Course on Wavelets

Multiresolution analysis
(Daubechies, §5.1)

Def. A *multiresolution analysis* of $L^2(\mathbb{R})$ is a sequence $(V_n)_{n\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ s.t.

(1) $\ldots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \ldots$

(2) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$, $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$

(3) $f \in V_j \iff f(2^j\cdot) \in V_0 \quad \forall j \in \mathbb{Z}$

(4) $f \in V_0 \iff f(\cdot - k) \in V_0 \quad \forall k \in \mathbb{Z}$

(5) $\exists \phi \in V_0$ s.t. $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is ON basis of V_0.

Assume (1) – (4). Let W_j be s.t.

$$V_{j-1} = V_j \oplus W_j \quad \text{(orthog. direct sum)}$$

Then:

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j \quad \text{(orthog. direct sum)}$$

$$f \in W_j \iff f(2^j\cdot) \in W_0 \quad \forall j \in \mathbb{Z}$$

$$f \in W_0 \iff f(\cdot - k) \in W_0 \quad \forall k \in \mathbb{Z}$$

For $f \in L^2(\mathbb{R})$ put

$$f_{j,k}(x) := 2^{-\frac{1}{2}j} f(2^{-j}x - k).$$

Assume (1) – (5). Then $\{\phi_{j,k}\}_{k\in\mathbb{Z}}$ is ON basis of V_j. We want $\psi \in W_0$ s.t. $\{\psi(\cdot - k)\}_{k\in\mathbb{Z}}$ is ON basis of W_0.

Then $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is ON basis of $L^2(\mathbb{R})$

Main results

Assume (1) – (5).

Prop. 1. $\exists\, (h_n)_{n\in\mathbb{Z}}$ with $\sum_{n\in\mathbb{Z}} |h_n|^2 = 1$ such that

$$\phi(x) = \sqrt{2} \sum_{n\in\mathbb{Z}} h_n \, \phi(2x-n) \quad \text{in } L^2(\mathbb{R}).$$

Put

$$m_0(\xi) := \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} h_n \, e^{-in\xi} \quad \text{in } L^2(\mathbb{R}/2\pi\mathbb{Z}).$$

Then

$$\hat{\phi}(2\xi) = m_0(\xi)\, \hat{\phi}(\xi) \quad \text{a.e.}$$

and

$$|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1 \quad \text{a.e.}.$$

Prop. 2. $\exists$ isometry of Hilbert spaces

$$f \longleftrightarrow \nu_f : W_0 \longrightarrow L^2(\mathbb{R}/2\pi\mathbb{Z})$$

such that

$$\hat{f}(\xi) = e^{\frac{1}{2}i\xi}\, \overline{m_0\left(\tfrac{1}{2}\xi + \pi\right)}\, \hat{\phi}\left(\tfrac{1}{2}\xi\right) \nu_f(\xi) \quad \text{a.e.}$$

Prop. 3. Let $\psi \in W_0$. Equivalent statements:

(a) $|\nu_\psi(\xi)| = 1$ a.e.

(b) $\{\psi(\cdot - k)\}_{k\in\mathbb{Z}}$ is ON system

(c) $\{\psi(\cdot - k)\}_{k\in\mathbb{Z}}$ is ON basis

In particular, to $\nu_\psi(\xi) := \rho_0 e^{im\xi}$ $(|\rho_0| = 1)$ corresponds

$$\psi(x) = -\rho_0\sqrt{2} \sum_{n\in\mathbb{Z}} (-1)^n\, \overline{h_{-n-2m-1}}\, \phi(2x-n)$$

in $L^2(\mathbb{R})$.

ad Prop. 1

Lemma Let $f, F \in L^2(\mathbb{R})$ and $\sum_{n\in\mathbb{Z}} |c_n|^2 < \infty$ such that

$$F(\xi) = \sum_{n\in\mathbb{Z}} c_n e^{-in\xi} f(\xi) \quad \text{in } L^2(\mathbb{R}).$$

Put $g(\xi) := \sum_{n\in\mathbb{Z}} c_n e^{-in\xi}$ in $L^2(\mathbb{R}/2\pi\mathbb{Z})$.

