Hypergroups made concrete for special orthogonal polynomials, a survey

Tom Koornwinder

Korteweg-de Vries Institute, University of Amsterdam
T.H.Koornwinder@uva.nl

lecture at Colloque *Orthogonal Polynomials and Hypergroups*, Toulouse, France, 18-19 June 2014

last modified: 23 June 2014

Toulouse and Thomas Stieltjes

Thomas J. Stieltjes, 1856–1894

until 1885 in the Netherlands;
assistant at Leiden observatory;
many letters exchanged with Hermite;
1885–1886 in Paris;
from 1886 in Toulouse:
1886–1889 maître de conférences;
1889–1894 professeur

His tomb is at the cemetery of Terre Cabade in Toulouse.

Stieltjes prize: an annual prize for the best PhD thesis in mathematics defended at a Dutch university, see
`http://testweb.science.uu.nl/WONDER/prizes.html`

Askey's sabbatical, Amsterdam, 1969–1970

Mathematisch Centrum, 2e Boerhaavestraat 49, Amsterdam
(earlier situation; in 1969 the top floor looked different)

http://beeldbank.amsterdam.nl/afbeelding/010003018854

Askey's sabbatical, Amsterdam, 1969–1970 (cntd.)

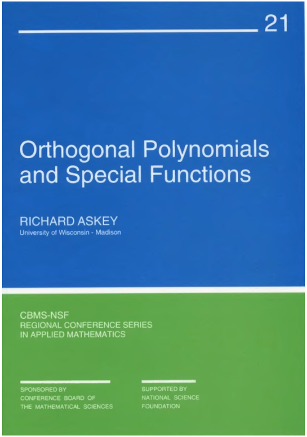
Dick Askey reading a math book in his Dutch home, Amstelveen, 1970

Askey's sabbatical, Amsterdam, 1969–1970 (cntd.)

In his lectures in Amsterdam, 1969–1970, Askey emphasized positivity properties of (special) orthogonal polynomials related to generalized translation and convolution. Thus he presented special hypergroups avant la lettre.

These results were presented with a much wider scope in his 1975 SIAM Lecture Notes.

Askey's 1975 SIAM Lecture Notes



Contents

Four canonical problems

$\{p_n\}$ and $\{q_n\}$ systems of orthogonal polynomials;

$$\int p_m(x)p_n(x)\,d\mu(x) = h_n\,\delta_{m,n}.$$

Often $p_n(x_0) = 1 = q_n(x_0)$ for some special x_0.

- product formula:
$$p_n(x)p_n(y) = \int p_n(z)K(x,y,z)\,d\mu(z).$$
- transmutation: $$q_n(x) = \int p_n(y)A(x,y)\,d\mu(y).$$
- linearization of products:
$$p_m(x)p_n(x) = \sum_{k=|m-n|}^{m+n} c_{m,n,k}\,p_k(x)/h_k.$$
- connection formula: $$q_n(x) = \sum_{k=0}^{n} a_{n,k}\,p_k(x)/h_k.$$

Find the integral and summation kernels explicitly and/or see when these kernels are nonnegative.

Four canonical problems (cntd.)

A product formula with nonnegative kernel is a necessary aspect of a hypergroup associated with orthogonal polynomials. Similarly, nonnegative linearization coefficients are necessary for a dual hypergroup associated with orthogonal polynomials. Transmutations and connection formulas with positive kernel have some hypergroup flavour, in particular when they connect with cosine functions or Chebyshev polynomials of the first kind.

Product formulas

$$p_n(x)\, p_n(y) = \int p_n(z)\, K(x, y, z)\, d\mu(z)$$

$$K(x, y, z) = \sum_{n=0}^{\infty} \frac{p_n(x)\, p_n(y)\, p_n(z)}{h_n}$$

Gegenbauer product formula

Jacobi polynomials $(\alpha,\beta>-1)$:

$$R_n^{(\alpha,\beta)}(x):=\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)},\quad d\mu_{\alpha,\beta}(x):=\frac{(1-x)^\alpha(1+x)^\beta\,dx}{\int_{-1}^1(1-x)^\alpha(1+x)^\beta\,dx},$$

$$\int_{-1}^1 R_m^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(x)\,d\mu_{\alpha,\beta}(x)=h_n^{(\alpha,\beta)}\,\delta_{m,n}.$$

Gegenbauer product formula $(\alpha>-\frac12)$: $R_n^{(\alpha,\alpha)}(x)R_n^{(\alpha,\alpha)}(y)$

$$=\int_{-1}^1 R_n^{(\alpha,\alpha)}\big(xy+(1-x^2)^{\frac12}(1-y^2)^{\frac12}t\big)\,d\mu_{\alpha-\frac12,\alpha-\frac12}(t).$$

For $\alpha=-\frac12$: $\cos n\phi\cos n\psi=\frac12\big(\cos n(\phi+\psi)+\cos n(\phi-\psi)\big)$.

