Riesz bases of scaling funtions

(Daubechies, §5.3.1)

Def. Let $\mathcal{H}$ be a separable Hilbert space. Let A be an index set. A system $\{x_\alpha\}_{\alpha\in A}$ in $\mathcal{H}$ is called a Riesz basis if $x_\alpha = Te_\alpha$ with $\{e_\alpha\}_{\alpha\in A}$ an ON basis and $T, T^{-1} \in B(\mathcal{H})$.

Let $\{x_\alpha\}_{\alpha\in A}$ be a Riesz basis. Then, for any complex $(c_\alpha)_{\alpha\in A}$ with $\sum_{\alpha\in A} |c_\alpha|^2 < \infty$ we have

$\sum_{\alpha\in A} c_\alpha x_\alpha$ converging unconditionally in $\mathcal{H}$, while any $x \in \mathcal{H}$ can be uniquely written as such a series $\sum_{\alpha\in A} c_\alpha x_\alpha$. There are also $C_1, C_2 > 0$ such that

$$C_1 \sum_{\alpha\in A} |c_\alpha|^2 \le \Big\| \sum_{\alpha\in A} c_\alpha x_\alpha \Big\|^2 \le C_2 \sum_{\alpha\in A} |c_\alpha|^2$$

whenever $\sum_{\alpha\in A} |c_\alpha|^2 < \infty$.

Let $(V_n)_{n\in\mathbb{Z}}$ satisfy the properties of a multiresolution analysis of $L^2(\mathbb{R})$ except that (5) is replaced by the property:

(5)' $\exists\, \varphi \in V_0$ s.t. $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis of V_0.

<u>Lemma 1.</u> $(V_0)^\wedge = \{ m\hat{\varphi} \mid m \in L^2(\mathbb{R}/2\pi\mathbb{Z})\}$.

Put $\omega(\xi) := \sum_{n\in\mathbb{Z}} |\hat{\varphi}(\xi + 2\pi n)|^2$.
Then $\omega \in L^1(\mathbb{R}/2\pi\mathbb{Z})$.

<u>Lemma 2.</u> $\exists\, C_1, C_2 > 0$ s.t.

$$C_1 \leq \omega(\xi) \leq C_2 \quad \text{a.e.}$$

<u>Prop.</u> Put $(\varphi^\#)^\wedge := \dfrac{\hat{\varphi}}{\omega^{\frac{1}{2}}}$. Then $\{\varphi^\#(\cdot - n)\}_{n\in\mathbb{Z}}$ is ON basis of V_0.

<u>Proof of Lemma 1.</u> Let $\sum_{n\in\mathbb{Z}} |c_n|^2 < \infty$.
Then:

$$f(x) = \sum_{n\in\mathbb{Z}} c_n \varphi(x-n) \quad \text{unconditionally in } L^2(\mathbb{R})$$

$$\Longleftrightarrow \quad \hat{f}(\xi) = \sum_{n\in\mathbb{Z}} c_n e^{-in\xi} \hat{\varphi}(\xi) \quad \text{unconditionally in } L^2(\mathbb{R})$$

$$\Longleftrightarrow \quad \hat{f}(\xi) = m(\xi)\, \hat{\varphi}(\xi) \quad \text{with}$$

$$m(\xi) = \sum_{n\in\mathbb{Z}} c_n e^{-in\xi} \quad \text{in } L^2(\mathbb{R}/2\pi\mathbb{Z}).$$

$\square$

Proof of Lemma 2

There are $C_1, C_2 > 0$ s.t., whenever $\sum_{n\in\mathbb{Z}} |c_n|^2 < \infty$,

$$C_1 \sum_{n\in\mathbb{Z}} |c_n|^2 \leq \int_{\mathbb{R}} \Big| \sum_{n\in\mathbb{Z}} c_n \phi(x-n) \Big|^2 dx \leq C_2 \sum_{n\in\mathbb{Z}} |c_n|^2$$

But, with $m(\xi) := \sum_{n\in\mathbb{Z}} c_n e^{-in\xi}$,

$$\int_{\mathbb{R}} \Big| \sum_{n\in\mathbb{Z}} c_n \phi(x-n) \Big|^2 dx$$
$$= \frac{1}{2\pi} \int_{\mathbb{R}} \Big| \sum_{n\in\mathbb{Z}} c_n e^{-in\xi} \hat{\phi}(\xi) \Big|^2 d\xi$$
$$= \frac{1}{2\pi} \int_{\mathbb{R}} |m(\xi)|^2 |\hat{\phi}(\xi)|^2 d\xi$$
$$= \frac{1}{2\pi} \int_0^{2\pi} |m(\xi)|^2 \omega(\xi)\, d\xi$$

