Charting the Askey and q-Askey schemes

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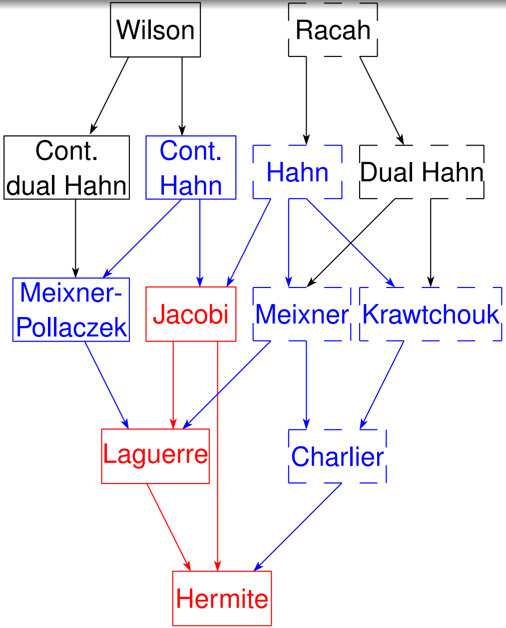
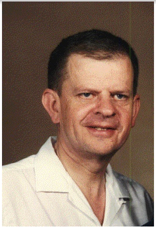
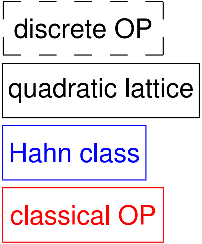
Askey scheme





Dick Askey
1933–2019



1. Uniform parameters for the Askey scheme

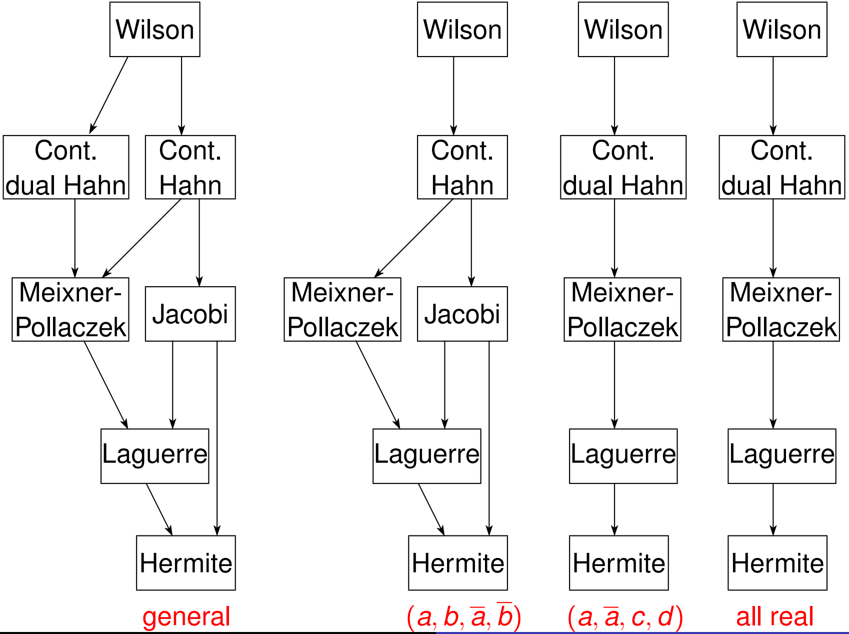
T. H. Koornwinder, The Askey scheme as a four-manifold with corners, *Ramanujan J.* **20** (2009), 409–439; arXiv:0909.2822.

Monic Wilson polynomial:

$$w_n(x^2; a, b, c, d) := \frac{(-1)^n (a+b)_n (a+c)_n (a+d)_n}{(n+a+b+c+d-1)_n} \times {}_4F_3\left(\begin{matrix} -n, n+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d \end{matrix}; 1\right)$$

Either two pairs of complex conjugate parameters
or one pair of complex conjugate parameters and two real ones
or four real parameters.

Wilson scheme and its subschemes



first Wilson subscheme

Rescaled monic Wilson polynomial in terms of new parameters a_1, a_2, a_3, a_4:

$$p_n(x; a_1, a_2, a_3, a_4) = \rho^n\, w_n(\rho^{-1}x - \sigma; a, b, \overline{a}, \overline{b}),$$

where

$$a = a_1^{-1} - \frac{1 - a_1^{1/2}\, a_2\, a_4}{2\, a_1^{3/2}\, a_2^2\, a_3\, a_4}\, \mathrm{i}, \quad b = a_1^{-1}\, a_2^{-1} - \frac{1 + a_1^{1/2}\, a_2\, a_4}{2\, a_1^{3/2}\, a_2^2\, a_3\, a_4}\, \mathrm{i},$$

