About the book by E. M. Stein & R. Shakarchi, *Fourier analysis, an introduction*

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These notes concern the book

E. M. Stein and R. Shakarchi, *Fourier analysis, an introduction*, Princeton University Press, 2003.

Here I will rephrase some definitions and therorems from the book more generally, by using spaces of Lebesgue integrable functions. Unless otherwise stated, our function spaces are taken over $\mathbb{C}$: they will contain complex-valued functions.

Function spaces

We will use the following notations (different from the usage in the book).

- $C_{2\pi}$ is the space of 2π-periodic continuous functions on $\mathbb{R}$ with norm $\|f\|_\infty := \sup\{|f(x)| \mid x \in \mathbb{R}\} = \max\{|f(x)| \mid x \in [-\pi,\pi]\}$.
- $C^k_{2\pi}$ is the space of k times continuously differentiable 2π-periodic functions on $\mathbb{R}$.
- $C^\infty_{2\pi}$ is the space of arbitrarily often continuously differentiable 2π-periodic functions on $\mathbb{R}$.
- $L^\infty_{2\pi}$ is the space of measurable 2π-periodic functions on $\mathbb{R}$ which are bounded outside a set of measure zero. The norm is $\|f\|_\infty := \operatorname{ess\,sup}\{|f(x)| \mid x \in \mathbb{R}\} = \inf\{a \ge 0 \mid \{x \in \mathbb{R} \mid |f(x)| > a\}$ has measure $0\}$.
- $L^p_{2\pi}$ $(1 \le p < \infty)$ is the space of measurable 2π-periodic functions on $\mathbb{R}$ for which

$$\|f\|_p := \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^p\,dx\Big)^{1/p} < \infty.$$

More precisely, the elements of the L^p spaces are equivalence classes of functions, where f and g are equivalent if they coincide outside a set of measure zero (see Appendix, Theorem 1.7).

In each function class the functions may also be considered while restricted to an interval $[a, a+2\pi]$ (usually $[-\pi,\pi]$). The restriction map will be bijective if we take into account that: (i) on $C^k_{2\pi}$ we have $f(a) = f(a+2\pi)$ and $f^{(j)}(a) = f^{j)}(a+2\pi)$ for all appropriate derivatives; and (ii) the integral of an integrable 2π-periodic function is the same on any interval of length 2π.

The spaces $C_{2\pi}$ and $L^p_{2\pi}$ $(1 \le p \le \infty)$ are Banach spaces w.r.t. their given norms. The space $L^2_{2\pi}$ is moreover a Hilbert space with inner product

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\overline{g(x)}\,dx.$$

We have inclusions and corresponding norm inequalities (use Hölder's inequality) as follows:

$$C_{2\pi} \subset L^\infty_{2\pi} \subset L^q_{2\pi} \subset L^p_{2\pi} \subset L^1_{2\pi}, \quad \|f\|_\infty \ge \|f\|_q \ge \|f\|_p \ge \|f\|_1 \qquad (1 < p < q < \infty).$$

$C_{2\pi}$ is dense in $L^p_{2\pi}$ if $1 \le p < \infty$, but is is not dense in $L^\infty_{2\pi}$. If $1 \le p < q \le \infty$ then $L^q_{2\pi}$ is dense in $L^p_{2\pi}$.

Let $\mathrm{Trig}_{2\pi}$ be the space of trigonometric polynomials, i.e., the space spanned by the functions $x \mapsto e^{inx}$ $(n \in \mathbb{Z})$. Then $\mathrm{Trig}_{2\pi}$ is dense in $C_{2\pi}$ (see Corollary 2.5.4), and hence also in $L^p_{2\pi}$ $(1 \le p < \infty)$.

Let $\mathcal{R}_{2\pi}$ be the space of Riemann integrable functions on $[-\pi, \pi]$ (extended to a 2π-periodic function on $\mathbb{R}$ whenever this is convenient). Then $f \in \mathcal{R}_{2\pi}$ iff $f \in L^\infty_{2\pi}$ and f is a.e. continuous.