Hence

$$\frac{C_1}{2\pi} \int_0^{2\pi} |m(\xi)|^2 d\xi \leq \frac{1}{2\pi} \int_0^{2\pi} |m(\xi)|^2 \omega(\xi)\, d\xi$$
$$\leq \frac{C_2}{2\pi} \int_0^{2\pi} |m(\xi)|^2 d\xi$$

($m \in L^2(\mathbb{R}/2\pi\mathbb{Z})$ arbitrarily

Hence $C_1 \leq \omega(\xi) \leq C_2$ a.e. $\square$

Proof of Prop.

$\hat{\phi} \in L^2(\mathbb{R}) \Longrightarrow (\phi^{\#})^{\wedge} \in L^2(\mathbb{R})$.

$\sum_{n\in\mathbb{Z}} |(\phi^{\#})^{\wedge}(\xi + 2n\pi)|^2 = 1$ a.e.

Hence $\{\phi^{\#}(\cdot - n)\}_{n\in\mathbb{Z}}$ is ON system.

$m \in L^2(\mathbb{R}/2\pi\mathbb{Z}) \Longleftrightarrow m\,\omega^{-\frac{1}{2}} \in L^2(\mathbb{R}/2\pi\mathbb{Z})$

Hence $\hat{V}_0 = \{ m \hat{\phi}^{\#} \mid m \in L^2(\mathbb{R}/2\pi\mathbb{Z}) \}$.

Hence $V_0 = \{ \sum_{n\in\mathbb{Z}} c_n \phi^{\#}(\cdot - n) \mid \sum |c_n|^2 < \infty \}$. $\square$

Observe:

$$\phi(\cdot - n) \xrightarrow{(2\pi)^{-\frac{1}{2}}\mathcal{F}} \frac{\hat{\phi}\, e^{-in\cdot}}{\sqrt{2\pi}} \xrightarrow{M} \frac{\hat{\phi}\, e^{-in\cdot}}{\sqrt{2\pi}\ \omega^{\frac{1}{2}}} \xrightarrow{\left((2\pi)^{-\frac{1}{2}}\mathcal{F}\right)^{-1}} \phi^{\#}(\cdot - n)$$

$(2\pi)^{-\frac{1}{2}}\mathcal{F}$ is unitary.

M is pos. def. self-adjoint.

Hence $\phi(\cdot - n) \overset{S}{\longmapsto} \phi^{\#}(\cdot - n)$ is pos. def. self-adjoint.

Recall: If $\mathcal{H}$ Hilbert space and $T, T^{-1} \in \mathcal{B}(\mathcal{H})$ then $\exists$ unique pos. def. s.a. S and unitary U s.t. $T = SU$ (polar decomposition).

Prop. Let $\{x_\alpha\}_{\alpha\in A}$ be a Riesz basis of $\mathcal{H}$. Then $\exists$ unique pos. def. s.a. S s.t. $\{S^{-1}x_\alpha\}_{\alpha\in A}$ is ON basis.

Proof $x_\alpha = Te_\alpha$ with $T, T^{-1} \in B(\mathcal{H})$ and $\{e_\alpha\}_{\alpha\in A}$ ON basis.

$T = SU$ (polar decomposition).

$f_\alpha := Ue_\alpha$. Then $\{f_\alpha\}_{\alpha\in A}$ ON basis and $x_\alpha = Sf_\alpha$.

Conversely, if $x_\alpha = S_1 g_\alpha$, S_1 pos. def., $\{g_\alpha\}$ ON basis then $g_\alpha = U_1 e_\alpha$ with U_1 unitary. Hence $S_1U_1e_\alpha = x_\alpha = SUe_\alpha$.

Hence $S_1U_1 = SU$. Hence $S_1 = S$. $\square$

The Meyer wavelet basis

(Daubechies, § 5.2)

Let $\hat\phi \in C^\infty(\mathbb{R})$ s.t.

