A distant mirror:
my joint paper with Aad after 35 years

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lecture on 24 February 2006
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23–24 February 2006

my 1971 paper with Aad

A. Dijksma & T. H. Koornwinder,

Spherical harmonics and the product of two Jacobi polynomials,

Proc. KNAW, Series A, 74 = Indag. Math. 33 (1971), 191–196

(communicated by prof. A. C. Zaanen)

Adriaan Zaanen, 1913–2003

Jacobi polynomials

Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are orthogonal polynomials on $[-1,1]$ with respect to the measure

$$(1-x)^\alpha(1+x)^\beta\,dx \quad (\alpha,\beta>-1),$$

and with normalization

$$P_n^{(\alpha,\beta)}(1)=\frac{(\alpha+1)_n}{n!}\,,$$

where $(a)_k := a(a+1)\ldots(a+k-1)$.

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Gegenbauer polynomials are special Jacobi polynomials

$$C_n^\lambda(x) := \frac{(2\lambda)_n}{(\lambda+\frac{1}{2})_n}\,P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x).$$

Dijksma-Koornwinder product formula

$$
\begin{aligned}
&P_n^{(\alpha,\beta)}(1-2x^2)\,P_n^{(\alpha,\beta)}(1-2y^2)\\
&\quad = \text{const.}\int_{-1}^{1}\int_{-1}^{1} C_{2n}^{\alpha+\beta+1}\Big(xyu+\sqrt{1-x^2}\,\sqrt{1-y^2}\,v\Big)\\
&\qquad\qquad \times (1-u^2)^{\alpha-\frac12}(1-v^2)^{\beta-\frac12}\,du\,dv \quad (\alpha,\beta>-\tfrac12).
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For $\alpha,\beta = 0,\frac{1}{2},1,\frac{3}{2},\ldots$ interpretation on spherical harmonics.

Spherical harmonics

Unit sphere S^{d-1} in $\mathbb{R}^d$.

$O(d)$ is group of orthogonal transformations of $\mathbb{R}^d$.

$O(d)$ acts on functions on S^{d-1}: $(Tf)(x) := f(T^{-1}x)$.

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The representation of $O(d)$ on $\mathcal{H}_n$ is irreducible.

$O(d-1)$ is the subgroup of $O(d)$ which leaves $e_1 = (1, 0, \ldots, 0)$ fixed.

$S^{d-1} = O(d)/O(d-1)$.

Zonal spherical harmonics

Theorem $f \in \mathcal{H}_n$ and $O(d-1)$-invariant

iff $f(x) = \mathrm{const.}\, C_n^{\frac{1}{2}d-1}(\langle x, e_1 \rangle)$.

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Lemma If $g \in \mathcal{H}_n$ and $f(x) := \int_{O(d-1)} g(Tx)\, dT$

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Corollary 1 Take $g(x) := (x_1 + ix_2)^n$.

$$C_n^{\frac{1}{2}d-1}(\cos\theta) = \text{const.} \int_{O(d-1)} (\cos\theta\, e_1 + i\sin\theta\, (Te_2)_2)^n\, dT$$
$$= \text{const.} \int_{-1}^{1} (\cos\theta + i\sin\theta\, v)^n\, (1-v^2)^{\frac{1}{2}d-2}\, dv.$$

Zonal spherical harmonic: product formula

Corollary 2 Take $g(x) := C_n^{\frac{1}{2}d-1}(\langle x, y\rangle) \quad (y \in S^{d-1})$.

$$C_n^{\frac{1}{2}d-1}(\langle x, e_1\rangle)\, C_n^{\frac{1}{2}d-1}(\langle y, e_1\rangle) = \text{const.} \int_{O(d-1)} C_n^{\frac{1}{2}d-1}(\langle Tx, y\rangle)\, dT,$$

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Analytic continuation w.r.t. d by Carlson's theorem.
For $\lambda > 0$:

$$C_n^{\lambda}(x)\, C_n^{\lambda}(y) = \text{const.} \int_{-1}^{1} C_n^{\lambda}(xy + \sqrt{1-x^2}\,\sqrt{1-y^2}\, v)\, (1-v^2)^{\lambda-1}\, dv.$$

Mathematisch Centrum

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$O(d') \times O(d'')$-invariant spherical harmonics

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door

Prof.Dr. B.L.J. Braaksma.

