Using the scaling function as a starting point (Daubechies, §5.3.2)

Prop. 1. Let $\phi \in L^2(\mathbb{R})$ such that

(a) $\exists\, C_1, C_2 > 0$ with

$$C_1 \leq \sum_{n \in \mathbb{Z}} |\hat{\phi}(\xi + 2\pi n)|^2 \leq C_2 \quad \text{a.e.}$$

(b) $\exists\, m_0 \in L^2(\mathbb{R}/2\pi\mathbb{Z})$ such that

$$\hat{\phi}(2\xi) = m_0(\xi)\,\hat{\phi}(\xi) \quad \text{a.e.}$$

Then $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a Riesz basis of $V_0 := \overline{\mathrm{Span}\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}}$ (ON basis iff $C_1 = C_2 = 1$)

Put $V_{-n} := \{ f(2^n \cdot) \mid f \in V_0 \}$.

Then properties (1), (3), (4) of multiresolution analysis are satisfied and moreover the second part of Property (2): $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.

What about prop. (2a): $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$?

Prop. 2. Assumptions as in Prop. 1.

(i) $\lim\limits_{j\to\infty} \frac{1}{2 \cdot 2^{-j}} \int_{-2^{-j}}^{2^{-j}} |\hat{\phi}(\xi)|^2 \, d\xi = C_2 \Longrightarrow (2a)$

$\Longrightarrow \liminf\limits_{j\to\infty} \frac{1}{2 \cdot 2^{-j}} \int_{-2^{-j}}^{2^{-j}} |\hat{\phi}(\xi)|^2 \, d\xi \geq C_1$.

(ii) $\liminf\limits_{j\to\infty} |\hat{\phi}(2^{-j}\xi)|^2 \geq C > 0$ a.e. $\Longrightarrow (2a)$

(iii) $\lim\limits_{\xi\to 0} |\hat{\phi}(\xi)|^2 = C > 0 \Longrightarrow (2a)$ and $C_1 \leq C \leq C_2$.

In (iv) and (v) below assume moreover $C_1 = C_2 = 1$.

Prop. 2 (continued), $C_1 = C_2 = 1$

(iv) $\displaystyle\lim_{j\to\infty} \frac{1}{2\cdot 2^{-j}} \int_{-2^{-j}}^{2^{-j}} |\hat{\varphi}(\xi)|^2 d\xi = 1 \Longleftrightarrow (2a)$

(v) $\displaystyle\lim_{\xi\to 0} |\hat{\varphi}(\xi)|^2 = C > 0 \Longrightarrow (2a)$ and $C = 1$.

Prop. 3 Assumption as in Prop. 1 and $C_1 = C_2 = 1$. Put

$$\hat{\psi}(\xi) := e^{\frac{1}{2}i\xi}\, \nu(\xi)\, \overline{m_0(\tfrac{1}{2}\xi + \pi)}\, \hat{\varphi}(\tfrac{1}{2}\xi).$$

Then

(i) $|\hat{\psi}(\xi)|^2 = |\hat{\varphi}(\tfrac{1}{2}\xi)|^2 - |\hat{\varphi}(\xi)|^2$

(ii) $\displaystyle (2a) \Longrightarrow \lim_{j\to\infty} \frac{1}{2\cdot 2^{-j}} \int_{-2^{-j}}^{2^{-j}} |\hat{\psi}(\xi)|^2 d\xi = 0$

(iii) $\displaystyle\lim_{\xi\to 0} |\hat{\varphi}(\xi)|^2 = C > 0 \Longrightarrow$

$$\Longrightarrow \begin{cases} (2d) \\ C = 1 \\ \displaystyle\lim_{\xi\to 0} |m_0(\xi)|^2 = 1 \\ \displaystyle\lim_{\xi\to\pi} \cancel{|m_0(\xi)|}\; m_0(\xi) = 0 \\ \displaystyle\lim_{\xi\to 0} \hat{\psi}(\xi) = 0. \end{cases}$$