Let $l^\infty(\mathbb{Z})$ the space of bounded functions on $\mathbb{Z}$ with norm $\|g\|_\infty := \sup_{n\in\mathbb{Z}} |g(n)|$. Let $c_0(\mathbb{Z})$ be its subspace of functions g on $\mathbb{Z}$ for which $\lim_{|n|\to\infty} g(n) = 0$, with the same norm $\|\,.\,\|_\infty$. These are Banach spaces.

Let $l^p(\mathbb{Z})$ $(1 < p < \infty)$ be the space of functions g on $\mathbb{Z}$ such that

$$\|g\|_p := \Big(\sum_{n\in\mathbb{Z}} |g|^p\Big)^{1/p} < \infty.$$

These are Banach spaces, and $l^2(\mathbb{Z})$ is moreover a Hilbert space with inner product

$$\langle g, h\rangle := \sum_{n\in\mathbb{Z}} g(n)\, \overline{h(n)}.$$

The embeddings of spaces and the norm inequalities are here in converse direction as for the function spaces on $[-\pi, \pi]$:

$$l^1(\mathbb{Z}) \subset l^p(\mathbb{Z}) \subset l^q(\mathbb{Z}) \subset c_0(\mathbb{Z}) \subset l^\infty(\mathbb{Z}), \quad \|g\|_1 \ge \|g\|_p \ge \|g\|_q \ge \|g\|_\infty \quad (1 < p < q < \infty).$$

All embeddings are dense, except for the last one: $c_0(\mathbb{Z})$ is a non-dense closed subspace of $l^\infty(\mathbb{Z})$.

Fourier coefficients and the Riemann-Lebesgue lemma

For $f \in L^1_{2\pi}$ write its *Fourier coefficients* as

$$\widehat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx \qquad (n \in \mathbb{Z}).$$

If $f \in L^1_{2\pi}$ then

$$\|\widehat{f}\|_\infty \le \|f\|_1 \quad \text{and} \quad \lim_{|n|\to\infty} \widehat{f}(n) = 0.$$

Then $f \mapsto \widehat{f}\colon L^1_{2\pi} \to c_0(\mathbb{Z})$ is continuous, but not surjective. The statement $\lim_{|n|\to\infty} \widehat{f}(n) = 0$ is the *Riemann-Lebesgue lemma*, This was stated for $f \in \mathcal{R}_{2\pi}$ in Theorem 3.1.4, as a corollary of Theorem 1.3. But the result extends to $L^1_{2\pi}$ by an argument which uses that $\mathcal{R}_{2\pi}$ (or already $C_{2\pi}$ or $\mathrm{Trig}_{2\pi}$) is dense in $L^1_{2\pi}$.

More generally than Theorem 1.3 we can state:

The map $f \mapsto \widehat{f}$ is a Hilbert space isomorphism from $L^2_{2\pi}$ onto $l^2(\mathbb{Z})$.

Re: Theorem 2.2.1

This can be formulated more generally as:
If $f \in L^1_{2\pi}$ and $\widehat{f} = 0$ then $f(x_0) = 0$ whenever f is continuous at x_0.

Re: §2.3 Convolutions

In connection with Proposition 2.3.1 we have (special cases of *Young's inequality*):

- $L^1_{2\pi} * L^p_{2\pi} \subset L^p_{2\pi}, \quad \|f * g\|_p \leq \|f\|_1 \, \|g\|_p \quad (1 \leq p \leq \infty);$
- $L^p_{2\pi} * L^q_{2\pi} \subset C_{2\pi}, \quad \|f * g\|_\infty \leq \|f\|_p \, \|g\|_q \quad (1 < p < \infty,\ p^{-1} + q^{-1} = 1);$
- $L^1_{2\pi} * C_{2\pi} \subset C_{2\pi}, \quad \|f * g\|_\infty \leq \|f\|_1 \, \|g\|_\infty.$

Re: §2.4 Good kernels

More generally we can call $\{K_n\}_{n=1}^\infty$ a *family of good kernels* if $K_n \in L^1_{2\pi}$ for all n such that:

a) $\widehat{K_n}(0) = 1$ for all n;

b) there exists $M > 0$ such that $\|K_n\|_1 \leq M$ for all n;

c) for every $\delta \in (0, \pi)$ we have $\|K_n(\chi_{[-\pi,-\delta]} + \chi_{[\delta,\pi]})\|_1 \to 0$ as $n \to \infty$.
(Here χ_E is the characteristic function of a set E.)