"Jacobi and Gegenbauer Polynomials as spherical harmonics."

$O(d') \times O(d'')$-invariant spherical harmonics

$$d = d' + d'', \quad \mathbb{R}^d = \mathbb{R}^{d'} \times \mathbb{R}^{d''}, \quad x = (x', x'').$$

$$(T', T'') = \begin{pmatrix} T' & 0 \\ 0 & T'' \end{pmatrix} \in O(d') \times O(d'') \subset O(d).$$

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Theorem (Braaksma & Meulenbeld, 1968)

$f \in \mathcal{H}_{2n}$ and $O(d') \times O(d'')$-invariant

iff $f(x) = \mathrm{const.}\, P_n^{(\frac{1}{2}d''-1, \frac{1}{2}d'-1)}(2|x'|^2 - 1)$.

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$O(d') \times O(d'')$-invariant spherical harmonics: integral representation

Corollary 1 (Braaksma & Meulenbeld, 1968)

Take $g(x) := (x_1 + i x_d)^{2n}$.

$$P_n^{(\frac{1}{2}d''-1,\frac{1}{2}d'-1)}(2|x'|^2-1)$$
$$= \text{const.} \int_{O(d')} \int_{O(d'')} \big((T'x')_1 + i(T''x'')_d\big)^{2n}\, dT'\, dT'',$$

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$$\times (1-u^2)^{\frac{1}{2}d'-\frac{3}{2}} (1-v^2)^{\frac{1}{2}d''-\frac{3}{2}}\, du\, dv.$$

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Corollary 2 (Dijksma & K, 1971)

Take $g(x) := C_{2n}^{\frac{1}{2}d-1}(\langle x, y\rangle) \quad \big(y \in S^{d-1}\big)$.

$$P_n^{(\frac{1}{2}d''-1,\frac{1}{2}d'-1)}(2|x'|^2-1)\,P_n^{(\frac{1}{2}d''-1,\frac{1}{2}d'-1)}(2|y'|^2-1)$$

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Analytic continuation w.r.t. d', d'' by Carlson's theorem.

Gasper's product formula (1971)

$$\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}\,\frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)}=\int_{-1}^{1}\frac{P_n^{(\alpha,\beta)}(z)}{P_n^{(\alpha,\beta)}(1)}\,K_{\alpha,\beta}(x,y,z)\,dz$$

for an explicit kernel $K_{\alpha,\beta}(x,y,z)\geq 0$ on $[-1,1]^3$ if $\alpha\geq\beta\geq-\frac{1}{2}$.

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For $\alpha=\beta$ this can be rewritten as the original product formula for Gegenbauer polynomials.

Question by Richard Askey
Is a similar rewriting of Gasper's product formula possible if $\alpha\neq\beta$?

Dick Askey

Complex spherical harmonics

Answer to Askey's question is yes.
One way is by using analogues of spherical harmonics on $U(d)/U(d-1)$ (K, 1972, independently from earlier work by Vilenkin and Šapiro).

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Definition A *harmonic* of type (m,n) on $S^{2d-1} \subset \mathbb{C}^d$ is the restriction to S^{2d-1} of a polynomial homogeneous of degree m in $z_1,\ldots,z_d$ and homogeneous of degree n in $\overline{z_1},\ldots,\overline{z_d}$ which is annihilated by $\frac{\partial^2}{\partial z_1 \partial \overline{z_1}} + \cdots + \frac{\partial^2}{\partial z_d \partial \overline{z_d}}$.
Let $\mathcal{H}_{m,n}$ be the space of all such harmonics.

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Let $\mathcal{H}_{m,n}$ be the space of all such harmonics.

$U(d)$ is the group of unitary $d \times d$ matrices. $U(d-1)$ is the subgroup which leaves the vector $e_1 = (1, 0, \ldots, 0)$ in $\mathbb{C}^d$ fixed. Then $S^{2d-1} = U(d)/U(d-1)$.