Then $F(\xi) = g(\xi) f(\xi)$ a.e.

Proof $\exists\ N_1 < N_2 < N_3 < \cdots$ s.t.

$$\lim_{k\to\infty} \sum_{|n|\le N_k} c_n e^{-in\xi} f(\xi) = F(\xi) \quad \text{a.e.}$$

$\exists\ k_1 < k_2 < k_3 < \cdots$ s.t.

$$\lim_{j\to\infty} \sum_{|n|\le N_{k_j}} c_n e^{-in\xi} = g(\xi) \quad \text{a.e.}$$

Hence

$$F(\xi) = \lim_{j\to\infty} \sum_{|n|\le N_{k_j}} c_n e^{-in\xi} f(\xi) = g(\xi) f(\xi).$$

$\square$

Proof that $\hat{\phi}(2\xi) = m_0(\xi)\hat{\phi}(\xi)$ a.e.

$$\phi(x) = \sqrt{2} \sum_{n\in\mathbb{Z}} h_n \phi(2x-n) \quad \text{in } L^2(\mathbb{R}).$$

Hence

$$\hat{\phi}(2\xi) = \sum_{n\in\mathbb{Z}} \frac{h_n}{\sqrt{2}} e^{-in\xi} \hat{\phi}(\xi) \quad \text{in } L^2(\mathbb{R})$$

$$= \underbrace{\left(\sum_{n\in\mathbb{Z}} \frac{h_n}{\sqrt{2}} e^{-in\xi} \right)}_{\substack{= m_0(\xi) \\ \text{in } L^2(\mathbb{R}/2\pi\mathbb{Z})}} \hat{\phi}(\xi)$$

$\square$

Lemma. Let $\varphi \in L^2(\mathbb{R})$.

Then:

$\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ ON system $\Longleftrightarrow$

$\Longleftrightarrow \sum_{\ell\in\mathbb{Z}} |\hat{\varphi}(\xi + 2\ell\pi)|^2 = 1$ a.e.

Proof. $\int_{\mathbb{R}} \varphi(x)\overline{\varphi(x-k)}\, dx$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{\varphi}(\xi)|^2 e^{ik\xi}\, d\xi$$

$$= \sum_{\ell=-\infty}^{\infty} \frac{1}{2\pi}\int_{2\ell\pi}^{2(\ell+1)\pi} |\hat{\varphi}(\xi)|^2 e^{ik\xi}\, d\xi$$

$$= \sum_{\ell=-\infty}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} |\hat{\varphi}(\xi+2\ell\pi)|^2 e^{ik\xi}\, d\xi$$

$$\overset{\text{by dominated convergence}}{=} \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\sum_{\ell=-\infty}^{\infty} |\hat{\varphi}(\xi+2\ell\pi)|^2}_{\in L^1(\mathbb{R}/2\pi\mathbb{Z})} e^{ik\xi}\, d\xi .$$

$\square$

Proof that $|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1$.

$\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ ON system and

$\hat{\varphi}(2\xi) = m_0(\xi)\,\hat{\varphi}(\xi)$. Hence

$$1 = \sum_{\ell\in\mathbb{Z}} |\hat{\varphi}(2\xi + 2\ell\pi)|^2 = \sum_{\ell\in\mathbb{Z}} |m_0(\xi+\ell\pi)|^2 \cdot |\hat{\varphi}(\xi+\ell\pi)|^2$$

$$= |m_0(\xi)|^2 \sum_{\ell\in\mathbb{Z}} |\hat{\varphi}(\xi+2\ell\pi)|^2$$

$$+ |m_0(\xi+\pi)|^2 \sum_{\ell\in\mathbb{Z}} |\hat{\varphi}(\xi+\pi+2\ell\pi)|^2 = |m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 \text{ a.e.}$$

ad Prop. 2

Lemma $\exists$ isometry of Hilbert spaces $f \mapsto \sqrt{2}\, m_f : V_{-1} \longrightarrow L^2(\mathbb{R}/2\pi\mathbb{Z})$

s.t. $\hat{f}(2\xi) = m_f(\xi)\,\hat{\varphi}(\xi)$ a.e.