$$\rho = 2^{3/2}\, a_1^2\, a_2^2\, a_3^2\, a_4,$$

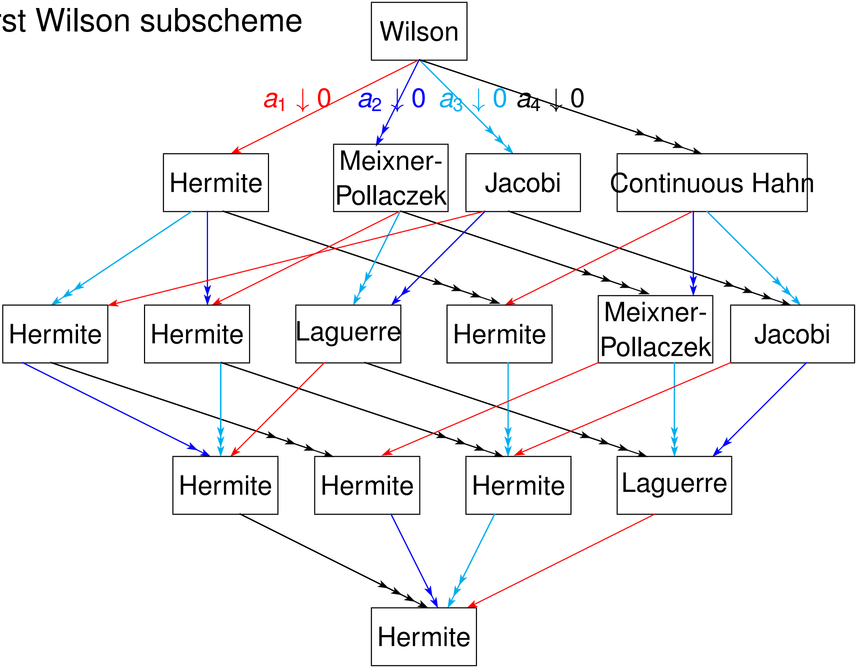
$$\sigma = -\frac{1}{4\, a_1^3\, a_2^4\, a_3^2\, a_4^2} + \frac{1 - a_2}{2\, a_1^{5/2}\, a_2^3\, (1 + a_2 - a_1\, a_2)\, a_3^2\, a_4}\,.$$

$p_n(x; a_1, a_2, a_3, a_4)$ satifies three-term recurrence relation

$$x\, p_n(x) = p_{n+1}(x) + B_n\, p_n(x) + C_n\, p_{n-1}(x)$$

with $C_n > 0$ and $B_n \in \mathbb{R}$ depending continuously on $a_1, a_2, a_3, a_4 \geq 0$.

first Wilson subscheme



2. Askey–Wilson polynomials

Askey–Wilson polynomial, monic in $z+z^{-1}$:

$$\frac{a^n\,(q^{n-1}abcd;q)_n}{(ab,ac,ad;q)_n}\,p_n^{\rm monic}(z+z^{-1};a,b,c,d;q)$$

$$={}_4\phi_3\left(\begin{matrix}q^{-n},q^{n-1}abcd,az,az^{-1}\\ ab,ac,ad\end{matrix};q,q\right)$$

$$=\sum_{k=0}^n\frac{(q^{-n},q^{n-1}abcd,az,az^{-1};q)_k}{(ab,ac,ad;q)_k\,(q;q)_k}\,q^k$$

$$=\sum_{k=0}^n\frac{q^k}{(ab,ac,ad;q)_k\,(q;q)_k}\,(q^{-n},q^{n-1}abcd;q)_k\,(az,az^{-1};q)_k\,.$$

Askey–Wilson polynomials, cntd.

$$p_n^{\mathrm{monic}}(z+z^{-1};a,b,c,d;q)=\frac{(ab,ac,ad;q)_n}{a^n\,(q^{n-1}abcd;q)_n}$$
$$\times\sum_{k=0}^{n}\frac{q^k}{(ab,ac,ad;q)_k\,(q;q)_k}\,(q^{-n},q^{n-1}abcd;q)_k\,(az,az^{-1};q)_k$$
$$=\left(\prod_{j=0}^{n-1}\frac{g_{j+1}}{h_n-h_j}\right)\sum_{k=0}^{n}\prod_{j=0}^{k-1}\frac{(h_n-h_j)\,(z+z^{-1}-x_j)}{g_{j+1}},$$

where

$$x_k=aq^k+a^{-1}q^{-k},\qquad h_k=abcdq^{k-1}+q^{-k},$$
$$g_k=a^{-1}q^{-2k+1}(1-abq^{k-1})(1-acq^{k-1})(1-adq^{k-1})(1-q^k)$$

Verde-Star's idea

$$p_n^{\mathrm{monic}}(x;a,b,c,d;q)=u_n(x)=\sum_{k=0}^{n}c_{n,k}\,v_k(x),$$

where $v_k(x)=(x-x_0)(x-x_1)\ldots(x-x_{k-1})$ (Newton basis)

and $c_{n,k}=\displaystyle\prod_{j=k}^{n-1}\frac{g_{j+1}}{h_n-h_j}$.