$U(d)$ acts irreducibly on $\mathcal{H}_{m,n}$.

Zonal complex spherical harmonics

Theorem $f \in \mathcal{H}_{m,n}$ and $U(d-1)$-invariant iff

$$f(re^{i\theta}, z_2, \ldots, z_d) = r^{|m-n|}\, e^{i(m-n)\theta}\, P_{m\wedge n}^{(d-2,|m-n|)}(\cos 2\theta).$$

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Disk polynomials:

$$R_{m,n}^{(\alpha)}(re^{i\theta}) := \mathrm{const.}\, r^{|m-n|}\, e^{i(m-n)\theta}\, P_{m\wedge n}^{(\alpha,|m-n|)}(\cos 2\theta).$$

The polynomials $R_{m,n}^{(\alpha)}(z)$ are orthogonal on the unit disk in $\mathbb{C}$ with respect to the weight function $(1-x^2-y^2)^\alpha$ $(z = x+iy)$.

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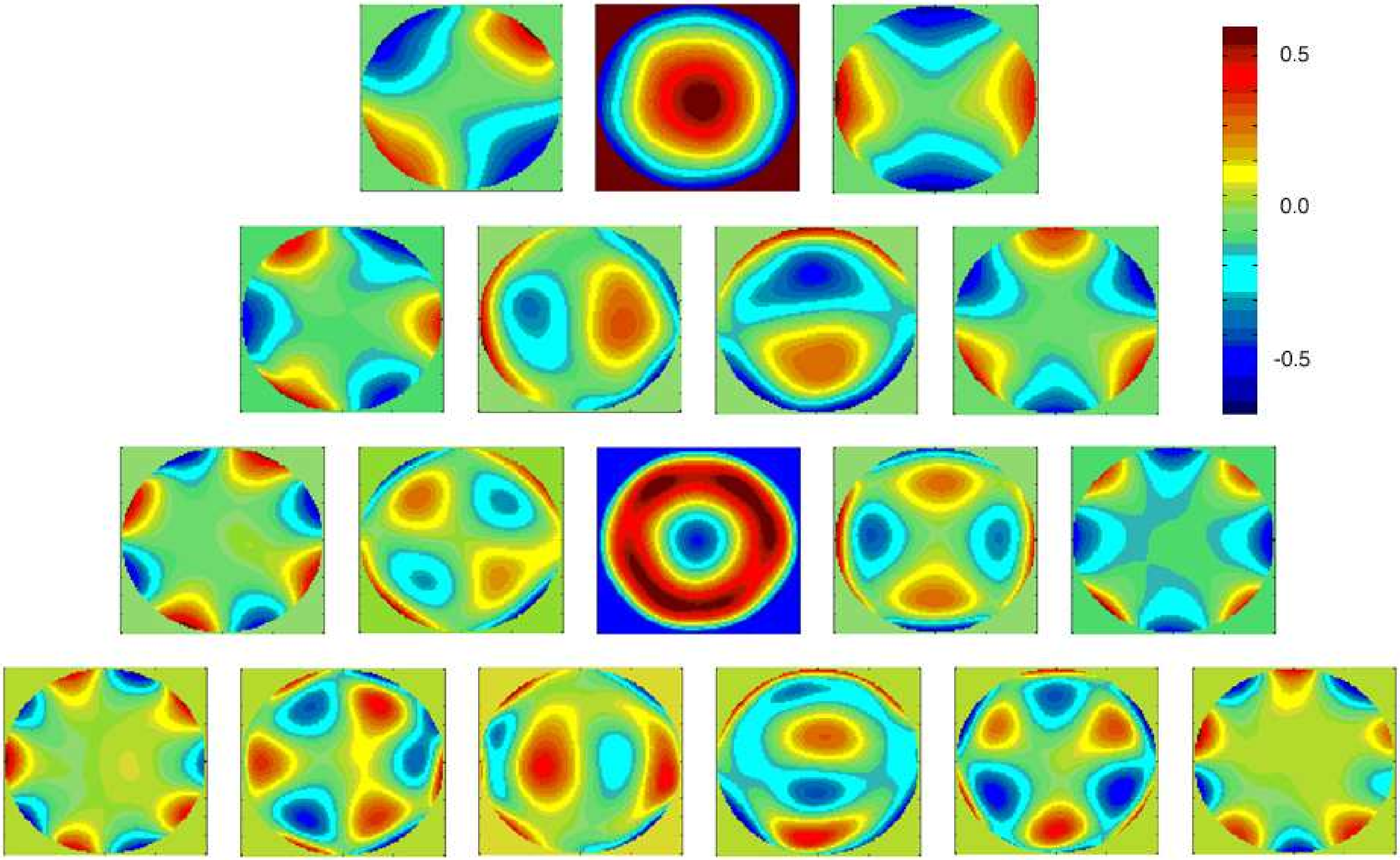
Essentially *Zernike polynomials*.

