*)$ ($\mathfrak{g}_0$ normal real form)

If $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra over $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ then, for $c \in \mathbb{F}\backslash\{0\}$, we obtain again a Lie bialgebra if we introduce a new Lie bracket $[\xi, \eta]' := c[\xi, \eta]$ on $\mathfrak{g}^*$. In particular, if $\mathbb{F} := \mathbb{C}$ and $c \notin \mathbb{R}$ then $(\mathfrak{g}, \mathfrak{g}^*)$ with modified Lie algebra structure on $\mathfrak{g}^*$ may allow an essentially different real form than $(\mathfrak{g}, \mathfrak{g}^*)$ with the original Lie algebra structure on $\mathfrak{g}^*$.

Example K20 (compact real form of the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ of Example K19)
Let for the moment $\mathfrak{g}$ be any complex Lie algebra, write $\mathfrak{g}_\mathrm{R}$ for $\mathfrak{g}$ considered as a real Lie algebra, let the real Lie algebra $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$, and define the real linear map $\tau \colon \mathfrak{g}_\mathrm{R} \to \mathfrak{g}_\mathrm{R}$ by $\tau(X + iY) := X - iY$ ($X, Y \in \mathfrak{g}_0$). We call τ the conjugation on $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. It follows immediately that τ is a real Lie algebra automorphism of $\mathfrak{g}_\mathrm{R}$ satisfying $\tau(zX) = \overline{z}\,\tau(X)$ ($z \in \mathbb{C}$, $X \in \mathfrak{g}$).

Let now $\mathfrak{g}$ be a complex semisimple Lie algebra. We will make contact between Examples K16 and K19. Let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}$ as in Corollary K14 and let τ be the conjugation on $\mathfrak{g}$ with respect to $\mathfrak{u}$. Then

$$\kappa(X, \tau(Y)) = \overline{\kappa(X, Y)} \qquad (X \in \mathfrak{u},\ Y \in \mathfrak{g}).$$

(For the proof write $Y = Y_1 + iY_2$ with $Y_1, Y_2 \in \mathfrak{u}$, and use that κ is a real form on $\mathfrak{u}$.) Hence

$$\kappa(X, \tfrac{1}{2i}(Y - \tau(Y))) = \operatorname{Im} \kappa(X, Y) \qquad (X \in \mathfrak{u},\ Y \in \mathfrak{g}).$$

In Example K16 we have seen that the pair of Lie algebras $(\mathfrak{u}, \mathfrak{b}_0)$ (both considered as real Lie subalgebras of $\mathfrak{g}$) is a real Lie bialgebra if we consider $\mathfrak{u}$ and $\mathfrak{b}_0$ as linear duals of each other via the pairing (K6). Thus we can also write this pairing as

$$\langle X, Y \rangle = \kappa(X, \tfrac{1}{2i}(Y - \tau(Y))) \qquad (X \in \mathfrak{u},\ Y \in \mathfrak{b}_0).$$

By the definitions of $\mathfrak{u}$ (Corollary K14) and $\mathfrak{b}_0$ (Example K16) we see that the map $Y \mapsto \frac{1}{2i}(Y - \tau(Y)) \colon \mathfrak{b}_0 \to \mathfrak{u}$ is a real linear bijection. Now identify $\mathfrak{u}$ with its linear dual $\mathfrak{u}^*$

via the (negative definite) Killing form on $\mathfrak{u}$ and make $\mathfrak{u}^*$ into a Lie algebra such that $Y \mapsto \frac{1}{2i}(Y-\tau(Y))\colon \mathfrak{b}_0 \mapsto \mathfrak{u}^*$ is a Lie algebra isomorphism, i.e.,

$$[\tfrac{1}{2i}(Y-\tau Y), \tfrac{1}{2i}(Y'-\tau Y')]_{\mathfrak{u}^*} := \tfrac{1}{2i}([Y,Y']_{\mathfrak{g}} - \tau[Y,Y']_{\mathfrak{g}}) \qquad (Y,Y' \in \mathfrak{b}_0).$$

Then the pair $(\mathfrak{u},\mathfrak{u}^*)$ will be a real Lie bialgebra isomorphic to $(\mathfrak{u},\mathfrak{b}_0)$.

Now we will show that

$$[X,X']_{\mathfrak{u}^*} = [X,X']'_{\mathfrak{g}^*} := 2i[X,X']_{\mathfrak{g}^*} \qquad (X,X' \in \mathfrak{u}), \tag{K7}$$

where $[\,.\,,\,.\,]_{\mathfrak{g}^*}$ is the Lie bracket on $\mathfrak{g}^*$ considered in Example K19. Note that the pair $(\mathfrak{g},\mathfrak{g}^*)$ is still a complex Lie bialgebra with respect to the Lie bracket $[\,.\,,\,.\,]'_{\mathfrak{g}^*}$ on $\mathfrak{g}^*$. So we obtain that the real Lie bialgebra $(\mathfrak{u},\mathfrak{u}^*)$ is a real form of the thus modified complex Lie bialgebra $(\mathfrak{g},\mathfrak{g}^*)$.

For the proof of (K7) write a general element Y of $\mathfrak{b}_0$ as $Y = Y_0 + Y_+$ ($Y_0 \in \mathfrak{h}_0$, $Y_+ \in \mathfrak{n}_+$). Then $\frac{1}{2i}(Y-\tau(Y)) = \frac{1}{i}Y_0 + \frac{1}{2i}Y_+ - \frac{1}{2i}\tau(Y_+)$, where the first term is in $\mathfrak{h}$, the second term in $\mathfrak{n}_+$ and the third term in $\mathfrak{n}_-$. Hence, by Example K19, we have for $Y,Y' \in \mathfrak{b}_0$ that

$$\begin{aligned}[\tfrac{1}{2i}(Y-\tau Y), \tfrac{1}{2i}(Y'-\tau Y')]_{\mathfrak{g}^*} &= -\tfrac14[Y,Y']_{\mathfrak{g}} + \tfrac14[\tau(Y),\tau(Y')]_{\mathfrak{g}} \\ &= -\tfrac14[Y,Y']_{\mathfrak{g}} + \tfrac14\tau[Y,Y']_{\mathfrak{g}} = \tfrac{1}{2i}\,[\tfrac{1}{2i}(Y-\tau Y), \tfrac{1}{2i}(Y'-\tau Y')]_{\mathfrak{u}^*}.\end{aligned}$$

Poisson-Hopf algebras See the definitions of a *Poisson algebra* and of a *Poisson-Hopf algebra* in C&P, §6.2A. Similarly, by omitting the antipode axiom, we can define a *Poisson bialgebra*. See the statements of Exercise 30. We also have in a Poisson-Hopf algebra $\mathcal{A}$ that

$$S(\{a,b\}) = -\{S(a),S(b)\} \qquad (a,b \in \mathcal{A}).$$

This will later be given as an exercise, with some hints.

Poisson-Hopf algebras in connection with Hopf algebras deforming a commutative algebra Let $\mathbb{C}[q,q^{-1}]$ be the commutative $\mathbb{C}$-algebra of Laurent polynomials in q. Let $\mathcal{A}_q$ be a $\mathbb{C}[q,q^{-1}]$-module. Put

$$\mathcal{A} := \mathcal{A}_q/((q-1)\mathcal{A}_q).$$

Then $\mathcal{A}$ is a $\mathbb{C}$-module. Also write

$$\overline{a} := a \bmod (q-1) \qquad (a \in \mathcal{A}_q),$$

where $\bmod\,(q-1)$ means $\bmod\,((q-1)\mathcal{A}_q)$. We call $\mathcal{A}_q$ a *deformation* (or rather a $\mathbb{C}[q,q^{-1}]$-deformation) of $\mathcal{A}$ if $\mathcal{A}_q$ and $\mathcal{A}[q,q^{-1}]$ are isomorphic as $\mathbb{C}[q,q^{-1}]$-modules.

