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Fernando Mário de Oliveira Filho formulates in his thesis [2, p.47] the following open problem.

Problem 1. Let $R_n^{(\alpha,\beta)}(x) := P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$ be a normalized Jacobi polynomial and let $x_{n,1}^{(\alpha,\beta)} < \ldots < x_{n,n}^{(\alpha,\beta)}$ be its successive zeros. For $\alpha \geq 0$ and $-1 < x < 1$ let k be such that

$$\min\{R_j^{(\alpha,\alpha)}(x) \mid j = 0, 1, \ldots\} = R_k^{(\alpha,\alpha)}(x) \tag{1}$$

(such k exists). Is it true that the sequence

$$R_0^{(\alpha,\alpha)}(x), R_1^{(\alpha,\alpha)}(x), \ldots, R_k^{(\alpha,\alpha)}(x) \tag{2}$$

is decreasing?

Remark 1. Because of the identity

$$(k+\alpha+1)(1-x)R_k^{(\alpha+1,\alpha)}(x) = (\alpha+1)\left(R_k^{(\alpha,\alpha)}(x) - R_{k+1}^{(\alpha,\alpha)}(x)\right),$$

a necessary condition for (1) to hold for given k is that $x_{k-1,k-1}^{(\alpha+1,\alpha)} \leq x \leq x_{k,k}^{(\alpha+1,\alpha)}$. But then the sequence (2) is decreasing. Hence, Problem 1 is equivalent to the question whether (1) is true for $x_{k-1,k-1}^{(\alpha+1,\alpha)} \leq x \leq x_{k,k}^{(\alpha+1,\alpha)}$.

Remark 2. As shown in [1, §7], [2, Theorem 3.8], formula (1) is true for $x = x_{k-1,k-1}^{(\alpha+1,\alpha+1)}$.

References

[1] C. Bachoc, G. Nebe, F. M. de Oliveira Filho and F. Vallentin, Lower bounds for measurable chromatic numbers, *Geom. Funct. Anal.* 19 (2009), 645–661;
arXiv:0801.1059v3 [math.CO]

[2] F. M. de Oliveira Filho, *New bounds for geometric packing and coloring via harmonic analysis and optimization*, PhD Thesis, University of Amsterdam, 2009;
http://homepages.cwi.nl/~fmario/thesis.pdf