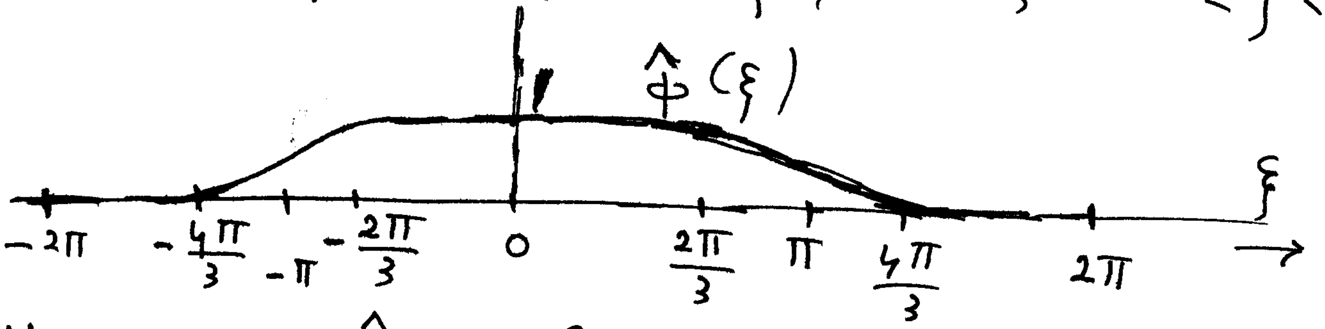
(i) $\operatorname{supp}(\hat\phi) \subset \left[-\frac{4\pi}{3}, \frac{4\pi}{3}\right]$;

(ii) $\hat\phi$ even;

(iii) $0 \le \hat\phi(\xi) \le 1$

(iv) $\hat\phi(\xi) = 1, \quad \xi \in \left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right]$

(v) $(\hat\phi(\xi))^2 + (\hat\phi(2\pi - \xi))^2 = 1, \quad 0 \le \xi \le 2\pi$



Hence $\hat\phi \in L^2(\mathbb{R})$,

$$\sum_{k\in\mathbb{Z}} |\hat\phi(\xi + 2k\pi)|^2 = 1,$$

$\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is ON basis of

$$V_0 := \overline{\operatorname{span}\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}}$$

Put $V_{-n} := \{f(2^n \cdot) \mid f \in V_0\}$

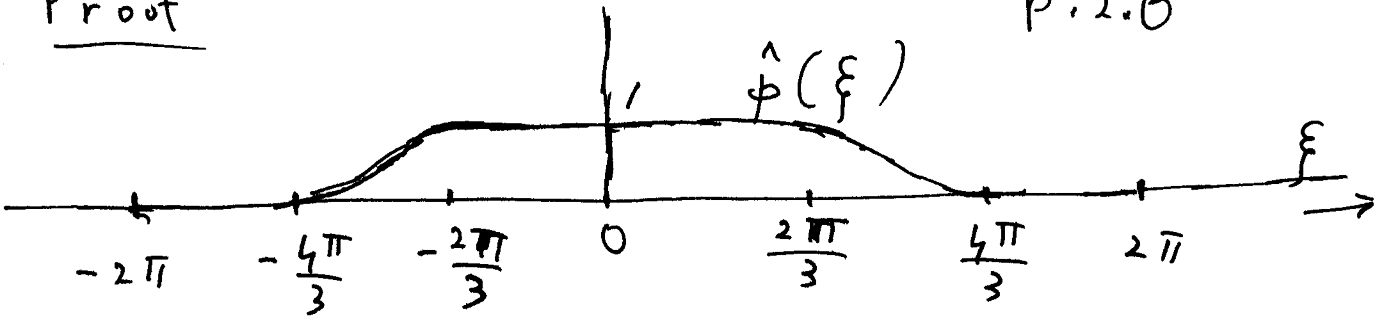
Put $m_0(\xi) := \sum_{k\in\mathbb{Z}} \hat\phi(2(\xi + 2\pi k))$

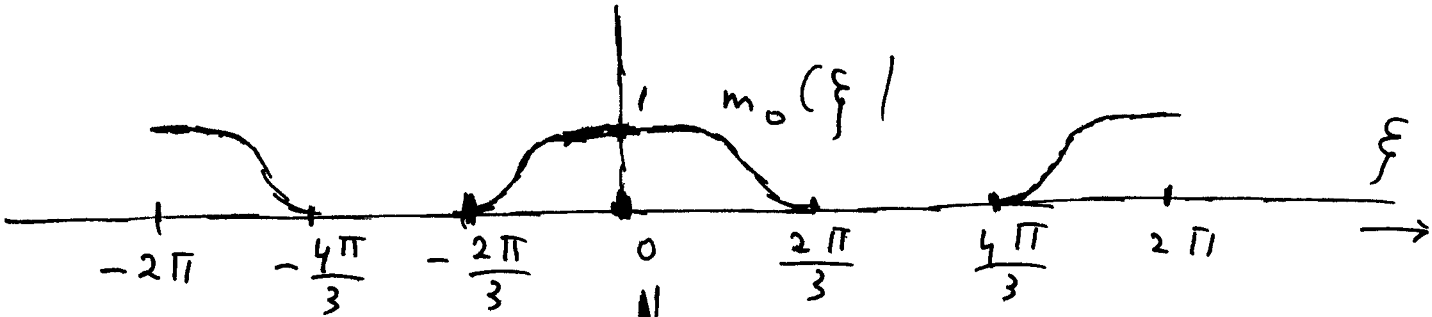
<u>Prop.</u> $\hat\phi(2\xi) = m_0(\xi)\,\hat\phi(\xi)$;

