More on Fourier integrals

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This note gives a minor extension to Chap. 5 of the book *Fourier analysis, an introduction* by E. M. Stein and R. Shakarchi.

Dense subspaces of $L^p(\mathbb{R})$

By a *simple function* on $\mathbb{R}$ we mean a finite linear combination of characteristic functions χ_E of measurable subsets E of $\mathbb{R}$. In particular, a simple function on $\mathbb{R}$ is integrable iff it is a finite linear combination of characteristic functions χ_E with $\lambda(E) < \infty$.
The Lebesgue measure λ on $\mathbb{R}$ is *regular*, i.e., for every measurable set $E \subset \mathbb{R}$ we have:

$$\lambda(E) = \inf\{\lambda(V) : E \subset V \text{ and } V \text{ open}\},$$
$$\lambda(E) = \sup\{\lambda(K) : K \subset E \text{ and } K \text{ compact}\}.$$

This follows from Theorem 2.18 in Rudin, *Real and complex analysis.*

Proposition Let $1 \le p < \infty$. The following spaces are dense in $L^p(\mathbb{R})$:

1. The space of integrable simple functions on $\mathbb{R}$.
2. The linear span of the characteristic functions of bounded intervals in $\mathbb{R}$.
3. The space of continuous functions on $\mathbb{R}$ of *compact support*, i.e., which vanish outside some bounded interval.

Proof We will prove these results for $p = 1$. The proof for other p is similar.
Proof of 1. Every $f \in L^1(\mathbb{R})$ can be written as $f = f_1 - f_2 + if_3 - if_4$ with f_1, f_2, f_3, f_4 nonnegative L^1 functions. So it is sufficient to prove that every nonnegative L^1 function f can be approximated in L^1 norm by integrable simple functions. There is an increasing sequence of nonnegative simple functions $t_n(x)$ which tend pointwise to f as $n \to \infty$. Then $\int t_n$ tends to $\int f$ as $n \to \infty$, so $\|f - t_n\|_1 \to 0$.

Proof of 2. By 1. it is sufficient to prove that, if $E \subset \mathbb{R}$ is measurable with $\lambda(E) < \infty$ then χ_E can be approximated in L^1 norm by finite linear combinations of characteristic functions of bounded intervals. Let $\varepsilon > 0$. By regularity of λ there is an open set $V \supset E$ such that $\lambda(V) < \lambda(E) + \frac{1}{2}\varepsilon < \infty$. Since V is a countable disjoint union of open intervals, there is a finite union $W \subset V$ of bounded open intervals such that $\lambda(W) > \lambda(V) - \frac{1}{2}\varepsilon$. Hence $\|\chi_E - \chi_W\| < \varepsilon$.

Proof of 3. Every characteristic function of a bounded interval can be approximated in L^1 norm by continuous functions of compact support. Now use 2. □

We can use part 2. of this Proposition in order to prove the *Riemann-Lebesgue Lemma* for the Fourier transform:

If $f \in L^1(\mathbb{R})$ then $\widehat{f}(\xi) \to 0$ as $\xi \to \pm\infty$.

Just observe that the statement is true for $f = \chi_{[a,b]}$.

Exercises

Exercise 1. (For this exercise use results from both Fourier series and Fourier integrals.) Below define $x^{-1} \sin x$ for $x = 0$ by continuity.

a) Let $t \in \mathbb{R}$. Show that for each $x \in (-\pi, \pi)$ we have

$$\sum_{n=-\infty}^{\infty} \frac{\sin(\pi(t-n))}{\pi(t-n)}\, e^{inx} = e^{ixt}$$

with pointwise convergence. What is the evaluation of the sum on the left-hand side for other real values of x?

b) Show that, for all $n, m \in \mathbb{Z}$, we have

$$\int_{-\infty}^{\infty} \frac{\sin(\pi(t-n))}{\pi(t-n)} \frac{\sin(\pi(t-m))}{\pi(t-m)}\, dt = \delta_{n,m},$$

where the integral converges absolutely.

c) Does there exist $f \in L^2(\mathbb{R})$ with $f \neq 0$ such that

$$\int_{-\infty}^{\infty} f(t)\, \frac{\sin(\pi(t-n))}{\pi(t-n)}\, dt = 0 \quad \text{for all } n \in \mathbb{Z}\,?$$

d) Let $f \in L^2([-\pi, \pi])$. Define $\widehat{f}$ as a function on $\mathbb{R}$ by

$$\widehat{f}(t) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-ixt}\, dx \qquad (t \in \mathbb{R}). \tag{1}$$

(For $t \in \mathbb{Z}$ this defines the Fourier coefficients of f; for general $t \in \mathbb{R}$ this defines the Fourier transform of a function on $\mathbb{R}$ which vanishes outside $[-\pi, \pi]$.) Show that

$$\widehat{f}(t) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)\, \frac{\sin(\pi(t-n))}{\pi(t-n)} \qquad (t \in \mathbb{R}) \tag{2}$$

with absolutely convergent sum.
(This shows in particular the following. Let g be an L^2 function on $\mathbb{R}$ which is the Fourier transform $g = \widehat{f}$ of an L^2 function f on $\mathbb{R}$ vanishing outside $[-\pi, \pi]$ (see (1)). So g is also continuous. Then g is completely determined by its restriction to $\mathbb{Z}$, with reconstruction formula given by (2).)

Hints to Problem 7 in Chapter 5 of Stein & Shakarchi

As for (b), show that, if f is continuous and of moderate decrease and if for all $k \in \mathbb{Z}_{\geq 0}$ we have

$$\int_{-\infty}^{\infty} f(y)\, e^{-y^2}\, y^k\, dy = 0 \tag{3}$$

then we have for all $x \in \mathbb{R}$ that

$$\int_{-\infty}^{\infty} f(y)\, e^{-y^2}\, e^{2xy}\, dy = 0,$$

and hence, by Ch.5, Exercise 8, $f = 0$. Now try to go from the assumption in (b) of Problem 7 to equation (3) above.

As for (c), a result from (a) is:

$$h_k(x) = (-1)^k e^{x^2/2} \left(\frac{d}{dx}\right)^k e^{-x^2}.$$

Show, by once differentiating this formula, that

$$h_{k+1}(x) = \left(x - \tfrac{d}{dx}\right) h_k(x). \tag{4}$$

Use (4) in (c) in order to prove the result there by induction with respect to k.

Now show, by applying the operator $x + \frac{d}{dx}$ to both sides of (4) and by using induction with respect to k, that

$$\left(x + \tfrac{d}{dx}\right) h_{k+1}(x) = 2(k+1) h_k(x). \tag{5}$$

Now it has to be proved in (d) that

$$(Lh_k)(x) := \left(x^2 - \tfrac{d^2}{dx^2}\right) h_k(x) = (2k+1) h_k(x).$$

Show this by expressing the operator L in terms of $x - \frac{d}{dx}$ and $x + \frac{d}{dx}$ and by using (4) and (5).