If $\mathcal{A}_q$ is a deformation of $\mathcal{A}$ then further structure (over $\mathbb{C}[q,q^{-1}]$) of $\mathcal{A}_q$ induces similar structure (over $\mathbb{C}$) of $\mathcal{A}$:

- If $\mathcal{A}_q$ is an associative algebra with unit 1 then $\mathcal{A}$ is an associative algebra with unit $\overline{1}$, where
$$\overline{a}\,\overline{b} := ab \bmod (q-1), \qquad \overline{1} := 1 \bmod (q-1).$$
- If $\mathcal{A}_q$ is a coassociative coalgebra with counit ε then $\mathcal{A}$ is an coassociative coalgebra with counit $\overline{\varepsilon}$, where
$$\Delta(\overline{a}) := \Delta(a) \bmod (q-1), \qquad \overline{\varepsilon}(\overline{a}) := \varepsilon(a) \bmod (q-1).$$
- If $\mathcal{A}_q$ is a bialgebra then $\mathcal{A}$ is a bialgebra.
- If $\mathcal{A}_q$ is a Hopf algebra with antipode S then $\mathcal{A}$ is a Hopf algebra with antipode $\overline{S}$, where
$$\overline{S}(\overline{a}) := S(a) \bmod (q-1).$$

Now suppose that $\mathcal{A}_q$ is an associative algebra with 1 and that $\mathcal{A}_q$ is a deformation of $\mathcal{A}$ and that $\mathcal{A}$ is a commutative algebra, i.e., $ab = ba \bmod (q-1)$ for all $a, b \in \mathcal{A}_q$. Then $(q-1)^{-1}(ab - ba)$ is a well-defined element of $\mathcal{A}_q$ for all $a, b \in \mathcal{A}_q$. Define:

$$\{\overline{a}, \overline{b}\} := (q-1)^{-1}(ab - ba) \bmod (q-1) \qquad (a, b \in \mathcal{A}_q).$$

Then $\{\,.\,,\,.\,\}$ is a well-defined $\mathbb{C}$-bilinear antisymmetric map of $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$. Moreover, this bracket satisfies the Leibniz rule and the Jacobi identity. Hence we have made $\mathcal{A}$ into a Poisson algebra. In a similar way we can make $\mathcal{A} \otimes \mathcal{A}$ into a Poisson algebra with

$$\{\overline{a_1} \otimes \overline{a_2}, \overline{b_1} \otimes \overline{b_2}\} := (q-1)^{-1}[a_1 \otimes a_2, b_1 \otimes b_2] \bmod (q-1) = \{\overline{a_1}, \overline{b_1}\} \otimes \overline{a_2}\,\overline{b_2} + \overline{a_1}\,\overline{b_1} \otimes \{\overline{a_2}, \overline{b_2}\}.$$

If, with the above assumptions, $\mathcal{A}_q$ is moreover a bialgebra (or Hopf algebra) then $\mathcal{A}$ becomes a Poisson bialgebra (or Poisson-Hopf algebra).

Exercises (submit 26 and 27)

Exercise 26 Use the notation of Example K19. Show that

$$\langle \tfrac{1}{2}(r_{12} + r_{21}), X \otimes X' \rangle = \tfrac{1}{2}\kappa(X, X').$$

Conclude that $\frac{1}{2}(r_{12} + r_{21}) = \frac{1}{2}t$, where t is the Casimir element defined by (K2) (with $\langle X, Y \rangle := \kappa(X, Y)$ in (K2)). Show also that

$$t = \tfrac{1}{2} \sum_{\alpha \in \Delta_+} \langle \alpha, \alpha \rangle \, (X_\alpha \otimes X_{-\alpha} + X_{-\alpha} \otimes X_\alpha) + \sum_{i=1}^{l} H_i \otimes H_i,$$

where the H_i form a basis of $\mathfrak{h}_0$ satisfying $\kappa(H_i, H_j) = \delta_{ij}$.

Exercise 27 Consider the real Lie bialgebra $(\mathfrak{u}, \mathfrak{b}_0)$ of Example K16. Show that this is a quasitriangular Lie bialgebra with r given by

$$r = \tfrac{1}{2} \sum_{\alpha \in \Delta_+} \langle \alpha, \alpha \rangle (X_\alpha - X_{-\alpha}) \wedge (iX_\alpha + iX_{-\alpha}) + it,$$

where t is the Casimir element given in Exercise 26.
Hint Use Exercise 26 and Example K20.

Exercise 28 Specify the structure of a general semisimple Lie algebra (as given on pp. 24, 25) for the case $\mathfrak{g} := sl(n, \mathbb{C})$:

(a) $\kappa(X, Y) = 2n \operatorname{tr}(XY) \quad (X, Y \in sl(n, \mathbb{C}))$.

(b) $\mathfrak{h} = \{\sum_{i=1}^n h_i E_{ii} \mid h_1, \ldots, h_n \in \mathbb{C},\ \sum_{i=1}^n h_i = 0\}$, where E_{ij} is the $(n \times n)$ matrix with 1 at place ij and with 0 elsewhere.

(c) $\mathfrak{h}^* = \{(\lambda_1, \ldots, \lambda_n) := \sum_{j=1}^n \lambda_j \varepsilon_j \mid \lambda_1, \ldots, \lambda_n \in \mathbb{C},\ \sum_{j=1}^n \lambda_j = 0\}$, where $\varepsilon_j \colon \sum_{i=1}^n h_i E_{ii} \mapsto h_j$.

(d) $\Delta = \{\varepsilon_i - \varepsilon_j \mid i, j = 1, \ldots, n,\ i \neq j\}$.

(e) $\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C}\, E_{ij}$.

(f) $\kappa(\sum_{i=1}^{n} h_i E_{ii}, \sum_{j=1}^{n} h'_j E_{jj}) = 2n \sum_{k=1}^{n} h_k h'_k$.

(g) $T_{(\lambda_1,\ldots,\lambda_n)} = \frac{1}{2n} \sum_{i=1}^{n} \lambda_i E_{ii}$.

(h) $\langle \lambda, \mu \rangle = \frac{1}{2n} \sum_{i=1}^{n} \lambda_i \mu_i$.

(i) $H_{\varepsilon_i - \varepsilon_j} = E_{ii} - E_{jj}$.

(j) $s_{\varepsilon_i - \varepsilon_j}$ exchanges the ith and jth coordinate in $(\lambda_1, \ldots, \lambda_n)$.

(k) $\Delta_+ = \{\varepsilon_i - \varepsilon_j\}_{i<j}$.

(l) $X_{\varepsilon_i - \varepsilon_j} = E_{ij}$.

(m) Let $i \neq j$, $k \neq l$. Then

$$[E_{ij}, E_{kl}] = \begin{cases} E_{il} & \text{if } j = k,\ i \neq l; \\ -E_{kj} & \text{if } i = l,\ j \neq k; \\ H_{\varepsilon_i - \varepsilon_j} & \text{if } i = l,\ j = k; \\ 0 & \text{otherwise.} \end{cases}$$

(n) $\kappa(E_{ij}, E_{ji}) = 2n$.

(o) A compact real form is $\mathfrak{u} = su(n)$ consisting of the skew-hermitian matrices of trace 0. A normal real form is $\mathfrak{g}_0 = sl(n, \mathbb{R})$.

Exercise 29 Specify Example K19 for the case that $\mathfrak{g} = sl(n, \mathbb{C})$. Thus show that $X = X_+ + X_0 + X_-$ is the decomposition of $X \in sl(n, \mathbb{C})$ into upper triangular matrices X_+, lower triangular matrices X_- and diagonal matrices X_0. Also show that

$$r = \tfrac{1}{2} t + \tfrac{1}{2n} \sum_{i<j} E_{ij} \wedge E_{ji}.$$

Exercise 30 Show that in a Poisson algebra $\mathcal{A}$ we have $\{1, a\} = 0$ for all $a \in \mathcal{A}$. Show also that in a Poisson-Hopf algebra $\mathcal{A}$ we have $\varepsilon(\{a, b\}) = 0$ for all $a, b \in \mathcal{A}$.

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Comment to pp.31,32 Suppose that $\mathcal{A}_q$ is a $\mathbb{C}[q,q^{-1}]$-algebra (or bialgebra or Hopf algebra) and that the algebra structure induced on $\mathcal{A} := \mathcal{A}_q/(q-1)\mathcal{A}_q$ is commutative. Then $\mathcal{A}$ becomes a Poisson algebra (or Poisson bialgebra or Poisson-Hopf algebra) as indicated at p.32. For this it is not necessary that $\mathcal{A}_q$ is a quantization of $\mathcal{A}$.

Bergman's diamond lemma (see G. M. Bergman, *The diamond lemma for ring theory*, Advances in Math. 29 (1978), 178–218).
The diamond lemma is a useful tool for finding explicit bases of algebras presented by generators and relations. We will formulate the lemma in the case of finitely many generators.