Generalized translation (Levitan, Bochner, Hirschman):

$$T_y[f](x):=\int_{-1}^1 f\big(xy+(1-x^2)^{\frac12}(1-y^2)^{\frac12}t\big)\,d\mu_{\alpha-\frac12,\alpha-\frac12}(t).$$

Positivity of generalized translation:

$$f(x)=\sum_{n=0}^\infty\frac{\widehat f(n)R_n^{(\alpha,\alpha)}(x)}{h_n}\geq0\Leftrightarrow T_y[f](x)=\sum_{n=0}^\infty\frac{\widehat f(n)R_n^{(\alpha,\alpha)}(x)R_n^{(\alpha,\alpha)}(y)}{h_n}\geq0$$

Gegenbauer product formula (cntd.)

Product formula and generalized translation in kernel form:

$$R_n^{(\alpha,\alpha)}(x)R_n^{(\alpha,\alpha)}(y) = \int_{-1}^{1} R_n^{(\alpha,\alpha)}(z)\,K_{\alpha,\alpha}(x,y,z)\,d\mu_{\alpha,\alpha}(z),$$

$$T_y[f](x) = \int_{-1}^{1} f(z)\,K_{\alpha,\alpha}(x,y,z)\,d\mu_{\alpha,\alpha}(z), \quad \text{where}$$

$$K_{\alpha,\alpha}(x,y,z) = \sum_{n=0}^{\infty} R_n^{(\alpha,\alpha)}(x)R_n^{(\alpha,\alpha)}(y)R_n^{(\alpha,\alpha)}(z)/h_n^{(\alpha,\alpha)}$$

$$= \frac{\Gamma(\alpha+1)^2}{\Gamma(\alpha+\frac{1}{2})\Gamma(\alpha+\frac{3}{2})}\,\frac{(1-x^2-y^2-z^2+2xyz)_+^{\alpha-\frac{1}{2}}}{\big((1-x^2)(1-y^2)(1-z^2)\big)^{\alpha}} \geq 0.$$

Convolution:

$$(f*g)(x) := \int_{-1}^{1} T_y[f](x)\,g(y)\,d\mu_{\alpha,\alpha}(y) = \int_{-1}^{1}\int_{-1}^{1} f(z)\,g(y)$$

$$\times\,K_{\alpha,\alpha}(x,y,z)\,d\mu_{\alpha,\alpha}(y)\,d\mu_{\alpha,\alpha}(z) = \sum_{n=0}^{\infty} \frac{\widehat{f}(n)\widehat{g}(n)R_n^{(\alpha,\alpha)}(x)}{h_n^{(\alpha,\alpha)}}\,.$$

Convolution algebra

Put $\|f\|_1 := \int_{-1}^{1} |f(x)|\, d\mu_{\alpha,\alpha}(x)$. Then we conclude:

$$\|f*g\|_1 \le \|f\|_1\, \|g\|_1, \quad \|f*g\|_\infty \le \|f\|_\infty\, \|g\|_1, \quad f,g \ge 0 \Rightarrow f*g \ge 0.$$

The same machinery would work for other orthogonal systems, provided we have a product formula with positive kernel.

Gegenbauer case $\alpha = \frac{1}{2}(d-3)$ **by group theory**:

$G = O(d)$, $K = O(d-1)$, $\Omega = G/K = S^{d-1}$ (Gelfand pair).

$$(f*g)(x) = (g*f)(x) = \int_G f(y)\, g(y^{-1}x)\, dy \quad (f, g\ K\text{-biinvariant}),$$

$$(F*G)(\langle x,y\rangle) = \int_\Omega F(\langle x,z\rangle)\, G(\langle z,y\rangle)\, d\omega(z) \quad (x,z \in \Omega),$$

$$\phi(x)\,\phi(y) = \int_K \phi(xky)\, dk \quad (x,y \in G,\ \phi(x) = R_n^{(\alpha,\alpha)}(\langle xe_1, e_1\rangle)\ ).$$

ϕ is spherical function (Gelfand); immediate positivity results; works also for certain other Jacobi parameters (Gangolli).