Proof $f \in V_{-1} \iff \exists (c_k)_{k\in\mathbb{Z}}$

s.t. $f(\frac{1}{2}x) = \sum_{k\in\mathbb{Z}} c_k \sqrt{2}\, \varphi(x-k)$ in $L^2(\mathbb{R})$.

The last identity is equivalent with

$$\hat{f}(2\xi) = \sum_{k\in\mathbb{Z}} \frac{c_k}{\sqrt{2}} e^{-ik\xi}\,\hat{\varphi}(\xi) \text{ in } L^2(\mathbb{R})$$

$$= \underbrace{\left(\sum_{k\in\mathbb{Z}} \frac{c_k}{\sqrt{2}} e^{-ik\xi}\right)}_{= m_f(\xi) \text{ in } L^2(\mathbb{R}/2\pi\mathbb{Z})} \hat{\varphi}(\xi)$$

$\square$

Observe: $m_{\varphi(\cdot - k)}(\xi) = m_0(\xi)\, e^{-2ik\xi}$

Lemma. Let $f \in V_{-1}$. Then:

$f \in W_0 \iff m_f(\xi)\,\overline{m_0(\xi)} + m_f(\xi+\pi)\,\overline{m_0(\xi+\pi)} = 0$ a.e.

Proof $f \perp \varphi(\cdot - k)\ \forall k \iff$

$\iff \forall k \quad 0 = \int_0^{2\pi} m_f(\xi)\,\overline{m_0(\xi)}\, e^{2ik\xi} d\xi$

$= \int_0^{\pi} + \int_{\pi}^{2\pi} \dots = \frac{1}{2} \int_0^{2\pi} m_f(\frac{1}{2}\xi)\,\overline{m_0(\frac{1}{2}\xi)}\, e^{ik\xi} d\xi$

$+ \frac{1}{2} \int_0^{2\pi} m_f(\frac{1}{2}\xi + \pi)\,\overline{m_0(\frac{1}{2}\xi+\pi)}\, e^{ik\xi} d\xi.$ $\square$

Proof of Prop. 2

Let $f \in V_{-1}$. Then:

$$f \in W_0$$

$$\Updownarrow$$

$$m_f(\xi)\overline{m_0(\xi)} + m_f(\xi+\pi)\overline{m_0(\xi+\pi)} = 0 \quad \text{a.e.}$$

$$\Updownarrow$$

$$(m_f(\xi), m_f(\xi+\pi)) \perp \underbrace{(m_0(\xi), m_0(\xi+\pi))}_{\text{unit vector}} \quad \text{a.e.}$$

$$\Updownarrow$$

$\exists$ L-meas. 2π-per. fct. χ s.t.

$$(m_f(\xi), m_f(\xi+\pi)) = \chi(\xi)\left(\overline{m_0(\xi+\pi)}, -\overline{m_0(\xi)}\right) \quad \text{a.e.}$$

$$\Updownarrow$$

$\exists$ L-meas. fct. χ s.t. $\chi(\xi+\pi) = -\chi(\xi)$

and $m_f(\xi) = \chi(\xi)\overline{m_0(\xi+\pi)}$ a.e.

$$\Updownarrow$$

$\exists$ L-meas. 2π-per. fct. ν s.t.