Then $Lu_n=h_n\,u_n$, where the operator L is determined by $Lv_n=h_nv_n+g_nv_{n-1}$.

Also the operator X of multiplication by x is determined by $Xv_n=x_nv_n+v_{n+1}$.

$$h_k=a_0+a_1q^k+a_2q^{-k},\quad x_k=b_0+b_1q^k+b_2q^{-k},$$
$$g_k=d_0+d_1q^k+d_2q^{-k}+d_3q^{2k}+d_4q^{-2k},\quad \textstyle\sum_{i=0}^4 d_i=0,$$
$$d_3=q^{-1}a_1b_1,\quad d_4=qa_2b_2.$$

Luis Verde-Star, Linear Algebra Appl. 627 (2021), 242–274

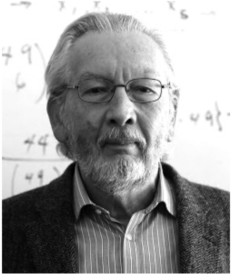
The a_i, b_i, d_i parametrize the q-Askey scheme.

3. The (q-)Verde-Star scheme

T. H. Koornwinder, *Charting the q-Askey scheme*, `arXiv:2108.03858v1`, 2021.

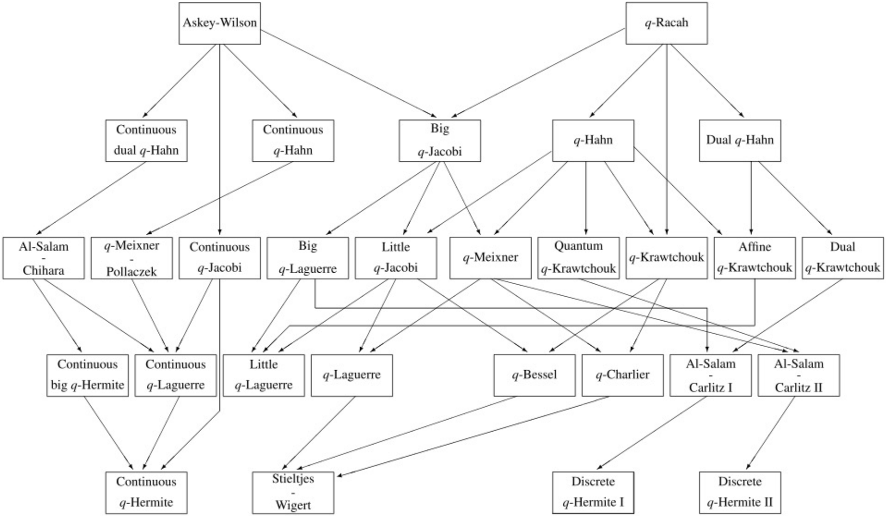
Inspired by the paper

L. Verde-Star, *A unified construction of all the hypergeometric and basic hypergeometric families of orthogonal polynomial sequences*, Linear Algebra Appl. 627 (2021), 242–274.



Luis Verde-Star

q-Askey scheme



q-Verde-Star scheme

$$u_n(x)=\sum_{k=0}^{n} c_{n,k}\, v_k(x),\quad v_k(x)=\prod_{j=0}^{k-1}(x-x_j),\ \ c_{n,k}=\prod_{j=k}^{n-1}\frac{g_{j+1}}{h_n-h_j}\,,$$

$$x_k=b_2q^{-k}+b_0+b_1q^k,\quad h_k=a_2q^{-k}+a_0+a_1q^k,$$

$$g_k=d_4q^{-2k}+d_2q^{-k}+d_0+d_1q^k+d_3q^{2k},\quad \textstyle\sum_{i=0}^4 d_i=0,$$

$$d_3=q^{-1}a_1b_1,\quad d_4=qa_2b_2.$$

Translation invariances: $x\to x+\sigma$, $x_k\to x_k+\sigma$, $h_k\to h_k+\tau$.
Dilation invariance: $u_n(x)\to\rho^n u_n(\rho^{-1}x)$, $v_k(x)\to\rho^k v_k(\rho^{-1}x)$, $x_k\to\rho x_k$, $g_k\to\rho g_k$.
Homogeneous of degree zero in h_k, g_k: $\quad h_k\to\mu h_k$, $g_k\to\mu g_k$.