Jacobi product formula

In the analogous Jacobi problem for $\alpha > \beta > -\frac{1}{2}$ Gasper (1971) showed that

$$R_n^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(y) = \int_{-1}^{1} R_n^{(\alpha,\beta)}(z)\, K_{\alpha,\beta}(x,y,z)\, d\mu_{\alpha,\beta}(z)$$

with $K_{\alpha,\beta}(x,y,z) \geq 0$ as a definite integral of an explicit nonnegative elementary function.

In fact he found this by combining two formulas in Watson's *Treatise on the theory of Bessel functions*, see there pages 411 and 413.

Jacobi product formula (cntd.)

Watson essentially has the same nonnegative kernel in the following two product formulas ($\alpha > \beta > -\frac{1}{2}$):

$$\frac{J_\alpha(x)}{x^\alpha}\,\frac{J_\beta(y)}{y^\beta} = \frac{1}{2^\alpha\Gamma(\alpha+1)}\int_0^\infty \frac{J_\beta(z)}{z^\beta}\,\widetilde{K}_{\alpha,\beta}(x,y,z)\,z^{2\beta+1}\,dz,$$

$$R_n^{(\alpha,\beta)}(\cos 2\theta_1)\,R_n^{(\alpha,\beta)}(\cos 2\theta_2) = \int_0^{\pi/2} R_n^{(\alpha,\beta)}(\cos 2\theta_3)$$

$$\times\,\widetilde{K}_{\alpha,\beta}(\sin\theta_1\sin\theta_2,\cos\theta_1\cos\theta_2,\cos\theta_3)\,(\cos\theta_3)^{2\beta+1}\,\sin\theta_3\,d\theta_3.$$

Askey's question

Rewrite the Gasper-Watson Jacobi product formula as something similar to the Gegenbauer product formula

$$R_n^{(\alpha,\alpha)}(x)R_n^{(\alpha,\alpha)}(y) = \int_{-1}^{1} R_n^{(\alpha,\alpha)}\left(xy+(1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}t\right)d\mu_{\alpha-\frac{1}{2},\alpha-\frac{1}{2}}(t).$$

For this purpose work with addition formulas and group theory.

Gegenbauer addition formula

The Gegenbauer product formula gives the constant term in the Gegenbauer addition formula:

$$R_n^{(\alpha,\alpha)}(xy+(1-x^2)^{\frac12}(1-y^2)^{\frac12}t) = \sum_{k=0}^{n} \frac{(-1)^k(-n)_k\,(n+2\alpha+1)_k}{2^{2k}((\alpha+1)_k)^2\,h_k^{(\alpha-\frac12,\alpha-\frac12)}}$$

$$\times(1-x^2)^{k/2}R_{n-k}^{(\alpha+k,\alpha+k)}(x)\,(1-y^2)^{k/2}R_{n-k}^{(\alpha+k,\alpha+k)}(y)\,R_k^{(\alpha-\frac12,\alpha-\frac12)}(t).$$

For $\alpha=\frac12(d-3)$ by group theory: $G=O(d)\supset K=O(d-1)\supset M=O(d-2)$, $A=SO(2)\subset G$ commuting with M;
ϕ spherical function for (G,K) and ψ_δ for (K,M) (Gelfand pairs):

$$\phi(a_1ka_2)=\sum_{\delta\in(K/M)^\wedge}\widehat{\phi}_{a_1,a_2}(\delta)\,d_\delta\,\psi_\delta(k)\quad(k\in K,\ a_1,a_2\in A).$$

Or as reproducing kernel for spherical harmonics of degree n:

$$R_n^{(\alpha,\alpha)}(\langle x,y\rangle)=\frac{1}{d_n}\sum_{k=1}^{d_n}Y_{n,k}(x)\,Y_{n,k}(y)\quad(x,y\in S^{d-1}=O(d)/O(d-1)).$$

disk polynomials

Gangolli: Jacobi polynomials $R_n^{(d-2,0)}$ are spherical functions on complex projective space $P^{d-1}(\mathbb{C}) = U(d)/(U(1) \times U(d-1))$ (a compact Riemannian symmetric space of rank one).