Proof of Prop. 1. Put

$$\omega(\xi) := \sum_{n\in\mathbb{Z}} |\hat{\varphi}(\xi + 2n\pi)|^2,$$

$$(\varphi^{\#})^\wedge := \omega^{-\frac{1}{2}} \hat{\varphi}, \quad m_0^{\#}(\xi) := \frac{\omega(\xi)}{\omega(2\xi)}\, m_0(\xi).$$

Then:

$\{\phi^{\#}(\cdot - n)\}_{n\in\mathbb{Z}}$ is ON basis of V_0 and $(\phi(\cdot - n))^\wedge = \omega^{\frac{1}{2}}(\phi^{\#}(\cdot - n))^\wedge$, hence $\{\phi(\cdot - n)\}_{n\in\mathbb{Z}}$ is Riesz basis of V_0. Also $(\phi^{\#})^\wedge(2\xi) = m_0^{\#}(\xi)(\phi^{\#})^\wedge(\xi)$.
Hence $\phi^{\#} \in V_{-1}$, so $V_0 \subset V_{-1}$.
This settles properties (1), (3), (4).

Proof that $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$:

Put $\phi_{j,k}(x) := 2^{-\frac{1}{2}j}\phi(2^{-j}x - k)$.
Put $P_j : L^2(\mathbb{R}) \longrightarrow V_j$ orthogonal projection. Then
$\{\phi_{j,k}\}_{k\in\mathbb{Z}}$ is Riesz-basis of V_j.
Hence (Exercise last time):
$\exists\, C_1, C_2 > 0$ s.t. $\forall f \in V_j$

$$C_1 \|f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, \phi_{j,k}\rangle|^2 \leq C_2 \|f\|^2.$$

Hence $\exists\, C_1, C_2 > 0$ s.t. $\forall f \in L^2(\mathbb{R})$

$$C_1 \|P_j f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, \phi_{j,k}\rangle|^2 \leq C_2 \|P_j f\|^2$$

We have to show that $\forall f \in L^2(\mathbb{R})$ $\lim_{j\to\infty} \|P_j f\| = 0$. Since

$\|P_j f\| \leq \|P_j \tilde{f}\| + \|f - \tilde{f}\|$, $\tilde{f} \in C_c(\mathbb{R})$,
it is sufficient to show that $\lim_{j\to\infty} \|P_j f\| = 0$ for $f \in C_c(\mathbb{R})$. Now show (Dau. pp. 141, 142): $\lim_{j\to\infty} \sum_{k\in\mathbb{Z}} |\langle f, \phi_{j,k}\rangle|^2 = 0$ if $f \in C_c(\mathbb{R})$. $\square$

For the proof of Prop. 2 we need:

Lemma 1. Let $f, g \in L^2(\mathbb{R})$ s.t. $\exists \varrho > 0, \exists C_0 > 0 : |\hat{f}(\xi)\,\hat{g}(\xi)| \leq \frac{C_0}{(1+|\xi|)^{1+\varrho}}$.

Then

$$\infty > \sum_{k\in\mathbb{Z}} \left| \int_{\mathbb{R}} f(x)\,\overline{g(x-k)}\,dx \right|^2$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{\ell\in\mathbb{Z}} \hat{f}(\xi+2\ell\pi)\,\overline{\hat{g}(\xi+2\ell\pi)} \right|^2 d\xi$$

$$= M + R,$$

where

$$M = \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(\xi)|^2\,|\hat{g}(\xi)|^2\,d\xi$$

$$= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x)\,\overline{g(x-y)}\,dx \right|^2 dy,$$

$$R = \sum_{\substack{\ell\in\mathbb{Z}\\ \ell\neq 0}} \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi)\,\overline{\hat{g}(\xi)}\;\overline{\hat{f}(\xi+2\ell\pi)} \cdot \hat{g}(\xi+2\ell\pi)\,d\xi,$$

and $0 \leq M < \infty$, $|R| < \infty$.