Let $X := \{x_1,\ldots,x_n\}$ be the set of *generators*. These generators are considered a prioiri as formal variables which do not commute with each other. Let $\langle X\rangle$ be the set of *monomials* $x_{i_1}x_{i_2}\ldots x_{i_l}$ in the generators. Here l is an arbitrary nonnegative integer and $i_1,\ldots,i_l \in \{1,2,\ldots,n\}$. We can multiply two monomials A and B by just putting the factors of B after the factors of A. This monomial is denoted by AB. The empty monomial ($l=0$) is denoted by 1. It acts as an identity for this multiplication. Thus $\langle X\rangle$ is a semigroup with 1: the free semigroup with 1 generated by X.

Let $\mathbf{k}$ be a commutative ring with 1. Typically this can be a field like $\mathbb{C}$ or an algebra of Laurent polynomials $\mathbb{C}[q,q^{-1}]$ or an algebra of formal power series $\mathbb{C}[[h]]$. Let $\mathbf{k}\langle X\rangle$ be the set of *polynomials* in the generators, i.e., the set of elements

$$\sum_{A\in\langle X\rangle} c_A\,A \quad \text{or more explicitly} \quad \sum_{l;\,i_1,\ldots,i_l} c_{i_1,\ldots,i_l}\,x_{i_1}x_{i_2}\ldots x_{i_l},$$

where the coefficients $c_A = c_{i_1,\ldots,i_l}$ are in the ring $\mathbf{k}$ and only finitely many coefficients are nonzero. Note that $\mathbf{k}\langle X\rangle$ is an associative algebra over $\mathbf{k}$ with 1: the free $\mathbf{k}$-algebra with 1 generated by X.

Next we take a set $S = \{(W_1,f_1),\ldots,(W_s,f_s)\}$, where each W_i is a monomial and each f_i is a polynomial. This set gives rise to *relations* $W_1 = f_1,\ldots,W_s = f_s$ and to the two-sided ideal J in $\mathbf{k}\langle X\rangle$ which is generated by the elements $W_1 - f_1,\ldots,W_s - f_s$. Let $\mathcal{A}$ be the quotient algebra: $\mathcal{A} := \mathbf{k}\langle X\rangle/J$, again an associative $\mathbf{k}$-algebra with 1. We are looking for a $\mathbf{k}$-basis of $\mathcal{A}$ consisting of suitable monomials.

We suppose that "$\le$" is a partial order on $\langle X\rangle$ such that:

(i) It satisfies the *descending chain condition*, i.e., there are no infinite descending chains $A_1 > A_2 > \ldots$;

(ii) It is a *semigroup partial ordering*, i.e., if B and B' are monomials with $B < B'$ then $ABC < AB'C$ for all monomials A, C.

A typical choice for such a partial order is to fix a linear order on X, say $x_1 < x_2 < \ldots < x_n$, to take the resulting lexicographic order on monomials of equal length, and to take $A < B$ if A has smaller length than B. We call this order on $\langle X\rangle$ the *standard order* corresponding to the given order on X. Assume also that

(iii) The partial order is *compatible with* S, i.e., for all $i \in \{1,\ldots,s\}$ the polynomial f_i is a linear combination of monomials less than W_i.

We say that a polynomial b is a *reduction* of a polynomial a if a contains a monomial of the form AW_iB wih nonzero coefficient and if b is obtained from a by replacing AW_iB by Af_iB. Clearly, $b = a \bmod J$ and, because of the assumptions (ii) and (iii), the monomials in Af_iB are $< AW_iB$. We say that a polynomial a is *irreducible* if no reduction of a is possible. Because of assumption (i), each polynomial is brought into irreducible form after finitely many steps. It follows that the algebra $\mathcal{A}$ is spanned as a k-module by the irreducible monomials (mod$J$). The diamond lemma will give sufficient conditions in order that the irreducible monomials (mod$J$) are linearly independent. These sufficient conditions concern the resolvability of two types of ambiguity:

(a) An *overlap ambiguity* is a monomial of the form ABC (with $A, B, C \in \langle X\rangle \backslash \{1\}$) such that $AB = W_i$ and $BC = W_j$ for certain $i, j \in \{1, \ldots, s\}$. Thus the polynomials f_iC and Af_j are reductions of ABC. The ambiguity is called *resolvable* if there are successive reductions of f_iC and successive reductions of Af_j which end at the same polynomial.

(b) An *inclusion ambiguity* is a monomial of the form ABC (with $A, B, C \in \langle X\rangle$) such that $W_i = B$ and $W_j = ABC$ for certain $i, j \in \{1, \ldots, s\}$ with $i \neq j$. Thus the polynomials Af_iC and f_j are reductions of ABC. The ambiguity is called *resolvable* if there are successive reductions of Af_iC and successive reductions of f_j which end at the same polynomial.

Theorem K21 (*diamond lemma*) Let $X, \mathbf{k}, S, J, \mathcal{A}$ be as above and let a partial order "$\leq$" on $\langle X\rangle$ satisfy conditions (i), (ii), (iii). If all overlap ambiguities and inclusion ambiguities are resolvable then the irreducible monomials (mod$J$) form a $\mathbf{k}$-basis of $\mathcal{A}$.

For the proof we refer to Bergman (*l.c*). In an application which is met quite often, one has the standard order on $\langle X\rangle$ corresponding to $x_1 < x_2 < \ldots < x_n$, and one has relations of the form

$$x_j x_i = \sum_{(k,l)<(j,i)} c_{j,i}^{k,l} x_k x_l + \sum_k b_{j,i}^k x_k + a_{j,i} \quad (i < j),$$

where the coefficients $c_{j,i}^{k,l}$, $b_{j,i}^k$, $a_{j,i}$ are in $\mathbf{k}$. Then there are no inclusion ambiguities. In each individual case one has to check if the overlap ambiguities are resolvable.

As a standard application of the diamond lemma one can thus prove the PBW theorem, see Exercise 32.

Example K22 Let $\mathbf{k} := \mathbb{C}[q, q^{-1}]$, $X := \{\alpha, \beta, \gamma, \delta\}$ and take the relations

$$\alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \beta\delta = q\delta\beta, \quad \gamma\delta = q\delta\gamma, \quad \beta\gamma = \gamma\beta, \quad \alpha\delta - \delta\alpha = (q - q^{-1})\beta\gamma.$$

This yields the bialgebra $\mathcal{A}_q(Mat(2, \mathbb{C}))$. If we add the relation $\alpha\delta - q\beta\gamma = 1$ then we obtain the Hopf algebra $\mathcal{A}_q(SL(2, \mathbb{C}))$ (quantized function algebra). In both cases comultiplication is given by

$$\Delta \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where the right-hand side has to be read in the sense of matrix multiplication, e.g., $\Delta(\alpha) = \alpha \otimes \alpha + \beta \otimes \gamma$.

Take for instance the standard order on $\langle X\rangle$ obtained from $\beta < \gamma < \alpha < \delta$. Accordingly the relations have to be rewritten slightly, with the highest term on the left-hand side, for instance $\delta\gamma = q^{-1}\gamma\delta$, $\delta\alpha = \alpha\delta - (q-q^{-1})\beta\gamma$. Straightforward application of the diamond lemma (see Exercise 33) now yields that $\mathcal{A}_q(Mat(2,\mathbb{C}))$ has a $\mathbb{C}[q,q^{-1}]$-basis of elements $\beta^i\gamma^j\alpha^k\beta^l$ (i,j,k,l nonnegative integers), and that $\mathcal{A}_q(SL(2,\mathbb{C}))$ has a $\mathbb{C}[q,q^{-1}]$-basis of elements $\beta^i\gamma^j$, $\beta^i\gamma^j\alpha^k$, $\beta^i\gamma^j\delta^k$ (i,j nonnegative integers, k positive integer).