But this is the space of $U(1)$-orbits on $S^{2d-1} = U(d)/U(d-1)$ (unit sphere in $\mathbb{C}^d$). Functions on $P^{d-1}(\mathbb{C})$ are $U(1)$-invariant functions on S^{2d-1}.

Moreover $(U(d), U(d-1))$ is Gelfand pair with Zernike's disk polynomials $R_{m,n}^{\alpha}(z)$ $(\alpha = d-2)$ as spherical functions.

$$R_{m,n}^{\alpha}(re^{i\phi}) := R_{\min(m,n)}^{(\alpha,|m-n|)}(2r^2-1)\, r^{|m-n|}\, e^{i(m-n)\phi},$$

$$\int_D R_{m,n}^{\alpha}(x+iy)\,\overline{R_{k,l}^{\alpha}(x+iy)}\,(1-x^2-y^2)^{\alpha}\,dx\,dy = 0$$

$$((m,n) \neq (k,l);\ D \text{ unit disk}).$$

Work with complex spherical harmonics on $\mathbb{C}^d$: refinement of ordinary spherical harmonics on $\mathbb{R}^{2d}$.

disk polynomials (cntd.)

The disk polynomials, introduced by the Dutch Nobel prize winner Zernike, find important applications at the Dutch world leading chip machine maker ASML.

Addition formula for disk polynomials

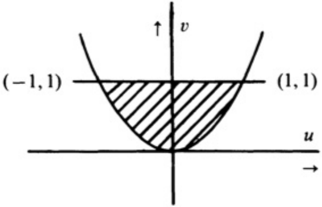
R. L. Šapiro (1968), K (1972):

$$R^{\alpha}_{m,n}\big(z_1 z_2 + (1-|z_1|^2)^{\frac12}(1-|z_2|^2)^{\frac12} w\big)$$
$$= \sum_{k=0}^{m}\sum_{l=0}^{n} c^{\alpha}_{m,n,k,l}\,(1-|z_1|^2)^{\frac12(k+l)} R^{\alpha+k+l}_{m-k,n-l}(z_1)$$
$$\times\,(1-|z_2|^2)^{\frac12(k+l)} R^{\alpha+k+l}_{m-k,n-l}(z_2)\,R^{\alpha-1}_{k,l}(w).$$

This yields an addition formula for Jacobi polynomials $R_n^{(\alpha,0)}$ and next, by differentiation and by analytic continuation in the parameters, an addition formula for Jacobi polynomials $R_n^{(\alpha,\beta)}$ $(\alpha > \beta > -\frac12)$. It involves an expansion in terms of orthogonal polynomials in two variables on a parabolic biangle.

$$R_{n,k}^{(\alpha,\beta)}(x,y) := R_k^{(\alpha,\beta+n-k+\frac{1}{2})}(2y-1)\,y^{\frac{1}{2}(n-k)}\,R_{n-k}^{(\beta,\beta)}(y^{-\frac{1}{2}}x).$$

$$\int_{y=0}^{1}\int_{x=-y^{\frac{1}{2}}}^{y^{\frac{1}{2}}} R_{n,k}^{(\alpha,\beta)}(x,y)R_{m,l}^{(\alpha,\beta)}(x,y) \times (1-y)^{\alpha}\,(y-x^2)^{\beta}\,dx\,dy = 0 \quad ((n.k)\neq(m,l)).$$



Parametrize this region by

$(x,y) = (r\cos\phi, r^2)$

$(0 \le r \le 1,\ 0 \le \phi \le \pi)$.

Put $$d\nu_{\alpha,\beta}(r,\phi) := \frac{r^{2\beta+2}(1-r^2)^{\alpha}\,(\sin\phi)^{2\beta+1}\,dr\,d\phi}{\int_{r=0}^{1}\int_{\phi=0}^{\pi} r^{2\beta+2}(1-r^2)^{\alpha}\,(\sin\phi)^{2\beta+1}\,dr\,d\phi}.$$