$3+3+5=11$ parameters, 3 constraints, 4 invariances:
Four essential parameters.

$q\leftrightarrow q^{-1}$ *exchange*: $\quad a_1\leftrightarrow a_2,\ b_1\leftrightarrow b_2,\ d_1\leftrightarrow d_2,\ d_3\leftrightarrow d_4.$

duality: $a_0\leftrightarrow d_0,\ a_1\leftrightarrow d_1,\ a_2\leftrightarrow d_2.$

q-Verde-Star scheme, cntd.

Represent by

$$
\begin{array}{llll}
x_k = & b_2 q^{-k} + b_0 + b_1 q^k & & b_2 \; b_0 \; b_1 \\
g_k = d_4 q^{-2k} + & d_2 q^{-k} + d_0 + d_1 q^k + d_3 q^{2k} & & d_4 \; d_2 \; d_0 \; d_1 \; d_3 \\
h_k = & a_2 q^{-k} + a_0 + a_1 q^k & & a_2 \; a_0 \; a_1
\end{array}
$$

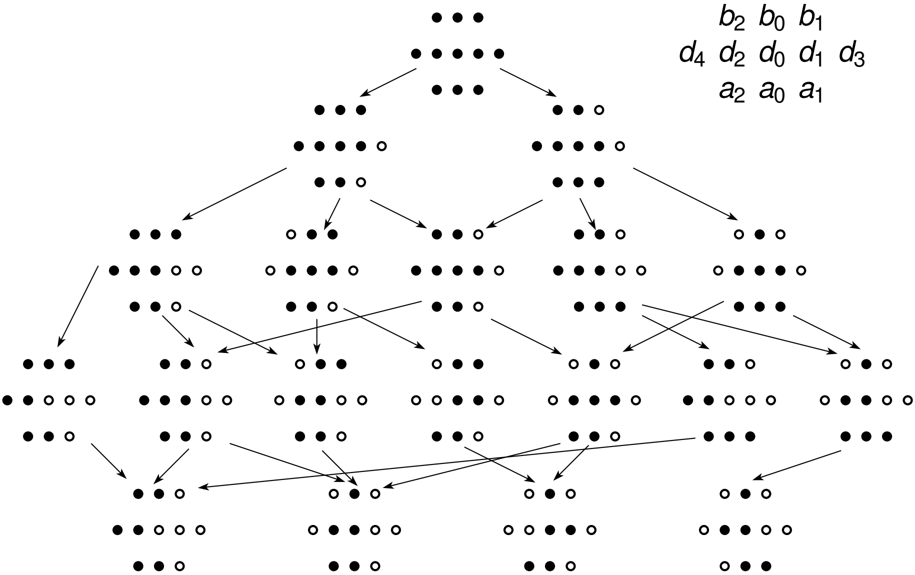
$\bullet$ denotes any parameter value and $\circ$ a zero parameter value.

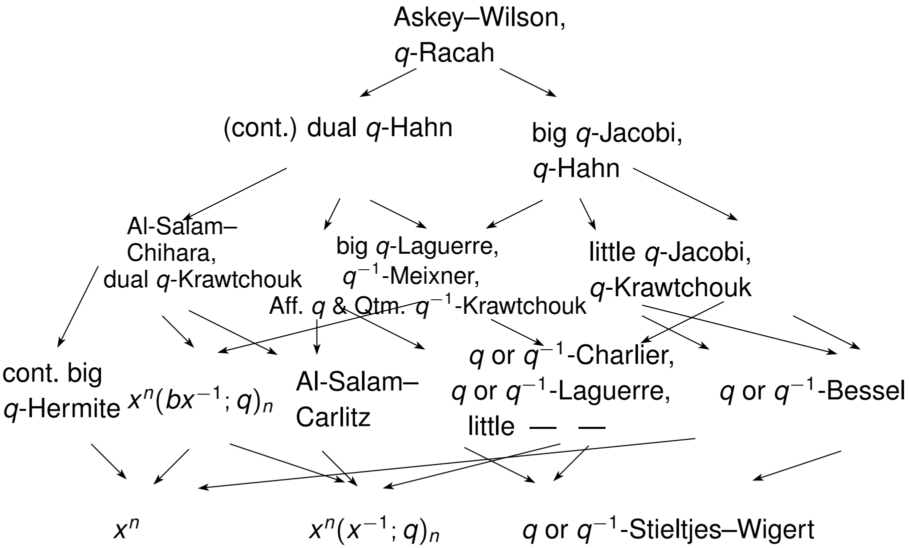
So Askey–Wilson corresponds to the symbol $\begin{array}{ccccc} & \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \end{array}$.