It follows that the bialgebra $\mathcal{A}_q(Mat(2,\mathbb{C}))$ is a $\mathbb{C}[q,q^{-1}]$-deformation of $\mathcal{A}(Mat(2,\mathbb{C}))$ (the commutative bialgebra of polynomial functions on the semigroup $Mat(2,\mathbb{C})$) and that $\mathcal{A}_q(SL(2,\mathbb{C}))$ s a $\mathbb{C}[q,q^{-1}]$-deformation of $\mathcal{A}(SL(2,\mathbb{C}))$ (the commutative Hopf algebra of polynomial functions on the group $SL(2,\mathbb{C})$),

Combination with the construction on top of p.32 immediately yields (see Exercise 34) the expressions for $\{\overline{\alpha},\overline{\beta}\}$, $\{\overline{\alpha},\overline{\gamma}\}$, etc., by which we get a structure of Poisson bialgebra for $\mathcal{A}(Mat(2,\mathbb{C}))$ (induced by $\mathcal{A}_q(Mat(2,\mathbb{C}))$) or a structure of Poisson-Hopf algebra for $\mathcal{A}(SL(2,\mathbb{C}))$ (induced by $\mathcal{A}_q(SL(2,\mathbb{C}))$). These are precisely the expressions given in Exercise 18.

Bialgebras defined by quadratic relations Let $\mathbf{k}$ be a commutative ring with 1. Let $\mathcal{A}$ be the $\mathbf{k}$-algebra generated by elements t_{ij} ($i,j=1,\ldots,n$) with relations

$$\sum_{k,l=1}^{n} R_{ij,kl}\, t_{kr}\, t_{ls} = \sum_{k,l=1}^{n} t_{jl}\, t_{ik}\, R_{kl,rs} \qquad (i,j,r,s=1,\ldots,n), \tag{K8}$$

where $(R_{ij,kl})$ is a $n^2\times n^2$ matrix over $\mathbf{k}$. Write $T := (t_{ij})_{i,j=1,\ldots,n}$. Then relations (K8) can be rewritten in a shorter and more symbolic notation as

$$RT_1T_2 = T_2T_1R. \tag{K9}$$

It can be shown (see Exercise 35) that the algebra $\mathcal{A}$ can be uniquely made into a bialgebra over $\mathbf{k}$ such that

$$\Delta(t_{ij}) = \sum_{k=1}^{n} t_{ik}\otimes t_{kj}, \quad \varepsilon(t_{ij}) = \delta_{ij} \qquad (i,j=1,\ldots,n). \tag{K10}$$

Now suppose that we work over the ring $\mathbb{C}[q,q^{-1}]$. Instead of R we write R_q (with matrix elements in $\mathbb{C}[q,q^{-1}]$) and instead of $\mathcal{A}$ we write $\mathcal{A}_q$ (a $\mathbb{C}[q,q^{-1}]$-bialgebra). Suppose that

$$R_q = I \bmod (q-1), \quad \text{i.e.,} \quad (R_q)_{ij,kl} = \delta_{ik}\delta_{jl} \bmod (q-1).$$

Let $\mathcal{A} := \mathcal{A}_q/(q-1)\mathcal{A}_q$. Then $\mathcal{A}$ is a commutative algebra with the t_{ij} as generators. Put

$$r := (q-1)^{-1}\,(R_q - I) \bmod (q-1), \quad \text{i.e.,} \quad r_{ij,kl} := (q-1)^{-1}\,((R_q)_{ij,kl} - \delta_{ik}\delta_{jl}) \bmod (q-1). \tag{K11}$$

The bialgebra structure of $\mathcal{A}_q$ induces a Poisson bialgebra structure on $\mathcal{A}$. For the Poisson bracket on the generators we derive from the identity

$$\begin{aligned}0 =& (q-1)^{-1}\,(R_qT_1T_2 - T_2T_1R_q) \bmod (q-1) = (q-1)^{-1}(T_1T_2 - T_2T_1) \bmod (q-1)\\ &+ rT_1T_2 - T_1T_2r\end{aligned}$$

that

$$\{T_1,T_2\} = T_1T_2r - rT_1T_2, \tag{K12}$$

which can be written in terms of matrix elements as

$$\{t_{ik}, t_{rs}\} = \sum_{k,l=1}^{n} t_{ik} t_{jl} r_{kl,rs} - \sum_{k,l=1}^{n} r_{ij,kl} t_{kr} t_{ls}. \tag{K13}$$

Note that (K12), (K13) precisely have the form (K5) at p.19, where we dealt with a linear coboundary Poisson-Lie group. Note also that there is no garuantee that $\mathcal{A}_q$ will be a quantization of $\mathcal{A}$. There is certainly no guarantee that $\mathcal{A} = Mat(n, \mathbb{C})$, i.e., that $\mathcal{A}$ is the algebra generated by the commuting variables t_{ij} without further relations.

It seems puzzling that, for a coboundary Poisson-Lie group G, r cannot be arbitrary but has to satisy the property that $[[r, r]]$ is Ad_G-invariant (see Proposition K6). On the other hand, in the derivation of (K12) we obtained r from a matrix R_q which is completely arbitrary, except that its zero order approximation at $q = 1$ equals I. However, note that the quadratic relations (K9) imply certain cubic relations:

$$\begin{aligned} R_{12} R_{13} R_{23}\, T_1 T_2 T_3 &= T_3 T_2 T_1\, R_{12} R_{13} R_{23}, \\ R_{23} R_{13} R_{12}\, T_1 T_2 T_3 &= T_3 T_2 T_1\, R_{23} R_{13} R_{12}. \end{aligned} \tag{K14}$$

If R is invertible then (K14) implies the relations

$$(R_{12} R_{13} R_{23})^{-1} R_{23} R_{13} R_{12}\, T_1 T_2 T_3 = T_1 T_2 T_3\, (R_{12} R_{13} R_{23})^{-1} R_{23} R_{13} R_{12},$$

which become trivial if R satisfies the *quantum Yang-Baxter equation* (QYBE)

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.$$

However, an arbitrary R will not satisfy the QYBE, and then relations (K14) may be nontrivial.

Suppose that $R = R_q = I \bmod (q - 1)$ and that r is given by (K11). Then (see Exercise 36) we have

$$(q-1)^{-2} \big((R_q)_{12} (R_q)_{13} (R_q)_{23} - (R_q)_{23} (R_q)_{13} (R_q)_{12} \big) = [[r, r]] \bmod (q - 1). \tag{K15}$$

Hence, by subtracting the two equations in (K14) from each other, dividing both sides of the resulting equation by $(q - 1)^2$ and then equating both sides $\mathrm{mod}(q - 1)$, we obtain in $\mathcal{A} := \mathcal{A}_q/(q - 1)\mathcal{A}_q$ the relations

$$[[r, r]]\, T_1 T_2 T_3 = T_1 T_2 T_3\, [[r, r]]. \tag{K16}$$

The generators t_{ij} of the algebra $\mathcal{A} = \mathcal{A}_q/(q - 1)A_q$ have to satisfy relations (K16). These relations are trivial if $[[r, r]]$ commutes with $Mat(n, \mathbb{C})^{\otimes 3}$, in particular if the classical Yang-Baxter equation (CYBE) $[[r, r]] = 0$ is satisfied. However, if these relations are nontrivial then the monomials in the generators t_{ij} will not form a $\mathbb{C}$-basis of $\mathcal{A}$. This will also imply that, for any order chosen on the generators t_{ij}, the monomials in the t_{ij} with factors in this order will not form a $\mathbb{C}[q, q^{-1}]$-basis of $\mathcal{A}_q$.

There will be no essential changes in the story of this section if we add further relations for the t_{ij} to the RTT relations (K9). (Of course, one has to verify that the additional relations are compatible with comultiplication.) On $\mathcal{A}$ one still obtains the Poisson structure (K12) on the generators, and the additional relations (K16) will be still valid. Of course, further relations on $\mathcal{A}_q$ may give rise to further relations on $\mathcal{A}$.

Example K23 Put $T := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Let

$$R = R_q := \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & q^{-1}-q & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}, \quad r := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \sigma := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{K17}$$

The relations defining $\mathcal{A}_q(Mat(2,\mathbb{C}))$ in Example K22 are equivalent to relations (K9) with R as above. The relations will not change if we add a scalar multiple of σ (the flip) to R. The resulting Poisson structure on $\mathcal{A}(Mat(2,\mathbb{C}))$ is given by (K12) with r as above. This Poisson structure does not change if we add scalar multiple of σ and of I to r. Compare with the r-matrix in Exercise 18.