Addition formula for Jacobi polynomials

$$\Lambda(x,y,r,\phi) := \tfrac{1}{2}(1+x)(1+y) + \tfrac{1}{2}(1-x)(1-y)\,r^2 + (1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}\,r\cos\phi - 1.$$

$$R_n^{(\alpha,\beta)}(\Lambda(x,y,r,\phi)) = \sum_{k=0}^{n}\sum_{l=0}^{k} c_{n,k,l}^{(\alpha,\beta)}\,(1-x)^{\frac{1}{2}(k+l)}(1+x)^{\frac{1}{2}(k-l)} \times R_{n-k}^{(\alpha+k+l,\beta+k-l)}(x)\,(1-y)^{\frac{1}{2}(k+l)}(1+y)^{\frac{1}{2}(k-l)}R_{n-k}^{(\alpha+k+l,\beta+k-l)}(y) \times R_{k,l}^{(\alpha-\beta-\frac{1}{2},\beta-\frac{1}{2})}(r\cos\phi, r^2).$$

Constant term in the expansion is the Jacobi product formula

$$R_n^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(y) = \int_{r=0}^{1}\int_{\phi=0}^{\pi} R_n^{(\alpha,\beta)}(\Lambda(x,y,r,\phi))\,d\nu_{\alpha-\beta-\frac{1}{2},\beta-\frac{1}{2}}(r,\phi).$$

Conversely, the product formula implies the addition formula by integration by parts and Rodrigues type formulas.

Laplace type integral representation

(Askey, 1974; K, 1974)

The Gegenbauer Laplace type integral representation is a degenerate case of the Gegenbauer product formula:

$$R_n^{(\alpha,\alpha)}(x) = \int_{-1}^{1} (x + i(1-x^2)^{\frac{1}{2}}\, t)^n \, d\mu_{\alpha-\frac{1}{2},\alpha-\frac{1}{2}}(t).$$

Combine this with a fractional integral or degenerate the Jacobi product formula for obtaining

Jacobi Laplace type integral representation: $R_n^{(\alpha,\beta)}(x) =$

$$\int_{r=0}^{1}\int_{\phi=0}^{\pi} \left(\tfrac{1}{2}(1+x) - \tfrac{1}{2}(1-x)r^2 + i(1-x^2)^{\frac{1}{2}}\, r\cos\phi\right)^n d\nu_{\alpha-\beta-\frac{1}{2},\beta-\frac{1}{2}}(r,\phi).$$

One can go back and forth between this integral representation and the Jacobi product formula by Bateman's bilinear sum and its inverse.

Bateman's bilinear sum and its inverse

$$(x+y)^n\,R_n^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right)=\sum_{k=0}^n a_{n,k}R_k^{(\alpha,\beta)}(x)R_k^{(\alpha,\beta)}(y),$$

$$\text{where}\quad (x+1)^n=\sum_{k=0}^n a_{n,k}R_k^{(\alpha,\beta)}(x);$$

$$R_n^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(y)=\sum_{k=0}^n b_{n,k}(x+y)^k\,R_k^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right),$$

$$\text{where}\quad R_n^{(\alpha,\beta)}(x)=\sum_{k=0}^n b_{n,k}(x+1)^k.$$

These connect

$$(x+y)^n\,R_n^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right)=\int_{r=0}^1\int_{\phi=0}^\pi(\Lambda(x,y,r,\phi)+1)^n\,d\nu_{\alpha-\beta-\frac12,\beta-\frac12}(r,\phi)$$

$$\text{and } R_n^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(y)=\int_{r=0}^1\int_{\phi=0}^\pi R_n^{(\alpha,\beta)}(\Lambda(x,y,r,\phi))\,d\nu_{\alpha-\beta-\frac12,\beta-\frac12}(r,\phi).$$

Hypergroups

The term *hypergroup* seems to have been used first in the paper

H. S. Wall, *Hypergroups*, Amer. J. Math. 59 (1937), 77–98.

It generalizes the definition of a group in the sense that the product of two elements is a sum of finitely many elements. Commutativity is not required and each element has an inverse.

Dunkl (TAMS, 1973) first defines a hypergroup as a locally compact space on which the space of finite regular Borel measures has a convolution structure preserving the probability measures. He requires commutativity and there is no inverse.

Similar but differently phrased definitions were given by Jewett (Adv. Math., 1975) and Spector (TAMS, 1978). An imported added axiom involves the so-called Michael topology of the collection of compact subsets of a locally compact space.

Hypergroups according to Dunkl, Jewett and Spector are sometimes called **DJS-hypergroups**.