For a given family of good kernels as above we can now formulate Theorem 4.1 as follows:

Let $f \in L^\infty_{2\pi}$. Then $\lim_{n\to\infty}(f * K_n)(x) = f(x)$ whenever f is continuous at x. If f is continuous everywhere, then this limit is uniform.

If we replace in c) of the definition of good kernel the L^1-norm by the L^p-norm and if we replace in the reformulation of Theorem 4.1 the assumption $f \in L^\infty_{2\pi}$ by $f \in L^q_{2\pi}$ ($p^{-1} + q^{-1} = 1$ and $1 \leq p \leq \infty$) then the Theorem remains valid. For the Fejér and the Poisson kernel we can make these changes for all p, in particular for $p = \infty$.

Theorem 4.1, in all its reformulated versions, has as an immediate Corollary:

Let $f \in L^r_{2\pi}$ ($1 \leq r < \infty$). Then $\lim_{n\to\infty} f * K_n = f$ in $L^r_{2\pi}$.

Prove it first for $f \in C_{2\pi}$ and then use density of $C_{2\pi}$ in $L^r_{2\pi}$.

From this it follows that the map $f \mapsto \widehat{f} \colon L^r_{2\pi} \to c_0(\mathbb{Z})$ is injective for $1 \leq r < \infty$, in particular for $r = 1$ (we knew it already for $r = 2$).

Re: §3.2.1 A local result

Theorem 3.2.1 can be formulated more generally as follows:

Let $f \in L^1_{2\pi}$, $x_0 \in \mathbb{R}$. Suppose that there are $\delta, M > 0$ such that $|f(x) - f(x_0)| \leq M\,|x - x_0|$ if $|x - x_0| < \delta$. Then $\lim_{N\to\infty}(S_N[f])(x_0) = f(x_0)$.

Theorem 3.2.2 can be formulated more generally as follows:

Let $f, g \in L^1_{2\pi}$, $x_0 \in \mathbb{R}$, and suppose that $f(x) = g(x)$ for x in some neigbourhood of x_0.

Then either $(S_N[f])(x_0)$ and $(S_N[g])(x_0)$ both converge as $N\to\infty$, while tending to the same limit, or both diverge as $N\to\infty$.

Re: Exercise 3.13, Fourier coefficients of C^∞-function

Exercise 3.13 can be formulated more generally as follows:
The map $f\mapsto\widehat{f}$ is a bijection from $C^\infty_{2\pi}$ onto the space of rapidly decreasing *functions g on $\mathbb{Z}$, i.e., g such that for all $k>0$ we have $g(n)=O(|n|^{-k})$ as $|n|\to\infty$.*

Re: §3.2.2 A continuous function with diverging Fourier series

Lemma 3.2.3 can be applied more generally to Fourier series, not just to the Fourier series (4) in §3.2.2:
If $f\in L^\infty_{2\pi}$ and $\widehat{f}(n)=O(|n|^{-1})$ as $|n|\to\infty$ then $\|S_N[f]\|_\infty=O(1)$ as $N\to\infty$.

Re: Exercises 3.19, 3.20, The Gibbs phenomenon

We can write

$$D_N(x)=\frac{\sin((N+\frac12)x)}{\sin(\frac12 x)}=\frac{2\sin(Nx)}{x}+\theta(x)\,\sin(Nx)+\cos(Nx)\qquad(0<|x|<2\pi),$$

where

$$\theta(x):=\cot(\tfrac12 x)-2x^{-1}.$$

Put $\theta(0):=0$. Then θ is an odd, differentiable and strictly decreasing function on $(-2\pi,2\pi)$ with $\theta'(0)=-\frac16$, $\theta(\pm\pi)=\mp2\pi^{-1}$ and $\max_{\pi\le x\le\pi}|\theta(x)|=2\pi^{-1}$. Thus

$$\begin{aligned}\sum_{n=1}^N\frac{\sin(nx)}{n}&=\tfrac12\int_0^x(D_N(t)-1)\,dt=\int_0^x\frac{\sin(Nt)}{t}\,dt+\tfrac12\int_0^x\theta(t)\,\sin(Nt)\,dt+\frac{\sin(Nx)}{2N}-\tfrac12 x\\&=\int_0^{Nx}\frac{\sin s}{s}\,ds-\frac{\theta(x)\cos(Nx)}{2N}+\frac{1}{2N}\int_0^x\theta'(t)\,\cos(Nt)\,dt+\frac{\sin(Nx)}{2N}-\tfrac12 x.\\&=\int_0^{Nx}\frac{\sin s}{s}\,ds-\tfrac12 x+O(N^{-1})=O(1)\quad\text{as }N\to\infty\text{, uniformly for }x\in[-\pi,\pi].\end{aligned}$$