Frits Zernike (1888–1966)

Zernike polynomials



http://bifano.bu.edu/tgbifano/Web/Images/zernike_bar.jpg

Product formula for Jacobi polynomials

From interpretation on $U(d)/U(d-1)$ a product and addition formula for disk polynomials.

Hence a product and addition formula for Jacobi polynomials $P_n^{(\alpha,0)}(x)$.

Hence by differentiation and analytic continuation an addition formula for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$.

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Hence by differentiation and analytic continuation an addition formula for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$.

Hence product formula for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ $(\alpha > \beta > -\frac{1}{2})$:

$$P_n^{(\alpha,\beta)}(\cos 2\theta_1)\, P_n^{(\alpha,\beta)}(\cos 2\theta_2) = \text{const.}$$

$$\times \int_{r=0}^{1} \int_{\phi=0}^{\pi} P_n^{(\alpha,\beta)}\Big(2|\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2\, re^{i\phi}|^2 - 1\Big)$$

$$\times\, (1-r^2)^{\alpha-\beta-1} r^{2\beta+1}\, (\sin\phi)^{2\beta}\, dr\, d\phi.$$

General theory: spherical functions

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π irreducible unitary representation of G on Hilbert space $\mathcal{H}$ of finite dimension d_π.

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Suppose there is a K-fixed unit vector e_K in $\mathcal{H}$, unique up to a constant factor.

Then $\quad \phi(x) := \langle \pi(x) e_K, e_K \rangle \quad$ is called a *spherical function*.

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Properties

$\phi(k_1 x k_2) = \phi(x) \quad \big(k_1, k_2 \in K\big), \qquad \phi(e) = 1, \quad |\phi(x)| \le 1.$

Theorem (Gelfand)

Let ϕ be continuous, not identically zero. Then ϕ is a spherical function for some irreducible representation iff

$$\phi(x)\phi(y) = \int_K \phi(xky)\, dk \quad (x, y \in G).$$

General theory: intertwining functions

G compact group with closed subgroups K and H.

π irreducible unitary representation of G on $\mathcal{H}$.

K-fixed resp. H-fixed unit vectors e_K, e_H in $\mathcal{H}$, unique up to constant factors.

$$\begin{aligned}
\phi_{KK}(x) &:= \langle \pi(x)e_K, e_K\rangle, \\
\phi_{HH}(x) &:= \langle \pi(x)e_H, e_H\rangle, \\
\phi_{HK}(x) &:= \langle \pi(x)e_K, e_H\rangle.
\end{aligned}$$

ϕ_{HK} is called an *intertwining function*.

$\phi_{HK}(e) = \langle e_K, e_H\rangle, \quad |\phi_{HK}(x)| \le 1, \quad |\phi_{HK}(e)| \le 1.$

$\phi_{KH}(x) = \overline{\phi_{HK}(x^{-1})}$.

Intertwining property of ϕ_{HK}

Put

$$f_{v,K}(x) := \langle \pi(x) e_K, v \rangle,$$
$$f_{v,H}(x) := \langle \pi(x) e_H, v \rangle.$$

Then

$$\int_G f_{v,H}(y)\, \phi_{HK}(y^{-1}x)\, dy = d_\pi^{-1} f_{v,K}(x),$$

so

$$f_{v,H} * \phi_{HK} = d_\pi^{-1} f_{v,K}.$$

$\phi_{H,K}$ is a convolution kernel which maps the representation space of π in $L^2(G/H)$ onto the representation space of π in $L^2(G/K)$, and which intertwines with the G-action.