Further developments

- Gasper (1972): Extension of Jacobi generalized translation to absolutely bounded (not necessarily positive) case.
- Dunkl: Addition formulas for Krawtchouk, Hahn and q-Hahn polynomials from interpretation on finite groups.
- D. Stanton: Similarly for q-Krawtchouk polynomials.
- K: Addition formula for little q-Legendre polynomials from quantum group interpretation.
- Floris: Addition formula for q-disk polynomials in non-commuting variables from quantum group interpretation.
- Koelink: addition formulas in many q-cases, both from quantum groups and analytic.
- Rahman: analytic proofs of q-addition formulas.
- K & A. Schwartz: positive convolution for orthogonal polynomials on triangle and simplex.
- Carlen, Geronimo & Loss (2011) proved Gssper's positivity of Jacobi generalized translation by probabilistic means.

Heckman-Opdam Jacobi polynomials

The big open problem: Show the positivity of convolution for Heckman-Opdam Jacobi polynomials.

Partial results by Rösler and by Remling & Rösler in BC_n case

Partial, yet unpublished results in A_2 case by Dominique Bakry and co-workers.

Transmutation

$$q_n(x) = \int p_n(y)\, A(x,y)\, d\mu(y)$$

$$A(x,y) = \sum_{n=0}^{\infty} \frac{q_n(x)\, p_n(y)}{h_n}$$

Fractional integrals

Riemann-Liouville:

$$(R_\mu f)(x) := \frac{1}{\Gamma(\mu)} \int_0^x f(y)\,(x-y)^{\mu-1}\,dy \quad (\mathrm{Re}\,\mu > 0).$$

Weyl:

$$(W_\mu f)(x) := \frac{1}{\Gamma(\mu)} \int_x^\infty f(y)\,(y-x)^{\mu-1}\,dy \quad (\mathrm{Re}\,\mu > 0).$$

Askey & Fitch (1969) emphasized Bateman's integral:

$$\frac{x^{c+\mu-1}}{\Gamma(c+\mu)}\,{}_2F_1\left(\begin{matrix} a,b \\ c+\mu \end{matrix};x\right) = \frac{1}{\Gamma(\mu)} \int_0^x \frac{y^{c-1}}{\Gamma(c)}\,{}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};y\right)\,(x-y)^{\mu-1}\,dy,$$

$(\mathrm{Re}\,\mu, \mathrm{Re}\,c > 0)$. Hence, for $\mathrm{Re}\,\mu > 0$:

$$\frac{(1-x)^{\alpha+\mu}}{\Gamma(\alpha+\mu+1)}\,R_n^{(\alpha+\mu,\beta-\mu)}(x) = \frac{1}{\Gamma(\mu)} \int_x^1 \frac{(1-y)^\alpha}{\Gamma(\alpha+1)}\,R_n^{(\alpha,\beta)}(y)\,(y-x)^{\mu-1}\,dy.$$

Transmutation property

Bateman's integral in kernel form:

$$R_n^{(\alpha+\mu,\beta-\mu)}(x) = \int R_n^{(\alpha,\beta)}(y)\,A(x,y)\,d\mu_{\alpha,\beta}(y),$$

where $\quad A(x,y) = \dfrac{2^{\alpha+\beta+1}\,\Gamma(\alpha+\mu+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)\,\Gamma(\mu)}\,\dfrac{(y-x)_+^{\mu-1}}{(1-x)^{\alpha+\mu}(1+y)^{\beta}}\,.$

Transmutation Theorem. Let $\{p_n\}$ and $\{q_n\}$ be complete orthogonal systems with respect to measures $d\mu$ and $d\nu$, respectively. Let D and E be operators having the p_n repectively the q_n as eigenfunctions with the same eigenvalue λ_n. Suppose that $q_n(x) = \int p_n(y)\,A(x,y)\,d\mu(y)$. Then the operator $\mathcal{A}$ given by $(\mathcal{A}f)(y) := \int f(x)\,A(x,y)\,d\nu(x)$ satisfies the transmutation property $\mathcal{A}\circ E = D\circ\mathcal{A}$.

Hence in case of Bateman's integral : $D = D_{\alpha,\beta}$, $E = D_{\alpha+\mu,\beta-\mu}$,
where $\quad D_{\alpha,\beta}R_n^{(\alpha,\beta)} = -n(n+\alpha+\beta+1)\,R_n^{(\alpha,\beta)}$.