Example K24 Instead of considering $\mathcal{A}_q(Mat(2,\mathbb{C}))$ as a deformation of an algebra of functions, we may consider it as a deformation of a universal enveloping algebra. First we rescale the relations for $\alpha,\beta,\gamma,\delta$ defining $\mathcal{A}_q(Mat(2,\mathbb{C}))$ in Example K22. We introduce new generators a,b,c,d by:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (q-1)\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In terms of a,b,c,d the relations are given by

$$\begin{aligned} &[a,b] = b + (q-1)ba, \quad [a,c] = c + (q-1)ca, \quad [b,d] = b + (q-1)db, \\ &[c,d] = c + (q-1)dc, \quad [b,c] = 0, \quad [a,d] = (q-q^{-1})bc. \end{aligned}$$

The comultiplication takes the form

$$\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} + (q-1)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Hence, if we consider this bialgebra mod$(q-1)$ then we obtain a bialgebra generated as an algebra by a,b,c,d satisfying relations which certainly include

$$[a,b] = b, \quad [a,c] = c, \quad [b,d] = b, \quad [c,d] = c, \quad [b,c] = 0, \quad [a,d] = 0,$$

and with a,b,c,d as primitive elements with respect to the comultiplication. So we get a non-commutative, cocommutative bialgebra. We will come back to this example later.

Exercises (submit 32 or 35)

Exercise 31 Prove that in a Poisson-Hopf algebra we have $S(\{a,b\}) = -\{S(a),S(b)\}$. This may be done in the following steps:

(a) $\sum_{(a),(b)} \Delta(\{a_{(1)},b_{(1)}\}) \otimes S(a_{(2)})S(b_{(2)}) = \sum_{(a),(b)} a_{(1)}b_{(1)} \otimes \{a_{(2)},b_{(2)}\} \otimes S(a_{(3)})S(b_{(3)}) + \sum_{(a),(b)} \{a_{(1)},b_{(1)}\} \otimes a_{(2)}b_{(2)} \otimes S(a_{(3)})S(b_{(3)})$.

(b) Apply first $S \otimes \mathrm{id} \otimes \mathrm{id}$ to both sides in (a) and next apply iterated multiplication to the threefold tensor products on both sides. This yields

$$0 = \sum_{(a),(b)} S(a_{(1)})S(b_{(1)}) \{a_{(2)}, b_{(2)}\} S(a_{(3)})S(b_{(3)}) + S(\{a,b\}).$$

(c) $0 = \sum_{(a)} a_{(1)} \{S(a_{(2)}), b\} + \sum_{(a)} \{a_{(1)}, b\} S(a_{(2)})$.

(d) Iterate (c) in order to obtain:

$$\sum_{(a),(b)} S(a_{(1)})S(b_{(1)}) \{a_{(2)}, b_{(2)}\} = \sum_{(a),(b)} \{S(a_{(1)}, S(b_{(1)})\} a_{(2)}b_{(2)}.$$

(e) Combine (b) and (d).

Exercise 32 Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field k (or over a commutative ring k with 1 such that $\mathfrak{g}$ is free as a k-module). Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$. The PBW theorem states that the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ has a basis $X_1^{i_1} \ldots X_n^{i_n}$ ($i_1, \ldots, i_n$ nonnegative integers). Prove this theorem by means of the diamond lemma, by writing $[X_i, X_j] = \sum_{r=1}^n c_{ij}^r X_r$ amd by taking for S the set of pairs $(X_jX_i, X_iX_j - \sum_{r=1}^n c_{ij}^r X_r) \quad (i < j)$.

Exercise 33 Prove the statements in Example K22 about the $\mathbb{C}[q, q^{-1}]$-bases.

Exercise 34 Prove the statements in Example K22 about the induced Poisson structures.

Exercise 35 Show that the algebra generated by the t_{ij} with relations (K8) becomes a bialgebra with comultiplication and counit given by (K10).
Hint Show that the comultiplication and counit are compatible with the relations. For instance, let J the two-sided ideal in $\mathbf{k}\langle T \rangle$ which is generated by the relations. Show that

$$\sum_{k,l=1}^n R_{ij,kl} \, \Delta(t_{kr}) \, \Delta(t_{ls}) - \sum_{k,l=1}^n \Delta(t_{jl}) \, \Delta(t_{ik}) \, R_{kl,rs} \in \mathbf{k}\langle T \rangle \otimes J + J \otimes \mathbf{k}\langle T \rangle.$$

Exercise 36 Prove equations (K15) and (K16).

Exercise 37 Let σ be the flip operator. Show that relations (K9) are equivalent to $(\sigma R)T_1T_2 = T_1T_2(\sigma R)$, and also to $(\sigma R \sigma)T_2T_1 = T_1T_2(\sigma R \sigma)$.

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co-Poisson-Hopf algebras First read C&P, §6.2A from Definition 6.2.2 onwards. Next assume notation of pp.31,32 of these notes on Poisson-Hopf algebras in connection with Hopf algebras deforming a commutative algebra. Instead of assuming a Hopf algebra $\mathcal{A}_q$ over $\mathbb{C}[q,q^{-1}]$ we may assume a topological Hopf algebra $\mathcal{U}_h$ over $\mathbb{C}[[h]]$ and we may put $\mathcal{U} := \mathcal{U}_h/(h\mathcal{U}_h)$, $\overline{u} := u \bmod h$. Then $\mathcal{U}$ is a Hopf algebra over $\mathbb{C}$.

Assume that $\mathcal{U}$ is a cocommutative Hopf algebra. Then $\mathcal{U}$ becomes a co-Poisson-Hopf algebra with co-Poisson bracket $\delta\colon \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}$ being defined by

$$\delta(\overline{u}) := h^{-1}\big(\Delta(u) - \sigma(\Delta(u))\big) \bmod h. \tag{K18}$$

Almost cocommutative Hopf algebras Suppose that $\mathcal{U}_h$ is an almost cocommutative Hopf algebra, i.e., with $R \in \mathcal{U}_h \otimes \mathcal{U}_h$ (in the topological completion of the tensor product) such that

$$R = 1 \otimes 1 \bmod h \tag{K19}$$

and

$$R\,\Delta(u) = \sigma(\Delta(u))\,R. \tag{K20}$$

It follows from (K19) that R is invertible, and (K19) and (K20) together imply that $\mathcal{U}$ is cocommutative. Put

$$r := h^{-1}(R - 1 \otimes 1) \bmod h. \tag{K21}$$

Then it follows that

$$\delta(\overline{u}) = \Delta(\overline{u})\,r - r\,\Delta(\overline{u}). \tag{K22}$$

It follows from (K20) that

$$X\,(\Delta \otimes \mathrm{id})\Delta(u) = (\Delta^{\mathrm{op}} \otimes \mathrm{id})\,\Delta^{\mathrm{op}}(u)\,X \tag{K23}$$

if X equals one of the following elements:

(1) $R_{12}\,(\Delta \otimes \mathrm{id})(R)$,

(2) $R_{23}\,(\mathrm{id} \otimes \Delta)(R)$,

(3) $R_{12}R_{13}R_{23}$,

(4) $R_{23}R_{13}R_{12}$.

Hence

$$Y\,(\Delta \otimes \mathrm{id})\Delta(u) = (\Delta \otimes \mathrm{id})\Delta(u)\,Y \tag{K24}$$

if Y is one of the following elements:

(i) $(R_{23}\,(\mathrm{id} \otimes \Delta)(R))^{-1}\,R_{12}\,(\Delta \otimes \mathrm{id})(R)$,

(ii) $(R_{23}R_{13}R_{12})^{-1}\,R_{12}R_{13}R_{23}$,

(iii) $(R_{13}R_{23})^{-1}\,(\Delta \otimes \mathrm{id})(R)$,

(iv) $(R_{13}R_{12})^{-1}\,(\mathrm{id} \otimes \Delta)(R)$.

It also follows from (K20) that

(v) $\sigma(R)\,R\,\Delta(u) = \Delta(u)\,\sigma(R)\,R$.

It follows from (i),(ii),(iii),(iv), respectively that

$$Z\,(\Delta\otimes\mathrm{id})\,\Delta(\overline{u}) = (\Delta\otimes\mathrm{id})\,\Delta(\overline{u})\,Z \tag{K25}$$

if Z is one of the following elements:
(a) $r_{12} + (\Delta\otimes\mathrm{id})(r) - r_{23} - (\mathrm{id}\otimes\Delta)(r)$,
(b) $[[r,r]]$,
(c) $(\Delta\otimes\mathrm{id})(r) - r_{13} - r_{23}$,
(d) $(\mathrm{id}\otimes\Delta)(r) - r_{13} - r_{12}$.
It follows from (v) that
(e) $(\sigma(r)+r)\,\Delta(\overline{u}) = \Delta(\overline{u})\,(\sigma(r)+r)$.

