Limits for *BC* Jacobi polynomials

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Partitions

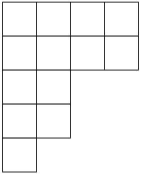
$\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$.

$|\lambda| := \sum_{i=1}^{n} \lambda_i$

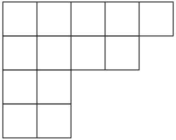
$\ell(\lambda) := \max\{j \mid \lambda_j > 0\}$

λ is called a *partition* of *weight* $|\lambda|$ and of *length* $\ell(\lambda)$ (with maximal length n).

A partition λ is often diagramatically displayed by a Young tableau, consisting of boxes (i, j) with $i = 1, \ldots, \ell(\lambda)$ and $j = 1, \ldots, \lambda_i$ for a given i. Its conjugate λ' is obtained by reflection of the tableau of λ with respect to the diagonal. For instance, $\lambda = (4, 4, 2, 2, 1)$ and $\lambda' = (5, 4, 2, 2)$ are displayed by



and



Partitions (cntd.)

Dominance partial order:

$\lambda \le \mu \;\Leftrightarrow\; \sum_{i=1}^m \lambda_i \le \sum_{i=1}^m \mu_i \quad (m = 1, \ldots, n)$.

Inclusion partial order: $\lambda \subseteq \mu \;\Leftrightarrow\; \lambda_i \le \mu_i \quad (i = 1, \ldots, n)$

(so the tableau of λ is a subset of the tableau of μ).

Note that $\lambda \subseteq \mu \;\Rightarrow\; \lambda \le \mu$.

Symmetrized monomials:

$$m_\lambda(x) := \sum_{\mu \in S_n \lambda} x^\mu \quad (x^\mu := x_1^{\mu_1} \ldots x_n^{\mu_n},\ S_n \text{ is symmetric group}).$$

The m_λ form a basis of the space of symmetric polynomials in n variables.

Jacobi polynomials

Jacobi polynomials $P_n^{(\alpha,\beta)}$ $(\alpha, \beta > -1)$ satisfy the orthogonality

$$\int_0^1 P_n^{(\alpha,\beta)}(1-2x)\, P_m^{(\alpha,\beta)}(1-2x)\, x^\alpha (1-x)^\beta\, dx = 0 \quad (n \neq m).$$

They are ${}_2F_1$ hypergeometric polynomials:

$$P_n^{(\alpha,\beta)}(1-2x) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; x\right),$$

where $(a)_k := a(a+1)\ldots(a+k-1)$ is the Pochhammer symbol and

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) := \sum_{k=0}^\infty \frac{(a)_k\,(b)_k}{(c)_k\, k!}\, z^k,$$

terminating after term with $k = n$ if $a = -n$ $(n = 0, 1, \ldots)$.

Orthogonal symmetric polynomials

Let $\{p_n\}_{n=0}^{\infty}$ be a system of *monic* polynomials p_n of degree n satisfying the orthogonality

$$\int_0^1 p_n(x)\,p_m(x)\,w(x)\,dx = 0 \quad (n \neq m)$$

for some integrable weight function $w \geq 0$ on $[0,1]$.

For $\lambda = (\lambda_1, \ldots, \lambda_n)$ a partition define

$$p_\lambda^0 := \sum_{\mu \in S_n \lambda} p_{\mu_1}(x_1) \ldots p_{\mu_n}(x_n).$$

Then $p_\lambda^0 = \sum_{\mu;\, \mu \subseteq \lambda} c_{\lambda,\mu} m_\mu$ with $c_{\lambda,\lambda} = 1$.

Orthogonality relation:

$$\int_{[0,1]^n} p_\lambda^0(x)\,p_\mu^0(x)\,w(x_1)\ldots w(x_n)\,dx_1 \ldots dx_n = 0 \quad (\lambda \neq \mu).$$

Orthogonal symmetric polynomials (cntd.)