This answers Exercise 3.19. For Exercise 3.20 we can write (f the sawtooth function):

$$(S_N[f])(x)-f(x)=\int_0^{Nx}\frac{\sin s}{s}\,ds-\tfrac12\pi+O(N^{-1})\quad\text{as }N\to\infty\text{, uniformly for }x\in(0,\pi].$$

Now use that the function $y\mapsto\int_0^y\frac{\sin s}{s}\,ds$ is nonnegative on $[0,\infty)$ and increasing on $[0,\pi]$, that it attains its absolute maximum (approximately $1.18\,\pi/2$) for $y=\pi$, and that it tends to $\pi/2$ as $y\to\infty$.

Re: §4.3 A continuous but nowhere differentiable function

Lemma 4.3.2 can be formulated more generally as follows:
For the Fejér kernel F_N we have:

$|F_N'(x)| = O(N^2)$ *as* $N \to \infty$, *uniformly for* $x \in [-\pi,\pi]$*;*

$|x^2 F_N'(x)| = O(1)$ *as* $N \to \infty$, *uniformly for* $x \in [-\pi,\pi]$*.*

If $g \in L^\infty_{2\pi}$ *and if* g *is differentiable at* x_0 *then* $(\sigma_N[g])'(x_0) = O(\log N)$ *as* $N \to \infty$*.*

Re: §5.1.2 Definition of the Fourier transform

More generally, the Fourier transform

$$\widehat{f}(\xi) := \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x\xi}\, dx$$

is well defined if $f \in L^1(\mathbb{R})$. In fact, $\widehat{f}$ is then continuous on $\mathbb{R}$ and $\widehat{f}(\xi) = o(1)$ as $|\xi| \to \infty$. These statements generalize Exercise 5.5(a). The first statement follows because $\widehat{f}(\xi) \to \widehat{f}(\xi_0)$ as $\xi \to \xi_0$ by dominated convergence. As for the second statement (Riemann-Lebesgue lemma for the Fourier transform on $\mathbb{R}$), prove it first if f is a characteristic function $\chi_{[a,b]}$ and next use density of the step functions in $L^1(\mathbb{R})$. Thus $f \mapsto \widehat{f}$ is a bounded linear map of $L^1(\mathbb{R})$ into $C_0(\mathbb{R})$ (the space of continuous functions on $\mathbb{R}$ which tend to 0 at $\pm\infty$):

$$\|\widehat{f}\|_\infty \le \|f\|_1.$$

Re: §5.1.4 The Fourier transform on $\mathcal{S}$

Proposition 5.1.2 remains valid for $f \in L^1(\mathbb{R})$. For part (iv) we have to add the additional condition that $f'(x)$ exists almost everywhere, that $f' \in L^1(\mathbb{R})$, and that $f(x) = \int_{-\infty}^x f'(y)\, dy$. For part (v) we have to add that $x \mapsto x f(x)$ is in $L^1(\mathbb{R})$. Part of the conclusion in (v) is then that $\widehat{f}$ is differentiable.

Proof of Theorem 5.1.4 We can also prove this by complex analysis: if $f(x) = e^{-\pi x^2}$ then

$$\begin{aligned}\widehat{f}(\xi) &= e^{-\pi\xi^2} \int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2}\, dx \\ &= e^{-\pi\xi^2} \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = e^{-\pi\xi^2},\end{aligned}$$

where we used in the second equality Cauchy's theorem and estimates on $\int e^{-\pi z^2}\, dz$ over contours $-\infty, -M, -M+i\xi, -\infty+i\xi$ and $\infty, M, M+i\xi, \infty+i\xi$.