Rules:

1. If b_1 or a_1 is $\circ$ then d_3 is $\circ$; if b_2 or a_2 is $\circ$ then d_4 is $\circ$.
2. b_0 and a_0 are always $\bullet$.
3. In the second row no $\circ$ between two $\bullet$ ones.
4. In the second and third row at least two $\bullet$ ones.
5. Flipping a $\bullet$ into a $\circ$ causes an arrow between the symbols.
6. Reflection w.r.t. central column means $q \leftrightarrow q^{-1}$.
7. Reflection w.r.t. middle row means $x - x_k \leftrightarrow h_n - h_k$ (duality).

q-Verde-Star scheme with symbols




Askey–Wilson, q-Racah
(cont.) dual q-Hahn
big q-Jacobi, q-Hahn
Al-Salam–Chihara, dual q-Krawtchouk
big q-Laguerre, q^{-1}-Meixner, Aff. q & Qtm. q^{-1}-Krawtchouk
little q-Jacobi, q-Krawtchouk
cont. big q-Hermite
$x^n(bx^{-1};q)_n$
Al-Salam–Carlitz
q or q^{-1}-Charlier, q or q^{-1}-Laguerre, little — —
q or q^{-1}-Bessel
x^n
$x^n(x^{-1};q)_n$
q or q^{-1}-Stieltjes–Wigert

q-Verde-Star scheme. Remarks

- Not just classification of families of OPs, but in combination with families of generalized monomials in which the OPs are expanded.
- Therefore same family of OPs may occur twice in the scheme. See big q-Laguerre, little q-Jacobi, q-Bessel.
- The scheme does not consider orthogonality w.r.t. a positive measure, but it classifies families of q-hypergeometric polynomials which are eigenfunctions of an operator in the x-variable and which can be seen to satify a three-term recurrence relation.
- One position in the scheme may contain both a continuous and a discrete family.
- The scheme also contains a few degenerate cases.
- The continuous q-Hermite polynomials are missing in the scheme, because they have only an expansion in terms of $x^n, x^{n-2}, \ldots$. The discrete q-Hermite I, II polynomials are subfamilies of the Al-Salam–Carlitz I, II polynomials.

Verde-Star scheme for $q = 1$

$$u_n(x) = \sum_{k=0}^{n} c_{n,k}\, v_k(x), \quad v_k(x) = \prod_{j=0}^{k-1}(x - x_j), \ \ c_{n,k} = \prod_{j=k}^{n-1} \frac{g_{j+1}}{h_n - h_j}\,,$$

$$x_k = b_2 k^2 + b_1 k + b_0, \quad h_k = a_2 k^2 + a_1 k + a_0,$$

$$g_k = d_4 k^4 + d_3 k^3 + d_2 k^2 + d_1 k + d_0, \quad d_0 = 0,$$

$$d_4 = a_2 b_1, \quad d_3 = a_1 b_2 + a_2 b_1 - 4 a_2 b_2.$$

Obtained from rescaled x_k, h_k, g_k in the q-case and then $q \to 1$:

$$x_k = \widetilde{b}_2 q^{-k} \left(\tfrac{1-q^k}{1-q}\right)^2 + \widetilde{b}_1 \tfrac{1-q^k}{1-q} + \widetilde{b}_0\,,$$

$$h_k = \widetilde{a}_2 q^{-k} \left(\tfrac{1-q^k}{1-q}\right)^2 + \widetilde{a}_1 \tfrac{1-q^k}{1-q} + \widetilde{a}_0\,,$$

$$g_k = \widetilde{d}_4 q^{-2k} \left(\tfrac{1-q^k}{1-q}\right)^4 + \widetilde{d}_3 q^{-k} \left(\tfrac{1-q^k}{1-q}\right)^3 + \widetilde{d}_2 q^{-k} \left(\tfrac{1-q^k}{1-q}\right)^2 + \widetilde{d}_1 \tfrac{1-q^k}{1-q}$$

(or variant of this if some original a_i, b_i, d_i vanish).

For $q = 1$ a similar but simpler scheme as in the q-case.
Hermite polynomials cannot be handled.