$\rho := (n-1, n-2, \ldots, 1, 0)$; $\varepsilon(\sigma)$ is the sign of $\sigma \in S_n$.

$$\Delta(x) := \prod_{1\le i<j\le n} (x_i - x_j) = \sum_{\sigma\in S_n} \varepsilon(\sigma)\, x^{\sigma\rho} \qquad \text{(Vandermonde)}.$$

$$s_\lambda(x) := \frac{1}{\Delta(x)} \sum_{\sigma\in S_n} \varepsilon(\sigma) x^{\sigma(\lambda+\rho)} \qquad \text{(Schur polynomial)}.$$

Then $s_\lambda = \sum_{\mu;\, \mu\le\lambda} c_{\lambda,\mu} m_\mu$ with $c_{\lambda,\lambda} = 1$.

$$p^1_\lambda(x) := \frac{1}{\Delta(x)} \sum_{\mu=\sigma(\lambda+\rho);\, \sigma\in S_n} \varepsilon(\sigma)\, p_{\mu_1}(x_1) \ldots p_{\mu_n}(x_n).$$

Then $p^1_\lambda = \sum_{\mu;\, \mu\subseteq\lambda} a_{\lambda,\mu} s_\mu = \sum_{\mu;\, \mu\le\lambda} b_{\lambda,\mu} m_\mu$ with $a_{\lambda,\lambda} = b_{\lambda,\lambda} = 1$.

Orthogonality relation:

$$\int_{[0,1]^n} p^1_\lambda(x)\, p^1_\mu(x)\, w(x_1)\ldots w(x_n)\, \Delta(x)^2\, dx_1 \ldots dx_n = 0 \quad (\lambda \ne \mu).$$

The cases p_λ^0 and p_λ^1 suggest to define for a coupling constant $\tau \geq 0$ a system $\{p_\lambda^\tau\}$ such that

(i) $p_\lambda^\tau = m_\lambda + \sum_{\mu;\, \mu<\lambda} c_{\lambda,\mu} m_\mu$;

(ii) $\int_{[0,1]^n} p_\lambda^\tau(x)\, m_\mu(x)\, w(x_1) \ldots w(x_n)\, |\Delta(x)|^{2\tau}\, dx_1 \ldots dx_n = 0$ if $\mu < \lambda$.

Then $\int_{[0,1]^n} p_\lambda^\tau(x)\, p_\mu^\tau(x)\, w(x_1) \ldots w(x_n)\, |\Delta(x)|^{2\tau}\, dx_1 \ldots dx_n = 0$ if $\mu < \lambda$, but generally not if $\mu \neq \lambda$ but unrelated by '$\leq$'.

One might also consider this definition with '$<$' being a lexicographic ordering. Then, of course, full orthogonality is achieved, but then the question is whether (i) already holds for some partial ordering with which the chosen lexicographic ordering is compatible. For general weight functions the answer is negative.

BC type Jacobi polynomials

Definition

For $w(x) := x^\alpha(1-x)^\beta$ let $p_\lambda^{\alpha,\beta,\tau}$ be equal to p_λ^τ as just defined. This is called a *BC* type Jacobi polynomial.

These polynomials, when rewritten as trigonometric polynomials and when reparametrized, become the Heckman-Opdam Jacobi polynomials for root system BC_n.

Let $\{\varepsilon_i\}_{i=1}^n$ be the standard basis of $\mathbb{R}^n$. The root system BC_n is the subset $\{\pm\varepsilon_i\}_{i=1}^n \cup \{\pm 2\varepsilon_i\}_{i=1}^n \cup \{\pm\varepsilon_i \pm \varepsilon_j\}_{1\le i<j\le n}$ of $\mathbb{R}^n$. Let the Weyl group invariant multiplicity function be such that ε_1, $2\varepsilon_1$, $\varepsilon_1+\varepsilon_2$ have multiplicities k_1, k_2, k_3 respectively.

Then for $x_i = \sin^2(\frac{1}{2}\theta_i)$, $\alpha = k_1+k_2-\frac{1}{2}$, $\beta = k_2-\frac{1}{2}$, $\tau = k_3$, $p_\lambda^{\alpha,\beta,\tau}(x)$ is equal to the Heckman-Opdam Jacobi polynomial for root system BC_n and multiplicities k_1, k_2, k_3, living on the torus $\{(e^{i\theta_1},\ldots,e^{i\theta_n})\}$.