Good kernels More generally than is written in the book after Corollary 5.1.5 we can define a family $\{K_\delta\}_{\delta>0}$ as a *family of good kernels* if $K_\delta \in L^1(\mathbb{R})$ and (i), (ii), (iii) are satisfied.
We call K a *very good kernel* if moreover:

(iv) For every $\eta > 0$ we have $\sup_{|x|\ge\eta} |K_\delta(x)| \to 0$ as $\delta \downarrow 0$.

For instance, K_δ given by Corollary 5.1.5 is a very good kernel.

Convolution Formula (5.7), i.e.,

$$(f * g)(x) := \int_{-\infty}^{\infty} f(x-t)\, g(t)\, dt,$$

defines more generally $(f*g)(x)$ for almost all x if $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ $(1 \le p \le \infty)$. Then $f * g \in L^p(\mathbb{R})$ and we have norm estimates

$$\|f * g\|_p \le \|f\|_1\, \|g\|_p,$$

in particular,

$$\|f * g\|_1 \le \|f\|_1\, \|g\|_1 \quad \text{and} \quad \|f * g\|_\infty \le \|f\|_1\, \|g\|_\infty.$$

Corollary 5.1.7 This can be formulated more generally as follows.
Let $\{K_\delta\}$ be a family of good kernels. If $f \in L^\infty(\mathbb{R})$ and f is continuous at x_0 then $(f * K_\delta)(x_0) \to f(x_0)$ as $\delta \downarrow 0$.
If $\{K_\delta\}$ is a family of very good kernels, if $f \in L^1(\mathbb{R})$ and f is continuous at x_0, then $(f * K_\delta)(x_0) \to f(x_0)$ as $\delta \downarrow 0$.
In both cases, if f is moreover uniformly continuous on $\mathbb{R}$ then $f * K_\delta \to f$ uniformly on $\mathbb{R}$ as $\delta \downarrow 0$.

Re: §5.1.5 The Fourier inversion

Proposition 1.8 holds more generally for $f, g \in L^1(\mathbb{R})$. For the proof use Fubini's theorem.

Proof of Theorem 1.9 (Fourier inversion) We can prove more generally:
If $f, \widehat{f}, g, \widehat{g} \in L^1(\mathbb{R})$ and if moreover f, g are continuous and bounded on $\mathbb{R}$ (for instance if $f, g \in \mathcal{S}(\mathbb{R})$), then

$$g(0) \int_{-\infty}^{\infty} \widehat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi = f(x) \int_{-\infty}^{\infty} \widehat{g}(y)\, dy.$$

This can be proved first for $x = 0$, by using dominated convergence as $\delta \downarrow 0$ in

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)\, g(\delta\xi)\, d\xi = \int_{-\infty}^{\infty} f(x)\, \delta^{-1} \widehat{g}(\delta^{-1} x)\, dx = \int_{-\infty}^{\infty} f(\delta y)\, \widehat{g}(y)\, dy.$$

Then we obtain

$$g(0) \int_{-\infty}^{\infty} \widehat{f}(\xi)\, d\xi = f(0) \int_{-\infty}^{\infty} \widehat{g}(y)\, dy.$$

Finally, put $f(x) := h(x+a)$, so $\widehat{f}(\xi) = e^{2\pi i a \xi} \widehat{h}(\xi)$, in order to get the result for $x := a$.

So the inversion formula of the Fourier transform for general f as above can be formulated in terms of a special g as above with $g(0) \ne 0$ as follows:

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi = C\, f(x) \quad \text{with} \quad C = \frac{1}{g(0)} \int_{-\infty}^{\infty} \widehat{g}(y)\, dy.$$

We already know (see Theorem 5.1.4) $g(x) := e^{-\pi x^2}$ with $\widehat{g} = g$ and $g(0) \ne 0$ (in fact $g(0) = 1$). So $C = \widehat{g}(0)/g(0) = g(0)/g(0) = 1$.