Verde-Star scheme for $q=-1$

$$u_n(x)=\sum_{k=0}^{n} c_{n,k}\, v_k(x),\quad v_k(x)=\prod_{j=0}^{k-1}(x-x_j),\ \ c_{n,k}=\prod_{j=k}^{n-1}\frac{g_{j+1}}{h_n-h_j}\,,$$

$$x_k=b_2k(-1)^k+b_1(-1)^k+b_0,\ \ h_k=a_2k(-1)^k+a_1(-1)^k+a_0,$$

$$g_k=d_4k^2+d_3k+d_2k(-1)^k+d_1(-1)^k+d_0,\quad d_0+d_1=0,$$

$$d_4=-a_2b_2,\quad d_3=-a_1b_2-a_2b_1+a_2b_2.$$

Obtained from rescaled x_k, h_k, g_k in the q-case and $q\to -1$:

$$x_k=\widetilde{b}_2q^{-k}\,\tfrac{1-q^{2k}}{1-q^2}+\widetilde{b}_1\,q^k+\widetilde{b}_0,\quad h_k=\widetilde{a}_2q^{-k}\,\tfrac{1-q^{2k}}{1-q^2}+\widetilde{a}_1q^k+\widetilde{a}_0,$$

$$g_k=\widetilde{d}_4q^{-2k}\left(\tfrac{1-q^{2k}}{1-q^2}\right)^2+\widetilde{d}_3\,\tfrac{1-q^{2k}}{1-q^2}+\widetilde{d}_2\,q^{-k}\,\tfrac{1-q^{2k}}{1-q^2}+\widetilde{d}_1(q^k-1)$$

(or variant of this if some original a_i, b_i, d_i vanish).

A Verde-Star scheme for $q=-1$ should be explored. Connect with the Bannai-Ito polynomials and many papers by Vinet et al.

4. The (q-)Zhedanov scheme

Inspired by the paper

Ya. I. Granovskii, I. M. Lutzenko and A. S. Zhedanov, *Mutual integrability, quadratic algebras, and dynamical symmetry*, Ann. Physics 217 (1992), 1–20.

Alexei Zhedanov

Zhedanov algebra

Let K_1 and K_2 be operators acting on sequences $f = \{f_n\}_{n=0}^{\infty}$:
$(K_1 f)_n = h_n f_n + g_{n+1} f_{n+1}, \quad (K_2 f)_n = x_n f_n + f_{n-1}$, where
$x_k = b_2 q^{-k} + b_0 + b_1 q^k, \quad h_k = a_2 q^{-k} + a_0 + a_1 q^k$,
$g_k = d_4 q^{-2k} + d_2 q^{-k} + d_0 + d_1 q^k + d_3 q^{2k}, \quad \sum_{i=0}^{4} d_i = 0$,
$d_3 = q^{-1} a_1 b_1, \quad d_4 = q a_2 b_2$.

Then

$$\begin{aligned}(q+q^{-1})K_2K_1K_2 - K_2^2K_1 - K_1K_2^2 \\ = A_1(K_1K_2 + K_2K_1) + A_2K_2^2 + C_1K_1 + DK_2 + G_1,\end{aligned}$$

$$\begin{aligned}(q+q^{-1})K_1K_2K_1 - K_1^2K_2 - K_2K_1^2 \\ = A_2(K_1K_2 + K_2K_1) + A_1K_1^2 + C_2K_2 + DK_1 + G_2.\end{aligned}$$

In addition a Casimir element Q: $(Qf)_n = \omega f_n$.

Assume $a_0 = b_0 = 0$. Then $A_1 = A_2 = 0$. Then the coefficients C_1, C_2, D, G_1, G_2 and the Casimir ω can be expressed in terms of the a_i, b_i, d_i.

Zhedanov algebra for $a_0 = b_0 = 0$

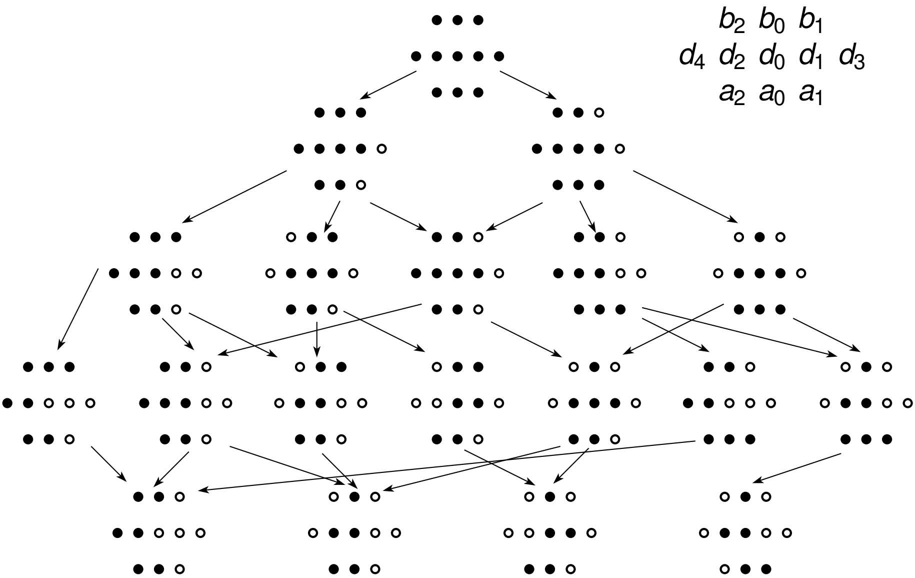
$$
\begin{aligned}
&(q+q^{-1})K_2K_1K_2 - K_2^2K_1 - K_1K_2^2 = C_1K_1 + DK_2 + G_1,\\
&(q+q^{-1})K_1K_2K_1 - K_1^2K_2 - K_2K_1^2 = C_2K_2 + DK_1 + G_2,\\
&Q = -\tfrac{1}{2}(q+q^{-1})(K_1K_2^2K_1 + K_2K_1^2K_2) + K_1K_2K_1K_2 + K_2K_1K_2K_1\\
&\qquad +\tfrac{1}{2}(q+q^{-1})(C_1K_1^2 + C_2K_2^2) + D(K_1K_2 + K_2K_1)\\
&\qquad +\tfrac{1}{2}(2+q+q^{-1})(G_1K_1 + G_2K_2) = \omega I,
\end{aligned}
$$