BC type Jacobi polynomials (cntd.)

The motivation for introducing these polynomials came from the theory of spherical functions on compact symmetric spaces. For each root system the corresponding Heckman-Opdam polynomials can be interpreted as spherical functions for some very special values of the parameters. Then full orthogonality is evident by Schur orthogonality of matrix elements of irreducible representations.

BC type Jacobi polynomials (cntd.)

In the general case full orthogonality was first proved by Heckman in 1987 by using deep results by Deligne. Later, in 1991, Heckman could give a much quicker proof of the full orthogonality by starting with non-symmetric Heckman-Opdam polynomials, where he extended the theory of Dunkl operators.

For the root system BC_2 the corresponding Jacobi polynomials were already introduced by K (1974), using lexicographic ordering. It was next proved by Sprinkhuizen in 1976 that then the expansion in symmetrized monomials only needed the dominance partial ordering.

Almost immediately after Heckman's paper of 1987, Macdonald already proved full orthogonality in the BC_n case as a limit case of a similar result in the q-case.

The second order differential operator

Recall that, for $w(x) := x^\alpha(1-x)^\beta$, the BC_n type Jacobi polynomial $p_\lambda^{\alpha,\beta,\tau}$ is defined by the two properties:

(i) $p_\lambda^{\alpha,\beta,\tau} = m_\lambda + \sum_{\mu;\ \mu<\lambda} c_{\lambda,\mu} m_\mu\,;$

(ii) $\displaystyle\int_{[0,1]^n} p_\lambda^{\alpha,\beta,\tau}(x)\, m_\mu(x)\, w(x_1)\ldots w(x_n)\, |\Delta(x)|^{2\tau}\, dx_1\ldots dx_n = 0$ if $\mu<\lambda$.

We get an equivalent definition if we replace (ii) by:

$(\mathrm{ii})'$ $D^{\alpha,\beta,\tau} p_\lambda^{\alpha,\beta,\tau} = d_\lambda^{\alpha,\beta,\tau} p_\lambda^{\alpha,\beta,\tau}$, where (with $(\partial_i := \partial/\partial x_i)$

$$D^{\alpha,\beta,\tau} = \sum_{i=1}^n (x_i^2 - x_i)\partial_i^2 + \sum_{i=1}^n ((2+\alpha+\beta)x_i - (\alpha+1))\partial_i + 2\tau \sum_{i,j;i\neq j} \frac{x_i^2 - x_i}{x_i - x_j}\,\partial_i$$

and $d_\lambda^{\alpha,\beta,\tau} = \sum_{i=1}^n \lambda_i(\lambda_i + \alpha + \beta + 1 + 2\tau(n-i))$.

The second order differential operator (cntd.)

That (i) and (ii) imply $(\mathrm{ii})'$ follows because

$$D^{\alpha,\beta,\tau} m_\lambda = d_\lambda^{\alpha,\beta,\tau} m_\lambda + \sum_{\mu:\mu<\lambda} a_{\lambda,\mu} m_\mu \quad \text{and}$$

$$\int_{[0,1]^n} (D^{\alpha,\beta,\tau} f)(x)\, g(x)\, w(x_1)\ldots w(x_n)\, |\Delta(x)|^{2\tau}\, dx_1 \ldots dx_n$$

$$= \int_{[0,1]^n} f(x)\, (D^{\alpha,\beta,\tau} g)(x)\, w(x_1)\ldots w(x_n)\, |\Delta(x)|^{2\tau}\, dx_1 \ldots dx_n$$

for all symmetric polynomials f, g.

That (i) and $(\mathrm{ii})'$ imply (ii) follows because moreover $d_\lambda^{\alpha,\beta,\tau} \neq d_\mu^{\alpha,\beta,\tau}$ if $\lambda > \mu$.