Re: §5.1.6 The Plancherel formula

In the proof of Proposition 5.1.11 it is sketched how to see that $f, g \in \mathcal{S}(\mathbb{R})$ implies $f * g \in \mathcal{S}(\mathbb{R})$. The inequality $\sup_x |x|^l\,|g(x-y)| \le A_l(1+|y|)^l$ is used. The hint for the proof of this inequality distinguishes two cases. For the case $|x| \ge 2|y|$ note that then $|x| \le |x-y| + |y| \le |x-y| + |x| - |y| \le 2|x-y|$. This inequality can also be proved by an application of the binomial formula:

$$|x|^l\,g(x-y)| \le \sum_{j=0}^{l} \binom{l}{j} |y|^{l-j}\,|x-y|^j\,|g(x-y)| \le A_l \sum_{j=0}^{l} \binom{l}{j} |y|^{l-j} = A_l(1+|y|)^l,$$

where $A_l := \sup\{|x|^j\,|g(x)| \mid x \in \mathbb{R},\ j = 0, 1, \ldots, l\}$.

Yet another proof that $f * g$ decreases faster than any inverse power if f and g do so, runs as follows.

$$\begin{aligned}|(f*g)(x)| &\le \int_{-\infty}^{\infty} |f(y)|\,|g(x-y)|\,dy = \int_{-\infty}^{\infty} |f(\tfrac{1}{2}x+y)|\,|g(\tfrac{1}{2}x-y)|\,dy \\ &\le C\int_{-\infty}^{\infty} \frac{1}{(1+(\frac{1}{2}x+y)^2)^l\,(1+(\frac{1}{2}x-y)^2)^l}\,dy \le C\int_{-\infty}^{\infty} \frac{1}{(1+\frac{1}{2}x^2+2y^2)^l}\,dy \\ &\le \frac{C}{(1+\frac{1}{2}x^2)^{l-1}} \int_{-\infty}^{\infty} \frac{dy}{1+2y^2}\,.\end{aligned}$$

Re: §5.1.7 Extension to functions of moderate decrease

There is a further extension of the Fourier transform, the Fourier inversion formula and the Plancherel formula to $L^2(\mathbb{R})$. In fact, since $\mathcal{F}\colon f \to \widehat{f}$ is a unitary transformation on $\mathcal{S}(\mathbb{R})$ (Theorem 5.1.12) and since $\mathcal{S}$ is dense in $L^2(\mathbb{R})$, $\mathcal{F}$ extends uniquely to a unitary transformation $\mathcal{F}_2\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$. So, if $f \in L^2(\mathbb{R})$, (f_n) is a sequence in $\mathcal{S}(\mathbb{R})$ and $f = \lim_{n\to\infty} f_n$ in $L^2(\mathbb{R})$, then $\mathcal{F}_2 f := \lim_{n\to\infty} \mathcal{F}(f_n)$ in $L^2(\mathbb{R})$, independent of the choice of the sequence (f_n). On the other hand, for $f \in L^1(\mathbb{R})$, $\mathcal{F}(f)$ can be defined as

$$\mathcal{F}(\xi) = \widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\,e^{-2\pi i x\xi}\,dx.$$

So on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ we have two definitions $\mathcal{F}$ and $\mathcal{F}_2$ of the Fourier transform. We will show that they agree.

Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Take $\varepsilon > 0$. Then there exists $M > 1$ such that $\|f\chi_{[-M,M]} - f\|_1 < \varepsilon$ and $\|f\chi_{[-M,M]} - f\|_2 < \varepsilon$. Then there exists $g \in \mathcal{S}(\mathbb{R})$ with support within $[-2M, 2M]$ such that $\|g - f\chi_{[-M,M]}\|_2 < (2M)^{-\frac{1}{2}}\varepsilon < \varepsilon$. Hence $\|g - f\chi_{[-M,M]}\|_1 < \varepsilon$. Hence $\|f-g\|_1 < 2\varepsilon$ and $\|f-g\|_2 < 2\varepsilon$. Hence $\|\mathcal{F}f - \mathcal{F}g\|_\infty < 2\varepsilon$ and $\|\mathcal{F}_2 f - \mathcal{F}g\|_2 = \|\mathcal{F}_2 f - \mathcal{F}_2 g\|_2 = \|f-g\|_2 < 2\varepsilon$. Since $\varepsilon > 0$ was taken arbitrarily, there is a sequence (g_n) in $\mathcal{S}(\mathbb{R})$ such that $\|\mathcal{F}f - \mathcal{F}g_n\|_\infty \to 0$ and $\|\mathcal{F}_2 f - \mathcal{F}g_n\|_2 \to 0$ as $n \to \infty$. Since a converging sequence in $L^2(\mathbb{R})$ has an almost everywhere converging subsequence, we see, by replacing (g_n) by a suitable subsequence again written as (g_n), that $\|\mathcal{F}f - \mathcal{F}g_n\|_\infty \to 0$ and $\mathcal{F}_2 f - \mathcal{F}g_n \to 0$ almost everywhere as $n \to \infty$. Thus $\mathcal{F}f = \mathcal{F}_2 f$ almost everywhere.