where

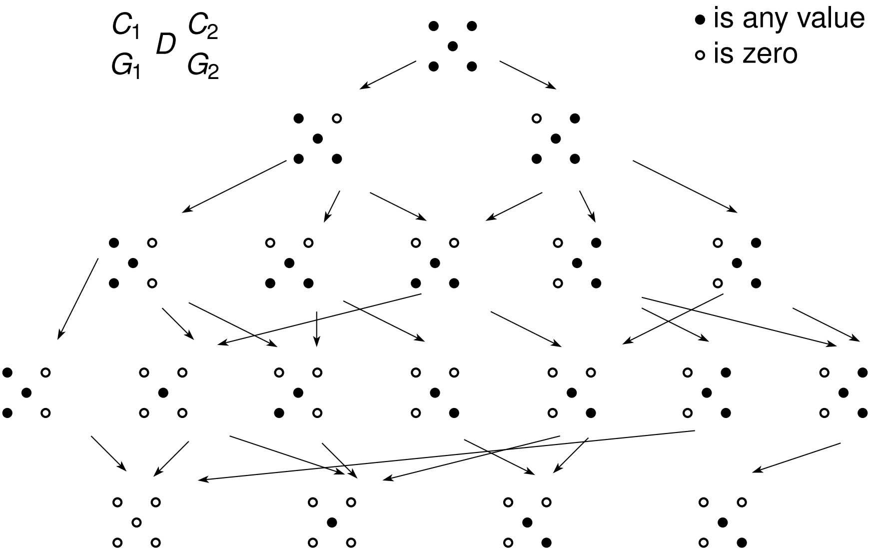
$$
\begin{aligned}
&C_1 = (q-q^{-1})^2 b_1b_2, \qquad C_2 = (q-q^{-1})^2 a_1a_2,\\
&D = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2\,(a_2b_1 + a_1b_2 + d_0),\\
&G_1 = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(q-q^{-1})(q^{-\frac{1}{2}}b_1d_2 + q^{\frac{1}{2}}b_2d_1),\\
&G_2 = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(q-q^{-1})(q^{-\frac{1}{2}}a_1d_2 + q^{\frac{1}{2}}a_2d_1), \quad \omega = \ldots.
\end{aligned}
$$

If $K_1 \to \lambda K_1$ and $K_2 \to \mu K_2$ then
$C_1, C_2, D, G_1, G_2, \omega \to \mu^2C_1, \lambda^2C_2, \lambda\mu D, \lambda\mu^2G_1, \lambda^2\mu G_2, \lambda^2\mu^2\omega$
Essentially four parameters.
Duality: $K_1 \leftrightarrow K_2$ corresponds to $C_1 \leftrightarrow C_2$, $G_1 \leftrightarrow G_2$, while D and ω remain invariant.