Jack polynomials

For $\lambda = (\lambda_1, \ldots, \lambda_n)$ a partition and $\tau \geq 0$ the Jack polynomial j_λ^τ is defined as a symmetric polynomial in $x_1, \ldots, x_n$, homogeneous of degree $|\lambda|$, such that

(i) $j_\lambda^\tau = m_\lambda + \sum_{\mu;\mu<\lambda} c_{\lambda,\mu} m_\mu$;

(ii) $D^\tau j_\lambda^\tau = d_\lambda^\tau j_\lambda^\tau$, where

$$D^\tau = \sum_{i=1}^n x_i^2 \partial_i^2 + 2\tau \sum_{i,j;i\neq j} \frac{x_i^2}{x_i - x_j} \partial_i \quad \text{and}$$

$$d_\lambda^\tau = \sum_{i=1}^n \lambda_i(\lambda_i - 1 + 2\tau(n-i)).$$

Jack polynomials were introduced by Jack in 1969. They were taken up later by Macdonald and Stanley.
They become A_{n-1} type Heckman-Opdam Jacobi polynomials when their homogeneity is divided out.

Jack polynomials (cntd.)

Jack polynomials degenerate for $n = 1$ to monomials x^l (no coupling constant τ for $n = 1$).

Elementary cases for $n > 1$ are $\quad j_\lambda^0 = m_\lambda; \quad j_\lambda^1 = s_\lambda$,

$j_\lambda^\infty := \lim_{\tau\to\infty} j_\lambda^\tau = e_{\lambda'}$, where $e_{\lambda'} := e_n^{\lambda_n} e_{n-1}^{\lambda_{n-1}-\lambda_n} \ldots e_1^{\lambda_1-\lambda_2}$

with e_i the i-th elementary symmetric polynomial.

Jack polynomials can be considered as multivariate symmetric τ-dependent analogues of the monomials in one variable.

For $c \in \mathbb{R}$ write $c^n := (c, c, \ldots, c) \in \mathbb{R}^n$.

Put $\quad j_\lambda^{*\tau}(x) := j_\lambda^\tau(x)/j_\lambda^\tau(1^n) \quad (j_\lambda^\tau(1^n)$ is explicitly known).

There is a generalized binomial formula of the form

$j_\lambda^{*\tau}(x + 1^n) = \sum_{\mu:\, \mu\subseteq\lambda} \binom{\lambda}{\mu}_\tau j_\mu^{*\tau}(x)$, which defines the

generalized binomial coefficients $\binom{\lambda}{\mu}_\tau$.

Expansion of *BC* Jacobi in Jack polynomials

Theorem (Macdonald, unpublished manuscript)

There are coefficients $c_{\lambda,\mu}^{\alpha,\beta,\tau}$ such that $p_\lambda^{\alpha,\beta,\tau}=\sum_{\mu;\mu\subseteq\lambda} c_{\lambda,\mu}^{\alpha,\beta,\tau} j_\mu^\tau$.

Of course, $c_{\lambda,\lambda}^{\alpha,\beta,\tau}=1$. Also write $p_\lambda^{\alpha,\beta,\tau}=\sum_{\mu;\mu\subseteq\lambda} C_{\lambda,\mu}^{\alpha,\beta,\tau} j_\mu^{*\tau}$. Then

$$C_{\lambda,\mu}^{\alpha,\beta,\tau}\sum_{i=1}^n(\lambda_i-\mu_i)(\lambda_i+\mu_i+\alpha+\beta+1+2\tau(n-i))$$
$$=-\sum_{i;\,\mu+\varepsilon_i\subseteq\lambda} C_{\lambda,\mu+\varepsilon_i}^{\alpha,\beta,\tau}\binom{\mu+\varepsilon_i}{\mu}_\tau(\mu_i+\alpha+1+\tau(n-i))\text{ with}$$

$$\binom{\mu+\varepsilon_i}{\mu}_\tau=(\mu_i+1+\tau(\ell(\mu+\varepsilon_i)-i))\prod_{j;\,j\neq i}\frac{\mu_i-\mu_j+1+\tau(j-i-1)}{\mu_i-\mu_j+1+\tau(j-i)}.$$

Since the coefficient of $C_{\lambda,\mu}^{\alpha,\beta,\tau}$ in the above recurrence is nonzero if $\mu\subset\lambda$, the recurrence determines the $C_{\lambda,\mu}^{\alpha,\beta,\tau}$ uniquely, up to a constant factor fixed by $C_{\lambda,\lambda}^{\alpha,\beta,\tau}=j_\lambda^\tau(1^n)$.