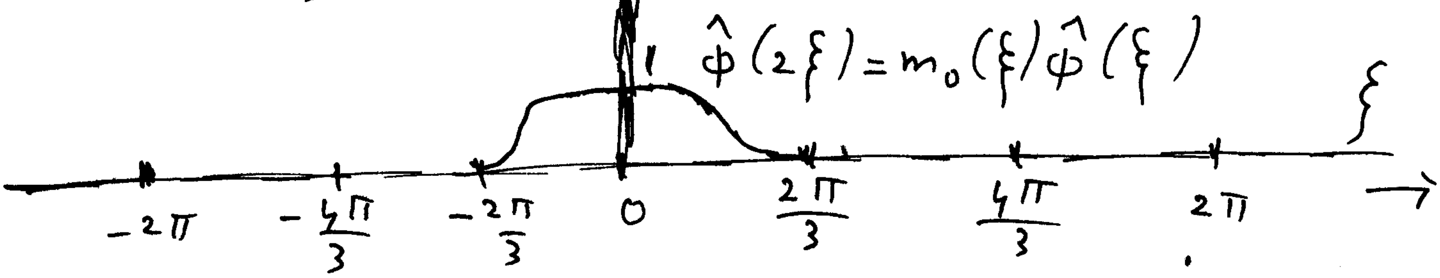
$\phi \in V_{-1}$;

$$\cdots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \cdots$$

Proof







$$m_0(\xi) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} h_n e^{-in\xi}$$

$$\hat{\phi}(2\xi) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} h_n e^{-in\xi} \hat{\phi}(\xi)$$

$$\phi(x) = \sqrt{2} \sum_{n\in\mathbb{Z}} h_n \phi(2x-n)$$

$$\phi \in V_{-1}. \qquad \square$$

Exercise $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R})$;

$$\bigcap_{n\in\mathbb{Z}} V_n = \{0\}.$$

Prop $\{\psi(\cdot - k)\}_{k\in\mathbb{Z}}$ is ON basis of W_0 iff,

$$\cancel{\hat{\psi}(2\xi) = e^{i\xi} r(2\xi)\left(\hat{\phi}(2\xi+2\pi) + \hat{\phi}(2\xi-2\pi)\right)\hat{\phi}(\xi)}$$

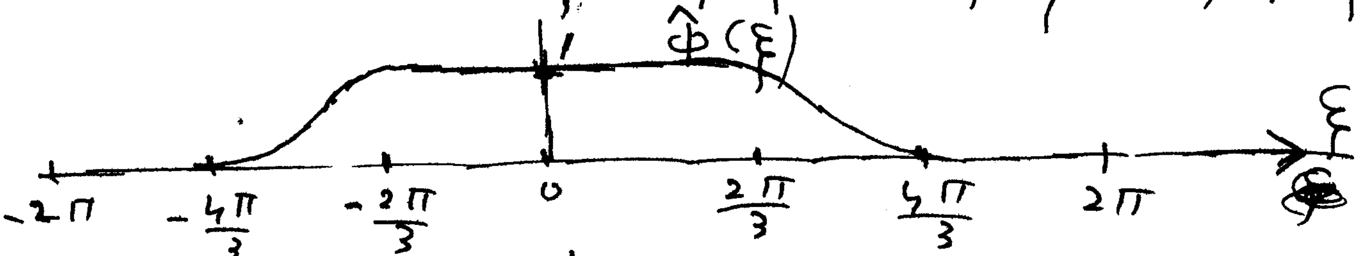
for some $r \in L^2(\mathbb{R}/2\pi\mathbb{Z})$ with $|r(\xi)| = 1$ a.e., we have :