$$m_f(\xi) = e^{i\xi}\nu(2\xi)\overline{m_0(\xi+\pi)}$$

$$\Downarrow$$

$$\int_{\mathbb{R}} |f(x)|^2 dx = 2\,\frac{1}{2\pi}\int_0^{2\pi} |m_f(\xi)|^2 d\xi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} |\nu(\xi)|^2 d\xi$$

$\square$

Proof of Prop. 3

Let $\psi \in W_0$. Then

$$r_{\psi(\cdot - k)}(\xi) = r_\psi(\xi)\, e^{-ik\xi}.$$ Now:

$\{\psi(\cdot - k)\}_{k \in \mathbb{Z}}$ ON system

$$\Updownarrow$$

$$\frac{1}{2\pi}\int_0^{2\pi} |r_\psi(\xi)|^2 e^{ik\xi}\, d\xi = \delta_{k,0} \quad \forall k \in \mathbb{Z}$$

$$\Updownarrow$$

$$|r_\psi(\xi)| = 1 \quad \text{a.e.}$$

If $|r_\psi(\xi)| = 1$ a.e and $f \in W_0$, $f \perp \psi(\cdot - k)$ $\forall k \in \mathbb{Z}$ then

$$\frac{1}{2\pi}\int_0^{2\pi} r_f(\xi)\overline{r_\psi(\xi)}\, e^{ik\xi}\, d\xi = 0 \quad \forall k \in \mathbb{Z},$$

hence $r_f(\xi)\, r_\psi(\xi) = 0$ a.e.,

hence $r_f(\xi) = 0$ a.e.,

hence $f = 0$. $\square$

Example: the Haar system

(Daubechies, §1.3.3, §5.2)

Let $j \in \mathbb{Z}$. A dyadic interval of length 2^j is any interval with endpoints $k2^j$, $(k+1)2^j$ for some $k \in \mathbb{Z}$.

$$V_j := \{ f \in L^2(\mathbb{R}) \mid f \text{ is constant on all dyadic intervals of length } 2^j \}.$$

Then $(V_j)_{j \in \mathbb{Z}}$ satisfies properties (1) – (4) of the def. of multiresolution analysis

Put
$$\phi(x) := \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Then $\{\phi(\cdot - k)\}_{k \in \mathbb{Z}}$ is ON basis of V_0.

$$\phi(x) = \phi(2x) + \phi(2x-1)$$

$$h_0 = h_1 = \frac{1}{\sqrt{2}}, \quad h_k = 0 \quad \text{if } k \neq 0, 1.$$

$$m_0(\xi) = \tfrac{1}{2}(1 + e^{-i\xi})$$

$$\hat{\phi}(\xi) = \frac{e^{-i\xi} - 1}{-i\xi}$$

$$\frac{e^{-2i\xi} - 1}{-2i\xi} = \frac{e^{-i\xi} + 1}{2} \, \frac{e^{-i\xi} - 1}{-i\xi}, \quad \text{i.e. } \hat{\phi}(2\xi) = m_0(\xi)\hat{\phi}(\xi)$$

$$1 = \sum_{l \in \mathbb{Z}} |\hat{\phi}(\xi + 2l\pi)|^2 = \sum_{l \in \mathbb{Z}} \frac{|e^{-i\xi} - 1|^2}{(\xi + 2l\pi)^2}$$

$$= \frac{\sin^2 \frac{1}{2}\xi}{\pi^2} \sum_{l \in \mathbb{Z}} \frac{1}{\left(\frac{\xi}{2\pi} + l\right)^2}.$$

$$\psi(x) = \phi(2x) - \phi(2x-1), \quad \nu_\psi(\xi) = -e^{-i\xi}$$

$$\hat{\psi}(\xi) = \frac{(e^{-\frac{1}{2}i\xi} - 1)^2}{2i\xi} = -e^{-\frac{1}{2}i\xi} \, \overline{m_0(\tfrac{1}{2}\xi + \pi)} \, \hat{\phi}(\tfrac{1}{2}\xi)$$