Expansion of *BC* Jacobi in Jack polynomials (cntd.)

The recurrence follows by applying $D^{\alpha,\beta,\tau}$ to both sides of

$$p_\lambda^{\alpha,\beta,\tau} = \sum_{\mu;\mu\subseteq\lambda} C_{\lambda,\mu}^{\alpha,\beta,\tau} j_\mu^{*\tau}, \quad \text{and next by writing}$$

$$D^{\alpha,\beta,\tau} = D^\tau - \widetilde{D}^\tau + (2+\alpha+\beta)\sum_{i=1}^n x_i\partial_i - (\alpha+1)\sum_{i=1}^n \partial_i\,, \quad \text{where}$$

$$\widetilde{D}^\tau := \sum_{i=1}^n x_i\partial_i^2 + 2\tau \sum_{i,j;i\neq j} \frac{x_i}{x_i-x_j}\,\partial_i\,, \quad \text{and by using that}$$

$$\sum_{i=1}^n \partial_i\, j_\mu^{*\tau} = \sum_{i;\,\mu-\varepsilon_i \text{ partition}} \binom{\mu}{\mu-\varepsilon_i}_\tau j_{\mu-\varepsilon_i}^{*\tau}\,, \quad \sum_{i=1}^n x_i\partial_i\, j_\mu^{*\tau} = |\mu|\, j_\mu^{*\tau}\,,$$

$$\widetilde{D}^\tau j_\mu^{*\tau} = \sum_{i;\,\mu-\varepsilon_i \text{ partition}} \binom{\mu}{\mu-\varepsilon_i}_\tau (\mu_i - 1 + \tau(n-i))\, j_{\mu-\varepsilon_i}^{*\tau}\,,$$

see Hallnäss, IMRN, 2009.

Two possible limit cases of BC_n Jacobi polynomials

For $n = 1$ we have

$$p_l^{\alpha,\beta}(x) = \frac{(-1)^l(\alpha+1)_l}{(l+\alpha+\beta+1)_l}\sum_{k=0}^{l}\frac{(-l)_k(l+\alpha+\beta+1)_k}{(\alpha+1)_k k!}\,x^k.$$

Let $p_l^a(x)$ be the limit of $p_l^{\alpha,\beta}(x)$ as $\alpha,\beta\to\infty$ such that $\frac{\alpha}{\alpha+\beta}\to a$ ($a\ne 0$). Then

$$p_l^a(x) = (-a)^l\sum_{k=0}^{l}\frac{(-l)_k}{k!}\,(a^{-1}x)^k = (x-a)^l.$$

Problem 1. Compute the limit $p_\lambda^{a,\tau}(x)$ of $p_\lambda^{\alpha,\beta,\tau}(x)$ as $\alpha,\beta\to\infty$ such that $\frac{\alpha}{\alpha+\beta}\to a$.

Problem 2. Compute the limit $p_\lambda^{\alpha,\beta,\infty}(x)$ of $p_\lambda^{\alpha,\beta,\tau}(x)$ as $\tau\to\infty$.

Generalized binomial coefficients

Recall $j^{*\tau}_{\lambda}(x+1^n)=\sum_{\mu:\,\mu\subseteq\lambda}\binom{\lambda}{\mu}_{\tau}j^{*\tau}_{\mu}(x)$.

By homogeneity: $j^{*\tau}_{\lambda}(x+a^n)=\sum_{\mu:\,\mu\subseteq\lambda}\binom{\lambda}{\mu}_{\tau}a^{|\lambda|-|\mu|}j^{*\tau}_{\mu}(x)$.