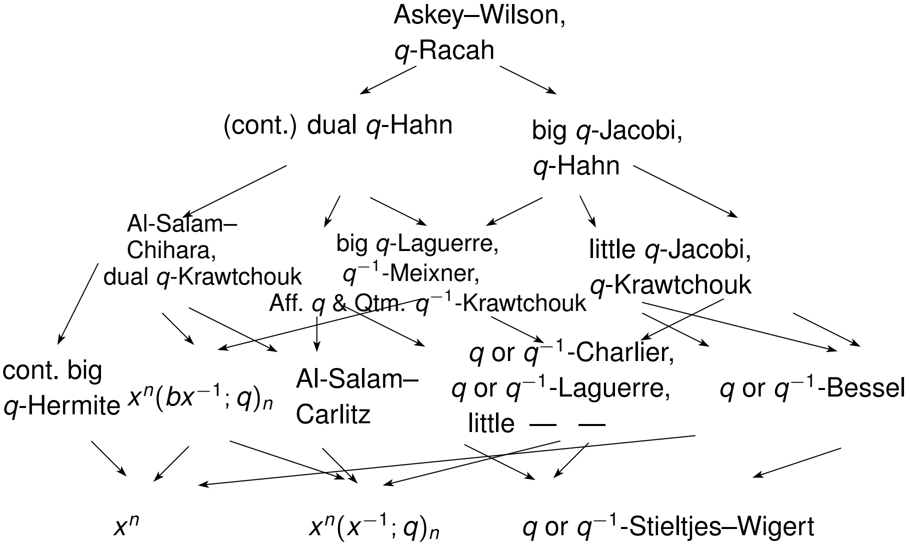
q-Verde-Star scheme, again



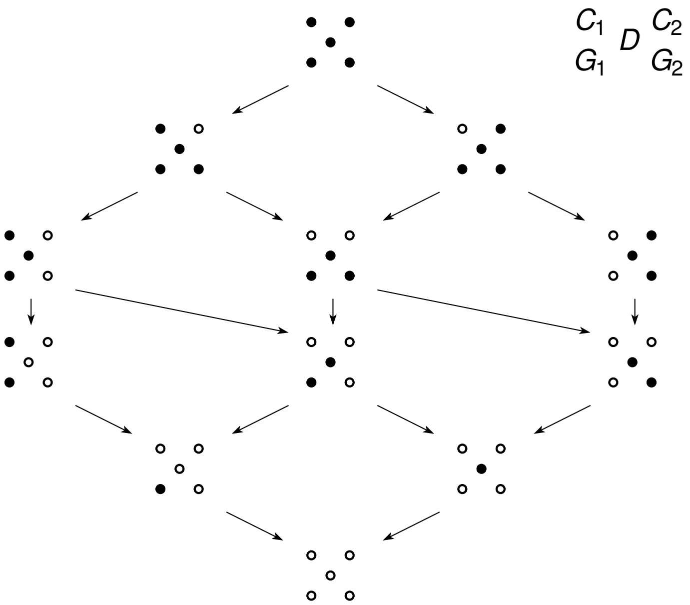
q-Verde-Star scheme with q-Zhedanov symbols



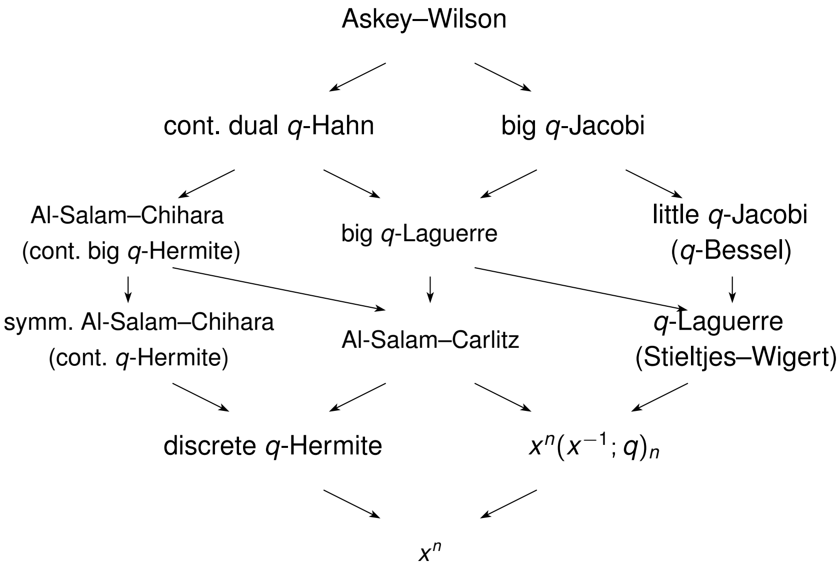
q-Verde-Star scheme, the families, again



The proper q-Zhedanov scheme



q-Zhedanov scheme, the (continuous) families



q-Zhedanov scheme. Remarks

- One family has split: Symmetric Al-Salam–Chihara now has a separate place in the scheme.
- Some families merge with another family as subfamily:
  Al-Salam–Chihara $\supset$ continuous big q-Hermite,
  symmetric Al-Salam–Chihara $\supset$ continuous q-Hermite,
  little q-Jacobi $\supset$ q-Bessel,
  q-Laguerre $\supset$ Stieltjes–Wigert.
- There is a similar Zhedanov scheme for $q = 1$.
- There should be a similar Zhedanov scheme for $q = -1$.

Thanks for listening.

Bar-le-Duc, 1984

Doron Lubinsky, Paul Nevai, Dick Askey, Tom Koornwinder