Feldheim-Vilenkin integral

Feldheim-Vilenkin integral
(not of the desired transmutation form):

$$\frac{(x-1)^{\alpha+\mu}}{\Gamma(\alpha+\mu+1)}\,x^{\frac{1}{2}n}\,R_n^{(\alpha+\mu,\alpha+\mu)}(x^{-\frac{1}{2}})$$
$$=\frac{1}{\Gamma(\mu)}\int_1^x \frac{(y-1)^{\alpha}}{\Gamma(\alpha+1)}\,y^{\frac{1}{2}n}\,R_n^{(\alpha,\alpha)}(y^{-\frac{1}{2}})\,(x-y)^{\mu-1}\,dy \quad (\mu>0).$$

Remark. Both the Bateman and Feldheim-Vilenkin integral can be obtained from spherical harmonics. For Bateman also use that

$$(x_1^2+\cdots+x_{q+p}^2)^n\,R_n^{(\frac{1}{2}p-1,\frac{1}{2}q-1)}\left(\frac{(x_1^2+\cdots+x_q^2)-(x_{q+1}^2+\cdots+x_{q+p}^2)}{x_1^2+\cdots+x_{q+p}^2}\right)$$

is an $O(q)\times O(p)$-invariant homogeneous harmonic polynomial of degree $2n$ on $\mathbb{R}^{q+p}$.

Transmutation in the non-compact case

Jacobi functions (surveyed by K, 1984). These form a continuous orthogonal system of Gauss hypergeometric functions. They are noncompact analogues of Jacobi polynomials. They have richer transmutation properties.

$$\phi_\lambda^{(\alpha,\beta)}(t) := {}_2F_1\left(\begin{matrix}\frac{1}{2}(\rho+i\lambda), \frac{1}{2}(\rho-i\lambda)\\ \alpha+1\end{matrix}; -\sinh^2 t\right), \quad \rho := \alpha+\beta+1;$$

$$\widehat{f}(\lambda) = \int_0^\infty f(t)\,\Delta_{\alpha,\beta}(t)\,dt, \qquad f(t) = \int_0^\infty \widehat{f}(\lambda)\,|c_{\alpha,\beta}(\lambda)|^{-2}\,d\lambda.$$

$$D_{\alpha,\beta}\,\phi_\lambda^{(\alpha,\beta)} = -\lambda^2\,\phi_\lambda^{(\alpha,\beta)}; \qquad \phi_\lambda^{(-\frac{1}{2},-\frac{1}{2})}(t) = \cos(\lambda t).$$

Transmutation: $\phi_\lambda^{(\alpha+\mu,\beta\pm\mu)}(t) = \int_0^t \phi_\lambda^{(\alpha,\beta)}(s)\,A(s,t)\,\Delta_{\alpha,\beta}(s)\,ds$

with $A(s,t)$ positive and elementary if $\mu > 0$. Relationship with Abel transform on noncompact semisimple Lie groups. Generalization to Chébli-Trimèche hypergroups.

Linearization of products

$$p_m(x)\,p_n(x) = \sum_{k=|m-n|}^{m+n} c_{m,n,k}\,p_k(x)/h_k$$

$$c_{m,n,k} = \int p_m(x)\,p_n(x)\,p_k(x)\,d\mu(x)$$

Linearization of products (cntd.)

Jacobi polynomials:

$$R_m^{(\alpha,\beta)}(x)\,R_n^{(\alpha,\beta)}(x)=\sum_{k=|m-n|}^{m+n} c_{m,n,k}^{(\alpha,\beta)}\,R_k^{(\alpha,\beta)}(x)/h_k^{(\alpha,\beta)}$$

Theorem (Gasper, 1970) (a) $\Longleftrightarrow$ (b) $\Longleftarrow$ (c)

(a) $c_{m,n,k}^{(\alpha,\beta)}\ge 0$ for all m,n,k.
(b) some quartic polynomial in α,β is nonnegative.
(c) $\alpha\ge\beta>-1$ and $\alpha+\beta>-1$.

General monic orthogonal polynomials p_n :

$$p_1(x)\,p_n(x)=p_{n+1}(x)+a_np_n(x)+b_np_{n-1}(x),$$
$$p_m(x)p_n(x)=\sum_{k=|m-n|}^{m+n} c_{m,n,k}\,p_k(x)/h_k.$$

Theorem (Askey, 1970)
$\forall n\quad a_n,b_n,a_{n+1}-a_n,b_{n+1}-b_n\ge 0\Longrightarrow\forall m,n,k\quad c_{m,n,k}\ge 0.$

This covers: If $\alpha\ge\beta$ and $\alpha+\beta\ge 1$ then $c_{m,n,k}^{(\alpha,\beta)}\ge 0$.