There is a deeper result (see Lassalle, CRAS, 1990):

$$\sum_{\kappa;|\kappa|=k,\ \mu\subseteq\kappa\subseteq\lambda}\binom{\lambda}{\kappa}_{\tau}\binom{\kappa}{\mu}_{\tau}=\binom{|\lambda|-|\mu|}{k-|\mu|}\binom{\lambda}{\mu}_{\tau}.$$

In particular, for $k=|\mu|+1$:

$$\sum_{i;\ \mu+\varepsilon_i\subseteq\lambda}\binom{\lambda}{\mu+\varepsilon_i}_{\tau}\binom{\mu+\varepsilon_i}{\mu}_{\tau}=(|\lambda|-|\mu|)\binom{\lambda}{\mu}_{\tau}.$$

Solution of Problem 1

Theorem (Beerends & K, unpublished; Rösler, K & Voit, 2012)

The limit $p_\lambda^{a,\tau}(x)$ of $p_\lambda^{\alpha,\beta,\tau}(x)$ as $\alpha,\beta\to\infty$ such that $\frac{\alpha}{\alpha+\beta}\to a$, is equal to $j_\lambda^\tau(x-a^n)$.

Proof (two different proofs are given in the cited papers).

Recall that $p_\lambda^{\alpha,\beta,\tau}=\sum_{\mu;\mu\subseteq\lambda} C_{\lambda,\mu}^{\alpha,\beta,\tau} j_\mu^{*\tau}$ with

$$C_{\lambda,\mu}^{\alpha,\beta,\tau}\sum_{i=1}^n(\lambda_i-\mu_i)(\lambda_i+\mu_i+\alpha+\beta+1+2\tau(n-i))$$

$$=-\sum_{i;\,\mu+\varepsilon_i\subseteq\lambda} C_{\lambda,\mu+\varepsilon_i}^{\alpha,\beta,\tau}\binom{\mu+\varepsilon_i}{\mu}_\tau(\mu_i+\alpha+1+\tau(n-i)).$$

Hence $p_\lambda^{a,\tau}=\sum_{\mu;\mu\subseteq\lambda} C_{\lambda,\mu}^{a,\tau} j_\mu^{*\tau}$ exists with

$$C_{\lambda,\mu}^{a,\tau}=-\frac{a}{|\lambda|-|\mu|}\sum_{i;\,\mu+\varepsilon_i\subseteq\lambda} C_{\lambda,\mu+\varepsilon_i}^{a,\tau}\binom{\mu+\varepsilon_i}{\mu}_\tau$$

Solution of Problem 1 (cntd.)

A solution of the recurrence

$$C_{\lambda,\mu}^{a,\tau} = -\frac{a}{|\lambda| - |\mu|} \sum_{i;\, \mu+\varepsilon_i \subseteq \lambda} C_{\lambda,\mu+\varepsilon_i}^{a,\tau} \binom{\mu+\varepsilon_i}{\mu}_\tau$$

is given by $C_{\lambda,\mu}^{a,\tau} = (-a)^{|\lambda|-|\mu|} \binom{\lambda}{\mu}$,

since this reduces to the true identity

$$\sum_{i;\, \mu+\varepsilon_i \subseteq \lambda} \binom{\lambda}{\mu+\varepsilon_i}_\tau \binom{\mu+\varepsilon_i}{\mu}_\tau = (|\lambda| - |\mu|) \binom{\lambda}{\mu}_\tau .$$

Hence

$$\begin{aligned} p_\lambda^{a,\tau}(x) &= \mathrm{const.} \sum_{\mu;\mu\subseteq\lambda} C_{\lambda,\mu}^{a,\tau} j_\mu^{*\tau}(x) \\ &= \mathrm{const.} \sum_{\mu;\mu\subseteq\lambda} (-a)^{|\lambda|-|\mu|} \binom{\lambda}{\mu}_\tau j_\mu^{*\tau}(x) \\ &= \mathrm{const.}\, j_\lambda^{*\tau}(x - a^n) = j_\lambda^\tau(x - a^n). \end{aligned}$$

□

Problem 1 (cntd.)

The just proved theorem is used in Rösler, K & Voit (2012) to obtain a similar limit result in the non-terminating case (by analytic continuation, using Carlson's theorem and suitable estimates). For $\tau = \frac{1}{2}, 1, 2$ and α, β suitably restricted this has an interpretation as limit of spherical functions on noncompact finite dimensional Grassmann manifolds to spherical functions on infinite dimensional Grassmann manifolds.