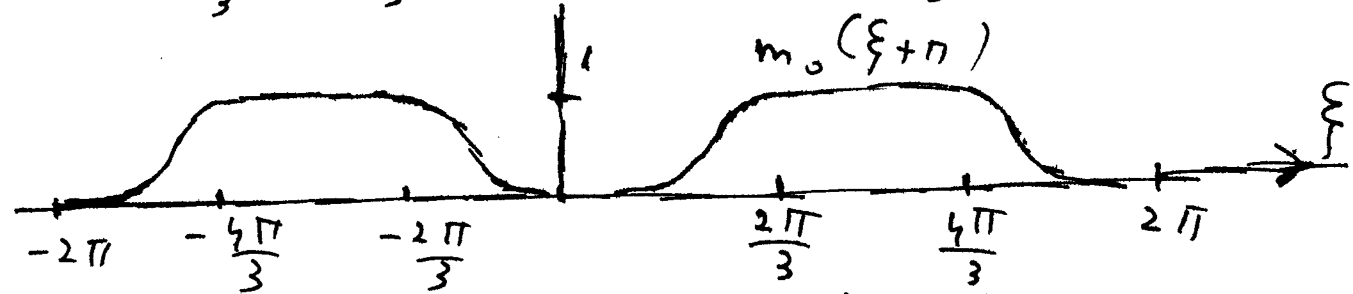
$$\hat{\psi}(2\xi) = e^{i\xi} r(2\xi) \begin{cases} \hat{\phi}(\xi), & \frac{2\pi}{3} \le |\xi| \le \frac{4\pi}{3} \\ \hat{\phi}(2\xi+2\pi), & -\frac{2\pi}{3} \le \xi \le -\frac{\pi}{3} \\ \hat{\phi}(2\xi-2\pi), & \frac{\pi}{3} \le \xi \le \frac{2\pi}{3} \\ 0, & \text{otherwise} \end{cases}$$

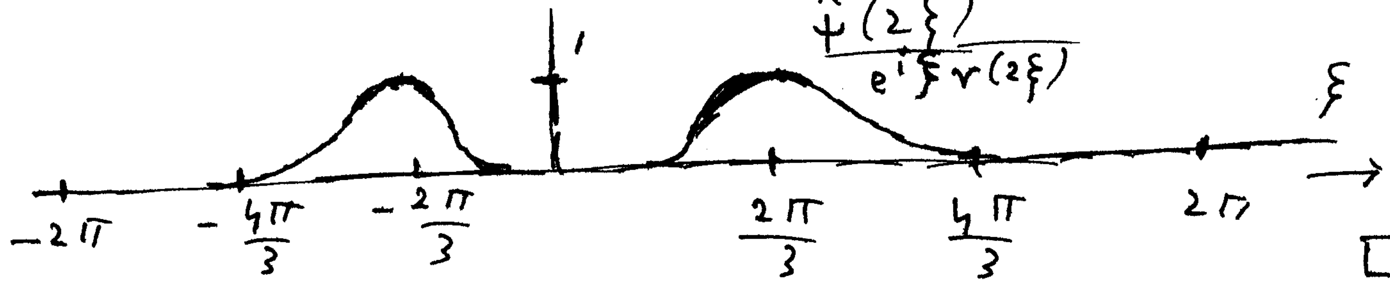
Proof

$$\hat{\psi}(2\xi) = e^{i\xi} r(2\xi)\, \overline{m_0(\xi+\pi)}\, \hat{\phi}(\xi)$$

$$= e^{i\xi} r(2\xi) \left(\hat{\phi}(2\xi+2\pi) + \hat{\phi}(2\xi-2\pi) \right) \hat{\phi}(\xi)$$

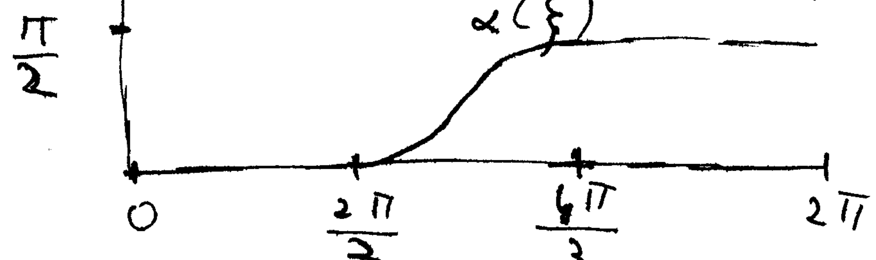






$\square$

Construction of $\hat{\phi}(\xi)$



$\alpha \in C^\infty([0, 2\pi])$

~~monoto~~ $\alpha'(\xi) \ge 0$

$\alpha(2\pi - \xi) = \frac{\pi}{2} - \alpha(\xi)$

$\alpha(\xi) = 0, \quad 0 \le \xi \le \frac{2\pi}{3}$

$\hat{\phi}(\xi) = \cos(\alpha(\xi)), \quad 0 \le \xi \le 2\pi.$