Linearization of products (cntd.)

Remark 1. A function f on a group G is called *positive definite* if for all $x_1, \ldots, x_k \in G$ and all $c_1, \ldots, c_k \in \mathbb{C}$

$$\sum_{i,j=1}^{k} f(x_i x_j^{-1}) c_i \overline{c_j} \geq 0.$$

If (G, K) is a Gelfand pair with G, K compact and with spherical functions ϕ_λ then $\phi_\lambda \phi_\mu = \sum_\nu c_{\lambda,\mu,\nu} \phi_\nu$ with $c_{\lambda,\mu,\nu} \geq 0$.

Indeed, spherical functions are elementary positive definite functions, a product of positive definite functions is again positive definite, and a K-biinvariant positive definite function is a nonnegative linear combination of spherical functions.

Thus for special parameter values the theorems of Gasper and Askey also follow from group theory.

Linearization of products (cntd.)

Remark 2. K (1978):
An addition formula obtained for a spherical function on a Gelfand pair carries the essential information making it positive definite and leading to nonnegative linearization coefficients.

This last information is preserved in an addition formula for other parameter values which do not come from group theory. The addition formula needs to have certain properties. In particular, the expansion coefficients in the addition formula should be nonnegative. Then it implies the nonnegativity of the linearization coefficients.

This works in the Jacobi case for $\alpha \geq \beta \geq -\frac{1}{2}$.

An application to Laguere polynomials

This works also for disk polynomials.
If these are rewritten in terms of Jacobi polynomials and next the limit to the Laguerre case is taken then:

$$\int_0^\infty L_k^\alpha(x)\,L_m^\alpha(\lambda x)\,L_n^\alpha((1-\lambda)x)\,x^\alpha\,e^{-x}\,dx \ge 0 \quad (\alpha \ge 0,\ \lambda \in [0,1]).$$

By iteration:

$$\int_0^\infty L_{n_1}^\alpha(x)\,L_{n_2}^\alpha(x)\,L_{n_3}^\alpha(x)\,L_{n_4}^\alpha(x)\,x^\alpha\,e^{-2x}\,dx > 0 \quad (\alpha > 0).$$

This leads to the four boxes paper by Askey, Ismail & K (1978).

Connection formula

$$q_n(x) = \sum_{k=0}^{n} a_{n,k}\, p_k(x)/h_k$$

$$a_{n,k} = \int q_n(x)\, p_k(x)\, d\mu(x)$$

Connection formula (cntd.)

$$R_n^{(\gamma,\delta)}(x) = \sum_{k=0}^{n} a_{n,k}\, R_k^{(\alpha,\beta)}(x)/h_k^{(\alpha,\beta)} \Longrightarrow a_{n,k} = \text{stuff} \times {}_3F_2(1).$$

In particular, $a_{n,k}$ is elementary and nonnegative in the cases

$$R_n^{(\gamma,\gamma)}(x) = \sum_{k=0}^{n} a_{n,k}\, R_k^{(\alpha,\alpha)}(x)/h_k^{(\alpha,\alpha)} \quad (\gamma > \alpha > -1),$$

$$R_n^{(\gamma,\beta)}(x) = \sum_{k=0}^{n} a_{n,k}\, R_k^{(\alpha,\beta)}(x)/h_k^{(\alpha,\beta)} \quad (\gamma > \alpha > -1).$$

Askey & Gasper (1971) give sufficient conditions for nonnegativity of $a_{n,k} = a_{n,k}^{(\gamma,\delta),(\alpha,\beta)}$. For given (α,β) this includes an infinite region in the (γ,δ) plane bounded by three lines with $(\gamma,\delta) = (2\alpha+1, 2\beta+1)$ as one of the vertices.

Askey (1968): Certain of these positivity cases from isometric embeddings of projective spaces.

Connection formula (cntd.)

Nevai (1979): Connection coefficients for p_n in terms of Chebyshev polynomials T_k are limits of linearization coefficients for p_n.

Lasser (1994): Under certain assumptions positivity of linearization coefficients implies positivity of connection coefficients with Chebyshev.

Further work by Szwarc.

It seems that certain conditions on the coefficients in the three-term recurrence relation can identify a class of orthogonal polynomials giving rise to the dual case of the Chébli-Trimèche hypergroups.