The case $\lambda = l^n$

Theorem (Macdonald, unpublished)

$$\frac{p_{l^n}^{\alpha,\beta,\tau}(x)}{p_{l^n}^{\alpha,\beta,\tau}(0)} = \sum_{\mu;\mu\subseteq\lambda} c_{l^n,\mu}^{\alpha,\beta,\tau} j_\mu^\tau(x) \text{ with}$$

$$c_{l^n,\mu}^{\alpha,\beta,\tau} = \frac{(-l;\tau)_\mu(l+\alpha+\beta+\tau(n-1)+1;\tau)_\mu}{(\alpha+\tau(n-1)+1;\tau)_\mu\, h_\tau^*(\mu)\,\tau^{|\mu|}}.$$

Here $(a;\tau)_\mu := \prod_{i=1}^{n}(a-\tau(i-1))_{\mu_i}$

(generalized Pochhammer symbol) and

$$h_\tau^*(\mu) := \prod_{(i,j)\in\mu}(\mu'_j - i + \tau^{-1}(\mu_i - j + 1))$$

(product of upper hook lengths).

We can compute the limit for $\tau\to\infty$ of $c_{l^n,\mu}^{\alpha,\beta,\tau}$.

The case $\lambda = l^n$ for $\tau \to \infty$

Recall: $c_{l^n,\mu}^{\alpha,\beta,\tau} = \dfrac{(-l)_\mu (l+\alpha+\beta+\tau(n-1)+1)_\mu}{(\alpha+\tau(n-1)+1)_\mu \, h_\tau^*(\mu)\, \tau^{|\mu|}}$. Use that

$$\lim_{\tau\to\infty} \frac{(l+\alpha+\beta+\tau(n-1)+1;\tau)_\mu}{(\alpha+\tau(n-1)+1;\tau)_\mu} = \frac{(n+\alpha+\beta+1)_{\mu_n}}{(\alpha+1)_{\mu_n}},$$

$$\lim_{\tau\to\infty} \frac{(-l;\tau)_\mu}{\tau^{\mu_2+\cdots+\mu_n}} = (-l)_{\mu_1}(-1)^{\mu_2+\cdots+\mu_n} 2^{\mu_3} \ldots (n-1)^{\mu_n},$$

$$\lim_{\tau\to\infty} \frac{1}{h_\tau^*(\mu)\tau^{\mu_1}} = \frac{2^{-\mu_3} 3^{-\mu_4} \ldots (n-1)^{-\mu_n}}{(\mu_1-\mu_2)! \ldots (\mu_{n-1}-\mu_n)!\, \mu_n!}.$$

Altogether, $c_{l^n,\mu}^{\alpha,\beta,\infty} := \lim_{\tau\to\infty} c_{l^n,\mu}^{\alpha,\beta,\tau}$

$$= \frac{(l+\alpha+\beta+1)_{\mu_n}}{(\alpha+1)_{\mu_n}} \, \frac{(-l)_{\mu_1}(-1)^{\mu_2+\cdots+\mu_n}}{(\mu_1-\mu_2)! \ldots (\mu_{n-1}-\mu_n)!\, \mu_n!}.$$

Recall: $j_\mu^\infty := \lim_{\tau\to\infty} j_\mu^\tau = e_1^{\mu_1-\mu_2} \ldots e_{n-1}^{\mu_{n-1}-\mu_n} e_n^{\mu_n}$.