On the other hand assume that $\mathcal{U}$ is a cocommutative Hopf algebra over $\mathbb{C}$ (not necessarily obtained from some $\mathcal{U}_h$) and define for some $r\in\mathcal{U}\otimes\mathcal{U}$ the map δ by (K22). Then it follows that $\delta(uv) = \delta(u)\Delta(v) + \Delta(u)\delta(v)$, while the above properties (e), (a), (b) are respectively equivalent to the antisymmetry of δ, the co-Leibniz identity and the co-Jacobi identity. Hence $\mathcal{U}$ together with δ defined by (K22) defines a co-Poisson-Hopf algebra iff property (e) holds and identity (K25) holds with Z given by (a) and by (b).

If $\mathfrak{g}$ is a Lie algebra and $\mathcal{U}(\mathfrak{g})$ is a co-Poisson-Hopf algebra with δ defined by (K22) for some $r\in\mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})$, then we would like to conclude that $r\in\mathfrak{g}\otimes\mathfrak{g}$. Possibly, this conclusion is not right in general. However it is right if the expression (a) equals 0 and $\sigma(r)+r=0$, see C&P, proof of Proposition 6.3.2. See also the definitions of coboundary Hopf algebra, quasi-triangular Hopf algebra and triangular Hopf algebra in C&P, Definition 4.2.6, and see the statement of Proposition 6.3.2.

By way of example consider $\mathcal{U}_h(sl(2,\mathbb{C}))$ (C&P, §6.4) and check the resulting quasi-triangular Lie bialgebra structure for $sl(2,\mathbb{C})$ with r obtained from the element $R=\mathcal{R}_h$ given in Proposition 6.4.8.

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Hochschild cohomology of associative algebras Let $\mathbf{k}$ be a field and let $\mathcal{A}$ be an associative algebra with 1 over $\mathbf{k}$. Let M be a linear space over $\mathbf{k}$ which is also an $\mathcal{A}$-bimodule, i.e., a left and right $\mathcal{A}$-module such that $(a.m).b = a.(m.b)$ for all $m \in M$, $a, b \in \mathcal{A}$. By definition, the space $C^n(\mathcal{A}, M)$ of *n-dimensional cochains* consists of all $\mathbf{k}$-linear maps $\widetilde{f}\colon \mathcal{A}^{\otimes(n+2)} \to M$ such that

$$\widetilde{f}(aa_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}b) = a.\widetilde{f}(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}).b \quad (a, a_0, \ldots, a_{n+1}, b \in \mathcal{A}).$$

Define a $\mathbf{k}$-linear map $d\colon C^n(\mathcal{A}, M) \to C^{n+1}(\mathcal{A}, M)$ by

$$(d\widetilde{f})(a_0 \otimes \cdots \otimes a_{n+2}) = \sum_{i=0}^{n+1} (-1)^i\, \widetilde{f}(a_0 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_{n+2}).$$

Then, by easy computation, $d \circ d = 0$.

As a $\mathbf{k}$-linear space, the space $C^n(\mathcal{A}, M)$ is isomorphic to $\mathrm{Hom}_{\mathbf{k}}(\mathcal{A}^{\otimes n}, M)$ by the linear bijection $f \leftrightarrow \widetilde{f}$ such that

$$\widetilde{f}(a_0 \otimes \cdots \otimes a_{n+1}) = a_0.f(a_1 \otimes \cdots \otimes a_n).a_{n+1}, \quad f(a_1 \otimes \cdots \otimes a_n) = \widetilde{f}(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1).$$

In this picture the map d becomes (again by easy computation):

$$\begin{aligned}(df)(a_1 \otimes \cdots \otimes a_{n+1}) =& a_1.f(a_2 \otimes \cdots \otimes a_{n+1}) + \sum_{i=1}^{n} (-1)^i\, f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots a_{n+1}) \\ &+ (-1)^{n+1}\, f(a_1 \otimes \cdots a_n).a_{n+1} \qquad (f \in C^n(\mathcal{A}, M)).\end{aligned}$$

We still have $d \circ d = 0$ in this picture. The n-cochains f which satisfy $df = 0$ are called *n-cocycles*, and the n-cochains f of the form dg for some $(n-1)$-cochain g, are called *n-coboundaries*. The space of n-cocycles modulo the space of n-coboundaries is denoted by $H^n_{\mathrm{alg}}(\mathcal{A}, M)$ (a cohomology group, or rather cohomology $\mathbf{k}$-space).

Clearly, the algebra $\mathcal{A}$ itself is an $\mathcal{A}$-bimodule, so we can speak about $H^n_{\mathrm{alg}}(\mathcal{A}, \mathcal{A})$. If, more generally, $\mathcal{A}$ is a bialgebra, then $\mathcal{A}^{\otimes j}$ is an $\mathcal{A}$-bimodule by the rule

$$a.(b_1 \otimes \cdots \otimes b_j) := \Delta^{(j)}(a)\,(b_1 \otimes \cdots \otimes b_j), \quad (b_1 \otimes \cdots \otimes b_j).a := (b_1 \otimes \cdots \otimes b_j)\,\Delta^{(j)}(a).$$

Here

$$\Delta^{(1)}(a) := a, \quad \Delta^{(2)}(a) := \Delta(a), \quad \Delta^{(j)}(a) := (\Delta \otimes \mathrm{id})\,(\Delta^{(j-1)}(a)).$$

For $\mathcal{A}$ a bialgebra write $C^{ij} := \mathrm{Hom}_{\mathbf{k}}(\mathcal{A}^{\otimes i}, \mathcal{A}^{\otimes j})$ and write d'_{ij} or d' for the map $d\colon C^{ij} \to C^{i+1,j}$. The map d' is then given in C&P, §6.1B, formula (12).

Bij way of example consider $\mathcal{A} := \mathcal{U}(\mathfrak{g})$, with $\mathfrak{g}$ a Lie algebra. Any $\mathcal{U}(\mathfrak{g})$-bimodule M is also a $\mathfrak{g}$-module in a natural way. Then we have:

$$\begin{aligned}H^n_{\mathrm{alg}}(\mathcal{U}(\mathfrak{g}), M) &\simeq H^n_{\mathrm{Liealg}}(\mathfrak{g}, M) \\ &\simeq H^n_{\mathrm{Liealg}}(\mathfrak{g}, \mathbf{k}) \otimes M^{\mathfrak{g}} \quad \text{if } \mathfrak{g} \text{ is semisimple}),\end{aligned}$$

see S. Shnider & S. Sternberg, *Quantum groups, from coalgebras to Drinfeld algebras, a guided tour*, International Press, 1993. Also recall the Whitehead lemmas: $H^n_{\mathrm{Liealg}}(\mathfrak{g}, M) = 0$ for $n = 1, 2$ if $\mathfrak{g}$ is semisimple.

Cohomology of coalgebras Let $\mathcal{A}$ now be a coassociative co-algebra over $\mathbf{k}$ with counit ε and let the $\mathbf{k}$-linear space M be an $\mathcal{A}$-co-bimodule, i.e., with $\mathbf{k}$-linear maps $\rho\colon M \to M \otimes \mathcal{A}$ and $\lambda\colon M \to \mathcal{A} \otimes M$ such that

$$
\begin{aligned}
&(\rho \otimes \mathrm{id}) \circ \rho = (\mathrm{id} \otimes \Delta) \circ \rho, \quad (\mathrm{id} \otimes \varepsilon) \circ \rho = \mathrm{id},\\
&(\mathrm{id} \otimes \lambda) \circ \lambda = (\Delta \otimes \mathrm{id}) \circ \lambda, \quad (\varepsilon \otimes \mathrm{id}) \circ \lambda = \mathrm{id},\\
&(\lambda \otimes \mathrm{id}) \circ \rho = (\mathrm{id} \otimes \rho) \circ \lambda.
\end{aligned}
$$

The definitions in the previous section can now be "dualized" for the case of a coalgebra.