Projection and product formulas

$$\begin{aligned}
\int_H \pi(h) e_K \, dh &= \langle e_K, e_H \rangle \, e_H, \\
\int_H \phi_{KK}(h) \, dh &= |\langle e_K, e_H \rangle|^2, \\
\int_H \phi_{KK}(xh) \, dh &= \phi_{HK}(e) \, \phi_{KH}(x), \\
\int_H \phi_{KK}(xhy) \, dh &= \phi_{KH}(x) \, \phi_{HK}(y) \\
&= \overline{\phi_{HK}(x^{-1})} \, \phi_{HK}(y).
\end{aligned}$$

The case of the Dijksma-K product formula

$G = O(d)$, $K = O(d-1)$, $H = O(d') \times O(d'')$ $\quad (d = d' + d'')$.
Assume $d' \le d''$. Let A be the one-parameter subgroup of G consisting of:

$$a_\theta := \begin{pmatrix} \cos\theta\, I_{d'} & -\sin\theta\, I_{d'} & 0 \\ \sin\theta\, I_{d'} & \cos\theta\, I_{d'} & 0 \\ 0 & 0 & I_{d''-d'} \end{pmatrix}.$$

Then $G = KAH$.

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$G = O(d)$, $K = O(d-1)$, $H = O(d') \times O(d'') \quad (d = d' + d'')$.
Assume $d' \le d''$. Let A be the one-parameter subgroup of G consisting of:

$$a_\theta := \begin{pmatrix} \cos\theta\, I_{d'} & -\sin\theta\, I_{d'} & 0 \\ \sin\theta\, I_{d'} & \cos\theta\, I_{d'} & 0 \\ 0 & 0 & I_{d''-d'} \end{pmatrix}.$$

Then $G = KAH$.

$$\phi_{HK}(K a_\theta H) = \langle e_K, e_H \rangle \, \frac{P_n^{(\frac12 d''-1, \frac12 d'-1)}(\cos 2\theta)}{P_n^{(\frac12 d''-1, \frac12 d'-1)}(1)}.$$

Hence $\quad |\phi_{HK}(x)| \le |\phi_{HK}(e)|$.

$$|\langle e_K, e_H \rangle|^2 = \frac{(\frac12 d'')_n \, (\frac12)_n}{(\frac12 d')_n \, (\frac12 (d-1))_n}.$$

The Flensted-Jensen condition (1973)

$G = O(d)$, $K = O(d-1)$, $H = O(d') \times O(d'')$, $d' \le d''$,

$$a_\theta = \begin{pmatrix} \cos\theta\, I_{d'} & -\sin\theta\, I_{d'} & 0 \\ \sin\theta\, I_{d'} & \cos\theta\, I_{d'} & 0 \\ 0 & 0 & I_{d''-d'} \end{pmatrix} \in A.$$

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Let $(T', T'') \in H$. Observe:

$$a_\theta \begin{pmatrix} T' & 0 \\ 0 & T'' \end{pmatrix} \subset a_\theta \begin{pmatrix} T' & 0 & 0 \\ 0 & T' & 0 \\ 0 & 0 & I_{d''-d'} \end{pmatrix} K$$

$$= \begin{pmatrix} T' & 0 & 0 \\ 0 & T' & 0 \\ 0 & 0 & I_{d''-d'} \end{pmatrix} a_\theta K \subset H a_\theta K.$$

Hence $aH \subset HaK$ for all $a \in A$

(Flensted-Jensen condition, 1973)

Flensted-Jensen condition: product formula

Theorem (Flensted-Jensen, 1973) If $aH \subset HaK$ then

$$\int_H \frac{\phi_{HK}(ahy)}{\phi_{HK}(e)}\, dh = \frac{\phi_{HK}(a)}{\phi_{HK}(e)}\, \frac{\phi_{HK}(y)}{\phi_{HK}(e)} \quad (y \in G).$$

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Corollary (K, 1973)

An alternative derivation of the product formula for Jacobi polynomials.

Citations of the Dijksma-K paper

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