The case $\lambda = l^n$ for $\tau \to \infty$ (cntd.)

$$\lim_{\tau\to\infty} \frac{p_{l^n}^{\alpha,\beta,\tau}(x)}{p_{l^n}^{\alpha,\beta,\tau}(0)} = \sum_{\mu;\mu\subseteq\lambda} c_{l^n,\mu}^{\alpha,\beta,\infty} j_\mu^\infty(x) = \sum_{\mu;\mu\subseteq\lambda} \frac{(l+\alpha+\beta+1)_{\mu_n}}{(\alpha+1)_{\mu_n}}$$

$$\times(-l)_{\mu_1}(-1)^{\mu_2+\cdots+\mu_n} \frac{e_1^{\mu_1-\mu_2}\ldots e_{n-1}^{\mu_{n-1}-\mu_n} e_n^{\mu_n}}{(\mu_1-\mu_2)!\ldots(\mu_{n-1}-\mu_n)!\,\mu_n!}$$

$$= \sum_{\mu_n=0}^{l} \frac{(-l)_{\mu_n}(l+\alpha+\beta+1)_{\mu_n}}{(\alpha+1)_{\mu_n}\,\mu_n!}\, e_n^{\mu_n} \sum_{\mu_1,\ldots,\mu_{n-1};\,\mu_n\le\mu_{n-1}\le\ldots\le\mu_1\le l}$$

$$\times(-1)^{\mu_1+\cdots+\mu_{n-1}} \frac{(l-\mu_n)!}{(l-\mu_1)!} \frac{e_1^{\mu_1-\mu_2}}{(\mu_1-\mu_2)!} \cdots \frac{e_{n-1}^{\mu_{n-1}-\mu_n}}{(\mu_{n-1}-\mu_n)!}$$

$$= \sum_{\mu_n=0}^{l} \frac{(-l)_{\mu_n}(l+\alpha+\beta+1)_{\mu_n}}{(\alpha+1)_{\mu_n}\,\mu_n!} ((-1)^n e_n)^{\mu_n}$$

$$\times(1-e_1+e_2-\cdots\pm e_{n-1})^{l-\mu_n}.$$

The case $\lambda = l^n$ for $\tau \to \infty$ (cntd.)

Theorem

$$\lim_{\tau\to\infty} \frac{p_{l^n}^{\alpha,\beta,\tau}(x)}{p_{l^n}^{\alpha,\beta,\tau}(0)} = (1 - e_1 + e_2 - \cdots \pm e_{n-1})^l \times {}_2F_1\left(\begin{matrix} -l, l+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{(-1)^n e_n}{1 - e_1 + e_2 - \cdots \pm e_{n-1}}\right).$$

The case $n = 2$

For $n = 2$ many things can be computed much more explicitly (K & Sprinkhuizen, 1978), with heavy use of the shift operators (which exist also for general n, see Opdam, 1988 and Heckman, 1991, but are explicitly given for $n = 2$).
In particular:

$$j^{\tau}_{\lambda_1,\lambda_2}(x_1,x_2) = \frac{(\lambda_1-\lambda_2)!}{(\tau)_{\lambda_1-\lambda_2}} \sum_{i=0}^{\lambda_1-\lambda_2} \frac{(\tau)_i(\tau)_{\lambda_1-\lambda_2-i}}{i!\,(\lambda_1-\lambda_2-i)!}\, x_1^{\lambda_1-i} x_2^{\lambda_2+i}$$

$$= \frac{2^{\lambda_1-\lambda_2}(\lambda_1-\lambda_2)!}{(2\tau+\lambda_1-\lambda_2)_{\lambda_1-\lambda_2}} (x_1x_2)^{\frac{1}{2}(\lambda_1+\lambda_2)} P^{(\tau-\frac{1}{2},\tau-\frac{1}{2})}_{\lambda_1-\lambda_2}\left(\frac{x_1+x_2}{2(x_1x_2)^{\frac{1}{2}}}\right),$$

$$\lim_{\tau\to\infty} \frac{p^{\alpha,\beta,\tau}_{\lambda_1,\lambda_2}(x_1,x_2)}{p^{\alpha,\beta,\tau}_{\lambda_1,\lambda_2}(0,0)} = (1-x_1-x_2)^{\lambda_1}$$

$$\times\, {}_2F_1\left(\begin{matrix}-\lambda_2,\lambda_2+\alpha+\beta+1\\ \alpha+1\end{matrix};\frac{-x_1x_2}{1-x_1-x_2}\right).$$

There are strong indications that the case $n = 2$ is not yet typical for the case of general n.

Some references

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