By definition, the space $C^n(M,\mathcal{A})$ of *n-dimensional cochains* consists of all $\mathbf{k}$-linear maps $\widetilde{f}\colon M \to \mathcal{A}^{\otimes(n+2)}$ such that

$$
(\mathrm{id} \otimes \widetilde{f})(\lambda(m)) = \Delta_0(\widetilde{f}(m)), \quad (\widetilde{f} \otimes \mathrm{id})(\rho(m)) = \Delta_{n+1}(\widetilde{f}(m)).
$$

Here Δ_r means Δ being applied to the rth factor in the tensor product, while id is applied to the other factors in the tensor product, with the factors being labeled from 0 to $n+1$. Define $d\colon C^n(M,\mathcal{A}) \to C^{n+1}(M,\mathcal{A})$ by

$$
(d\widetilde{f})(m) := \sum_{i=0}^{n+1} (-1)^i \, \Delta_i(\widetilde{f}(m)).
$$

Then $d \circ d = 0$.

As a $\mathbf{k}$-linear space, the space $C^n(M,\mathcal{A})$ is isomorphic to $\mathrm{Hom}_{\mathbf{k}}(M,\mathcal{A}^{\otimes n})$ by the linear bijection $f \leftrightarrow \widetilde{f}$ such that

$$
\widetilde{f}(m) = (\mathrm{id} \otimes f \otimes \mathrm{id})((\lambda \otimes \mathrm{id}) \circ \rho(m)), \quad f(m) = (\varepsilon \otimes \mathrm{id} \otimes \varepsilon)(\widetilde{f}(m)).
$$

In this picture the map d becomes

$$
(df)(m) = (\mathrm{id} \otimes f)(\lambda(m)) + \sum_{i=1}^{n} (-1)^i \, \Delta_i(f(m)) + (-1)^{n+1} \, (f \otimes \mathrm{id})(\rho(m)).
$$

We still have $d \circ d = 0$ in this picture. As always, with such a complex, the n-cochains f which satisfy $df = 0$ are called *n-cocycles*, and the n-cochains f of the form dg for some $(n-1)$-cochain g, are called *n-coboundaries*. The space of n-cocycles modulo the space of n-coboundaries is denoted by $H^n_{\mathrm{coalg}}(M,\mathcal{A})$.

The co-algebra $\mathcal{A}$ itself is an $\mathcal{A}$-co-bimodule, so we can speak about $H^n_{\mathrm{coalg}}(\mathcal{A},\mathcal{A})$. If, more generally, $\mathcal{A}$ is a bialgebra, then $\mathcal{A}^{\otimes i}$ is an $\mathcal{A}$-co-bimodule by the rule

$$
\begin{aligned}
\lambda(a_1 \otimes \cdots \otimes a_i) &:= \sum_{(a_1),\ldots,(a_i)} (a_1)_{(1)} \ldots (a_i)_{(1)} \otimes (a_1)_{(2)} \otimes \cdots \otimes (a_i)_{(2)},\\
\rho(a_1 \otimes \cdots \otimes a_i) &:= \sum_{(a_1),\ldots,(a_i)} (a_1)_{(1)} \otimes \cdots \otimes (a_i)_{(1)} \otimes (a_1)_{(2)} \ldots (a_i)_{(2)}.
\end{aligned}
$$

For $\mathcal{A}$ a bialgebra recall our notation $C^{ij} := \mathrm{Hom}_{\mathbf{k}}(\mathcal{A}^{\otimes i},\mathcal{A}^{\otimes j})$. Write d''_{ij} or d'' for the map $d\colon C^{ij} \to C^{i,j+1}$. The map d'' is then given in C&P, §6.1B, formula (13).

Algebra deformations Let $\mathcal{A}$ be a $\mathbf{k}$-algebra and let $\mathcal{A}_h$ be the topological $\mathbf{k}[[h]]$-linear space given by $\mathcal{A}[[h]]$. Thus $\mathcal{A}_h$ consists of elements $a_h = a_0 + a_1 h + a_2 h^2 + \cdots$ with $a_0, a_1, a_2, \ldots \in \mathcal{A}$ and we have $\mathcal{A} \simeq \mathcal{A}_h/(h\mathcal{A}_h)$ as $\mathbf{k}$-linear spaces. Denote the (associative) multiplication on $\mathcal{A}$ by $\mu\colon a\otimes b \mapsto ab\colon \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$. Let $\mu_h\colon \mathcal{A}_h \otimes \mathcal{A}_h \to \mathcal{A}_h$ be a continuous $\mathbf{k}[[h]]$-linear map deforming μ, i.e.,

$$\mu_h(a\otimes b) = \mu(a\otimes b) + \mu_1(a\otimes b)\, h + \mu_2(a\otimes b)\, h^2 + \cdots \quad (a, b \in \mathcal{A}),$$

where $\mu_1, \mu_2, \ldots$ are $\mathbf{k}$-linear maps of $\mathcal{A}\otimes\mathcal{A}$ to $\mathcal{A}$. We call μ_h *associative* $\bmod h^n$ if

$$\mu_h((\mu_h(a\otimes b)\otimes c) = \mu_h(a\otimes \mu_h(b\otimes c)) \bmod h^n \quad (a, b, c \in \mathcal{A}).$$

An easy calculation shows that μ_h is associative $\bmod h^2$ iff

$$a\,\mu_1(b\otimes c) - \mu_1(ab\otimes c) + \mu_1(a\otimes bc) - \mu_1(a\otimes b)\, c = 0,$$

i.e., iff the cochain $\mu_1 \in C^2(\mathcal{A},\mathcal{A})$ is a 2-cocycle, i.e., iff $d\mu_1 = 0$.

Two deformations μ_h and μ'_h of μ are called *equivalent* $\bmod h^n$ if there is a continuous $\mathbf{k}[[h]]$-linear bijection $f_h\colon \mathcal{A}_h \to \mathcal{A}_h$ of the form

$$f_h(a) = a + f_1(a)\, h + f_2(a)\, h^2 + \cdots$$

such that

$$\mu'_h(f_h(a)\otimes f_h(b)) = f_h(\mu_h(a\otimes b)) \bmod h^n \quad (a, b \in \mathcal{A}).$$

An easy calculation shows that μ_1 and μ'_1 are equivalent $\bmod h^2$ under f_h iff

$$\mu_1(a\otimes b) - \mu'_1(a\otimes b) = a\, f_1(b) - f_1(ab) + f_1(a)\, b,$$

i.e., iff $\mu_1 - \mu'_1 = df_1$. It follows that the space of equivalence classes $\bmod h^2$ of deformations of μ which are associative $\bmod h^2$ can be identified with $H^2_{\mathrm{alg}}(\mathcal{A},\mathcal{A})$.

Suppose that μ_h is associative $\bmod h^2$, i.e., $d\mu_1 = 0$. Define the 3-cochain ν_1 by

$$\nu_1(a\otimes b\otimes c) := \mu_1(\mu_1(a\otimes b)\otimes c) - \mu_1(a\otimes \mu_1(b\otimes c)).$$

A somewhat longer computation shows that ν_1 is a 3-cocycle, i.e., $d\nu_1 = 0$. An easy calculation shows that μ_h is associative $\bmod h^3$ iff $d\mu_2 = \nu_1$. Hence, if $H^3_{\mathrm{alg}}(\mathcal{A},\mathcal{A}) = 0$ then, for any μ_1 which makes μ_h associative $\bmod h^2$, we can find μ_2 which makes μ_h associative $\bmod h^3$. For $H^3_{\mathrm{alg}}(\mathcal{A},\mathcal{A})$ not necessarily 0 we call $H^3_{\mathrm{alg}}(\mathcal{A},\mathcal{A})$ the *obstruction* for extending μ_h from being associative $\bmod h^2$ to being associative $\bmod h^3$.

Suppose now that μ_h is associative $\bmod h^n$. Put

$$\nu_{n-1}(a\otimes b\otimes c) := \sum_{k=1}^{n-1} \big(\mu_k(\mu_{n-k}(a\otimes b)\otimes c) - \mu_k(a\otimes \mu_{n-k}(b\otimes c))\big).$$

A tedious computation shows that $d\nu_{n-1} = 0$. An easy computation shows that μ_h is associative $\bmod h^{n+1}$ iff $d\mu_n = \nu_{n-1}$. Hence, if $H^3_{\mathrm{alg}}(\mathcal{A},\mathcal{A}) = 0$ then, for any μ_1 which makes μ_h associative $\bmod h^2$, we can find $\mu_2, \mu_3, \ldots$ such that μ_h is an associative multiplication on $\mathcal{A}_h$. Also, for any n, $H^3_{\mathrm{alg}}(\mathcal{A},\mathcal{A})$ (not necessarily 0) is the obstruction for extending μ_h from being associative $\bmod h^n$ to being associative $\bmod h^{n+1}$.