Replace in Lemma 1 f by $2^{-\frac{j}{2}} f(2^{-j}\,\cdot)$ and g by φ. Then:

~~Assume $f, \varphi \in$~~

Lemma 2. Let $f, \varphi \in L^2(\mathbb{R})$, $\hat{\varphi} \in L^\infty(\mathbb{R})$, $|\hat{f}(\xi)| \le \frac{C_0}{(1+\xi^2)^{1+\varepsilon}}$ for some $C_0 > 0$, $\varepsilon > 0$. Then

$$\sum_{k\in\mathbb{Z}} |\langle f, \varphi_{-j,k}\rangle|^2 = \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{f}(\xi)|^2 \, |\hat{\varphi}(2^{-j}\xi)|^2 \, d\xi + O(2^{-j}) \quad \text{as } j \to \infty$$

Proof of Prop. 2 C_1, C_2 as in Prop. 1.

Then $|\hat{\varphi}(\xi)|^2 \le C_2$ and

$$C_1 \|P_{-j} f\|^2 \le \sum_{k\in\mathbb{Z}} |\langle f, \varphi_{-j,k}\rangle|^2 \le C_2 \|P_{-j} f\|^2$$

for $f \in L^2(\mathbb{R})$.

By Lemma 2, if $|\hat{f}(\xi)| \le \frac{C_0}{(1+\xi^2)^{1+\varepsilon}}$ for some $C_0 > 0$, $\varepsilon > 0$, then:

$$C_1 \|P_{-j} f\|^2 \le \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{f}(\xi)|^2 \, |\hat{\varphi}(2^{-j}\xi)|^2 \, d\xi + O(2^{-j}) \le C_2 \|P_{-j} f\|^2 \quad \text{as } j \to \infty.$$

ad (i) Assume (2a). Take $\hat{f} := \chi_{[-1,1]}$

Then $C_1 \|P_{-j} f\|^2 \le \frac{1}{2\pi}\int_{-1}^{1} |\hat{\varphi}(2^{-j}\xi)|^2 \, d\xi + O(2^{-j})$

Let $j \to \infty$. Then

$$\frac{1}{\pi} C_1 = C_1 \|f\|^2 \le \liminf_{j\to\infty} \frac{1}{2\pi}\int_{-1}^{1} |\hat{\varphi}(2^{-j}\xi)|^2 \, d\xi$$

$$= \frac{1}{\pi} \liminf_{j\to\infty} \frac{1}{2\cdot 2^{-j}} \int_{-2^{-j}}^{2^{-j}} |\hat{\varphi}(\xi)|^2 \, d\xi.$$

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Now assume;

$$\lim_{j\to\infty} \frac{1}{2.2^{-j}} \int_{-2^{-j}}^{2^{-j}} |\hat{\varphi}(\xi)|^2 \, d\xi = C_2 .$$

Then, for any $M>0$:

$$\lim_{j\to\infty} \int_{-M}^{M} \underbrace{\left(C_2 - |\hat{\varphi}(2^{-j}\xi)|^2 \right)}_{\geq 0} d\xi = 0 .$$

Hence, if $\hat{f} \in C_c(\mathbb{R})$ then let $j\to\infty$ in

$$\frac{1}{2\pi} \int_{-M}^{M} |\hat{f}(\xi)|^2 \, |\hat{\varphi}(2^{-j}\xi)|^2 \, d\xi + O(2^{-j}) \leq C_2 \|P_{-j} f\|^2$$

and obtain

$$C_2 \|f\|^2 \leq C_2 \lim_{j\to\infty} \|P_{-j} f\|^2 ,$$

so $\|f\|^2 = \lim\limits_{j\to\infty} \|P_{-j} f\|^2$.

Now use density of $(C_c(\mathbb{R}))^\wedge$ in $L^2(\mathbb{R})$.

<u>ad (ii)</u> Let $\hat{f} \in C_c(\mathbb{R})$. Let $j\to\infty$ in $\frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \, |\hat{\varphi}(2^{-j}\xi)|^2 \, d\xi + O(2^{-j})$

$$\leq C_2 \|P_{-j} f\|^2 .$$

Then

$$C\|f\|^2 \leq \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \liminf_{j\to\infty} |\hat{\varphi}(2^{-j}\xi)|^2 \, d\xi$$

$$\leq \liminf_{j\to\infty} \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 |\hat{\varphi}(2^{-j}\xi)|^2 \, d\xi \leq C_2 \lim_{j\to\infty} \|P_{-j} f\|^2$$