Thus we are entitled to use the notation $\mathcal{F}f = \widehat{f}$ unambiguously for $f \in L^1(\mathbb{R}) \cup L^2(\mathbb{R})$.

Let $f \in L^2(\mathbb{R})$. Since $f\chi_{[-n,n]} \to f$ in $L^2(\mathbb{R})$ as $n \to \infty$, we have $\mathcal{F}(f\chi_{[-n,n]}) \to \mathcal{F}(f)$ in $L^2(\mathbb{R})$ as $n \to \infty$. Thus $\widehat{f}$ is the function in $L^2(\mathbb{R})$ which is almost everywhere determined by

$$\int_{-\infty}^{\infty} \left| \widehat{f}(\xi) - \int_{-n}^{n} f(x)\, e^{-2\pi i x\xi}\, dx \right|^2 d\xi \to 0 \quad \text{as } n \to \infty.$$

The inverse Fourier transform on $L^2(\mathbb{R})$ can be treated in a similar way. For $f \in L^2(\mathbb{R})$ we get:

$$\int_{-\infty}^{\infty} \left| \widehat{f}(x) - \int_{-n}^{n} f(\xi)\, e^{2\pi i x\xi}\, d\xi \right|^2 dx \to 0 \quad \text{as } n \to \infty.$$

Re: §5.3.1 Theta and zeta functions

Theorem 5.3.1 applied to

$$\Theta(z \mid \tau) := \sum_{n=-\infty}^{\infty} e^{i\pi n^2 \tau}\, e^{2\pi i n z} \quad (\mathrm{Im}\,(\tau) > 0,\ z \in \mathbb{C})$$

gives

$$\Theta(z \mid \tau) = (-i\tau)^{-\frac{1}{2}}\, e^{-i\pi z^2/\tau}\, \Theta(z/\tau \mid -\tau^{-1}).$$

For this we need that the Fourier transform of $f(x) := e^{i\pi\tau x^2}$ is equal to $\widehat{f}(\xi) = (-i\tau)^{-\frac{1}{2}}\, e^{-i\pi\xi^2/\tau}$ if $\mathrm{Im}\,(\tau) > 0$. This can be reduced to the known case $i\tau < 0$ by deforming the contour defining the Fourier integral, where Cauchy's theorem and suitable estimates have to be used.

We see from the above two formulas that $\Theta(z \mid \tau)$ is complex analytic in z with period 1 and that $e^{i\pi z^2/\tau}\Theta(z \mid \tau)$ is complex analytic in z with period τ. The quotient

$$\frac{\Theta(z \mid \tau)}{\Theta(z + \frac{1}{2} \mid \tau)} = e^{i\pi/(4\tau)}\, e^{i\pi z/\tau}\, \frac{\Theta(z/\tau \mid -\tau^{-1})}{\Theta((z + \frac{1}{2})/\tau \mid -\tau^{-1})}$$

has periods 1 and 2τ, but it will have poles. It is essentially an *elliptic function.*

These poles are more visible from Jacobi's triple product formula. Note that $\Theta(z \mid \tau)$ is equal to the classical theta function $\theta_3(z \mid \tau) = \theta_3(z, q)$ with $q = e^{i\pi\tau}$, so $q \in \mathbb{C}$, $|q| < 1$. See A. Erdélyi, *Higher transcendental functions, Vol. 2*, formula 13.19 (8). With the notation $(a; q)_\infty := \prod_{k=0}^{\infty}(1 - aq^k)$ we have *Jacobi's triple product formula*

$$\theta_3(z, q) = (q^2; q^2)_\infty\, (-qe^{2\pi i z}; q^2)_\infty\, (-qe^{-2\pi i z}; q^2)_\infty,$$

see A. Erdélyi, *Higher transcendental functions, Vol. 2*, formula 13.19 (16).