Co-algebra deformations Let $\mathcal{A}$ be a $\mathbf{k}$-co-algebra and let $\mathcal{A}_h$ be the topological $\mathbf{k}[[h]]$-linear space given by $\mathcal{A}[[h]]$. Denote the (co-associative) co-multiplication on $\mathcal{A}$ by Δ. Let $\Delta_h\colon \mathcal{A}_h \to \mathcal{A}_h \otimes \mathcal{A}_h$ be a continuous $\mathbf{k}[[h]]$-linear map deforming Δ, i.e.,

$$\Delta_h(a) = \Delta(a) + \Delta_1(a)\,h + \Delta_2(a)\,h^2 + \cdots \quad (a \in \mathcal{A}),$$

where $\Delta_1, \Delta_2, \ldots$ are $\mathbf{k}$-linear maps of $\mathcal{A}$ to $\mathcal{A} \otimes \mathcal{A}$. We call Δ_h *co-associative* $\bmod h^n$ if

$$(\Delta_h \otimes \mathrm{id})(\Delta_h(a)) = (\mathrm{id} \otimes \Delta_h)(\Delta_h(a)) \bmod h^n \quad (a \in \mathcal{A}).$$

Then Δ_h is co-associative $\bmod h^2$ iff

$$(\mathrm{id} \otimes \Delta_1)(\Delta(a)) - (\Delta \otimes \mathrm{id})(\Delta_1(a)) + (\mathrm{id} \otimes \Delta)(\Delta_1(a)) - (\Delta_1 \otimes \mathrm{id})(\Delta(a)) = 0 \qquad (a \in \mathcal{A}),$$

i.e., iff $d\Delta_1 = 0$. Also,

$$(f_h \otimes f_h)(\Delta_h(a)) = \Delta'_h(f_h(a)) \bmod h^2$$

iff

$$\Delta'_1(a) - \Delta_1(a) = (\mathrm{id} \otimes f_1)(\Delta(a)) - \Delta(f_1(a))) + (f_1 \otimes \mathrm{id})(\Delta(a)),$$

i.e., iff $\Delta_1 - \Delta'_1 = -df_1$.

Suppose Δ_h is co-associative $\bmod h^2$. Put

$$\Gamma_1 := -(\Delta_1 \otimes \mathrm{id}) \circ \Delta_1 + (\mathrm{id} \otimes \Delta_1) \circ \Delta_1.$$

Then $d\Gamma_1 = 0$. Furthermore, Δ_h is co-associative $\bmod h^3$ iff $d\Delta_2 = -\Gamma_1$.

Gerstenhaber-Schack cohomology of bialgebras Let $\mathcal{A}$ be a bialgebra over $\mathbf{k}$. Recall that $C^{ij} := \mathrm{Hom}_{\mathbf{k}}(\mathcal{A}^{\otimes i}, \mathcal{A}^{\otimes j})$. We can identify the elements $f \in C^{ij}$ in a linear way with elements $\widetilde{f} \in \mathrm{Hom}_{\mathbf{k}}(\mathcal{A}^{\otimes(i+2)}, \mathcal{A}^{\otimes(j+2)})$ such that

$$\widetilde{f}(aa_0 \otimes a_1 \otimes \cdots \otimes a_i \otimes a_{i+1}b) = \Delta^{(j+2)}(a)\,\widetilde{f}(a_0 \otimes \cdots \otimes a_{i+1})\,\Delta^{(j+2)}(b)$$

and

$$\sum_{(a_0),\ldots,(a_{i+1})} (a_0)_{(1)} \ldots (a_{i+1})_{(1)} \otimes \widetilde{f}((a_0)_{(2)} \otimes \cdots \otimes (a_{i+1})_{(2)}) \otimes (a_0)_{(3)} \ldots (a_{i+1})_{(3)}$$
$$= (\Delta_0 \otimes \Delta_{j+1})(\widetilde{f}(a_0 \otimes \cdots \otimes a_{i+1})).$$

The linear bijection $\widetilde{f} \mapsto f$ is given by

$$f(a_1 \otimes \cdots \otimes a_i) = (\varepsilon \otimes \mathrm{id} \otimes \varepsilon)(\widetilde{f}(1 \otimes a_1 \otimes \cdots \otimes a_i \otimes 1)).$$

Recall the maps $d' = d'_{ij}\colon C^{ij} \to C^{i+1,j}$ and $d'' = d''_{ij}\colon C^{ij} \to C^{i,j+1}$ we defined earlier. When acting on $\widetilde{f}$ these become:

$$(d'_{ij}\widetilde{f})(a_0 \otimes \cdots \otimes a_{i+2}) = \sum_{r=0}^{i+1} (-1)^r\, \widetilde{f}(a_0 \otimes \cdots \otimes a_{r-1} \otimes a_r a_{r+1} \otimes a_{r+2} \otimes \cdots \otimes a_{i+2}),$$

$$(d''_{ij}\widetilde{f})(a_0 \otimes \cdots \otimes a_{i+1}) = \sum_{s=0}^{j+1} (-1)^s\, \Delta_s(\widetilde{f}(a_0 \otimes \cdots \otimes a_{i+1})).$$

Then it follows immediately that d' and d'' commute, i.e., $d''_{i+1,j} \circ d'_{ij} = d'_{i,j+1} \circ d''_{ij}$.

Put now $C^n := C^{n,1} \oplus C^{n-1,2} \oplus \cdots \oplus C^{1,n}$ and

$$df := d'_{ij} f + (-1)^i\, d''_{ij} f \qquad (f \in C^{ij}).$$

Then $d\colon C^n \to C^{n+1}$ and $d \circ d = 0$. Put $H^n(\mathcal{A}, \mathcal{A}) := d^{-1}\{0\}_{C^{n+1}} / d(C^{n-1})$.

Bialgebra deformations Let $\mathcal{A}$ be a bialgebra over $\mathbf{k}$ with multiplication μ and comultiplication Δ and let $\mathcal{A}_h$, μ_h, Δ_h and f_h be as before. We have seen earlier that μ_h is associative modh^2 iff $d'\mu_1 = 0$ and that Δ_h is co-associative modh^2 iff $d''\Delta_1 = 0$. An easy computation shows that Δ_h is an algebra homomorphism modh^2, i.e.,

$$\Delta_h(ab) = \Delta_h(a)\,\Delta_h(b) \bmod h^2 \quad (a, b \in \mathcal{A}),$$

iff $d'\Delta_1 + d''\mu_1 = 0$. We can summarize this by saying that $(\mathcal{A}_h, \mu_h, \Delta_h)$ is a bialgebra mod h^2 iff $d(\mu, \Delta) = 0$. Also, (μ_h, Δ_h) and (μ'_h, Δ'_h) are equivalent modh^2 by f_h iff $(\mu_1, \Delta_1) - (\mu'_1, \Delta'_1) = df_1$.

Read now C&P, Proposition 6.1.3, Corollary 6.1.5, Proposition 6.1.6 and Theorem 6.1.8(ii). Theorem 6.2.8 should be skipped, since this result has been withdrawn by the author mentioned there. Instead consider the paper P. Etingof & D. Kazhdan, *Quantization of Lie bialgebras, I*", (preprint in q-alg, 1995). Read also Remarks [1] after Corollary 6.5.4 in C&P.

Remark A good reference for the first part of this course is: I. Vaisman, *Lectures on the geometry of Poisson manifolds*, Birkhäuser, 1994.

Topics for a final piece of work for this course Extend the idea of quantization of s Poisson Lie group to the quantization of a Poisson homogeneous space. A very concrete example of this can be studied in

A, J.-L. Sheu, *Quantiztion of the Poisson $SU(2)$ and its Poisson homogeneous space — The 2-sphere*, Comm. Math. Phys. 135 (1991), 217–232.

A much more general case can be found in:

S. Khoroshkin, A. Radul & V. Rubtsov, *A family of Poisson structures on Hermitian symmetric spaces*, Comm. Math. Phys. 152 (1993), 299–315.