6. Hypergeometric series evaluation by Zeilberger's algorithm

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Suppose we have a ${}_rF_s$ hypergeometric function, where some upper index equals $-n$ (n running through the non-negative integers) and the other upper and lower indices are linear forms over $\mathbb{Z}$ in n and a fixed number of parameters. The argument of this hypergeometric function is usually keeped fixed (often equal to 1). Denote this hypergeometric function expression by $S(n)$ (emphasizing its dependence on n). A special case of Zeilberger's algorithm [2] determines if $S(n)/S(n-1)$ depends rationally over $\mathbb{Q}$ on n and the parameters. In case of positive answer, it gives this ratio explicitly. An explicit expression of $S(n)$ then follows by iteration.

Problem. If Zeilberger's algorithm succeeds, can $S(n)/S(n-1)$ then always be factorized as a quotient of products of linear forms over $\mathbb{Z}$ in n and the parameters?

Comments. Whenever a factorization as above is possible, it will be possible to write $S(n)$ as an nth power times a quotient of products of shifted factorials of the form $(a)_n$ (a not depending on n). This is the type of explicit summation formula one usually meets in the formula books.

A rather weak counterexample can be found in the analogous q-case (where an adaptation of Zeilberger's algorithm also works). Consider the terminating case of the q-Kummer sum (cf. Gasper & Rahman [1, (II.9)]):

$$S(n) := {}_2\phi_1(a, q^{-n}; aq^{n+1}; q, -q^{n+1}) = \frac{(-q;q)_n\,(aq;q)_n}{(a^{1/2}q;q)_n\,(-a^{1/2}q;q)_n}\,.$$

Thus

$$S(n)/S(n-1) = (1+q^n)\,(1-aq^n)/(1-aq^{2n}),$$

which cannot be written as a quotient of linear factors in q^n and a. If one wants to handle this case by computer algebra in an algebraic, square roots avoiding way, then one has to replace a by a^2.

I would like to have a counterexample, where $S(n)/S(n-1)$ has higher degree factors which are irreducible in a much more essential way.

Finally, the formulation of the conditions on the indices and argument in the hypergeometric function may be adapted somewhat in interaction with examples or counterexamples being found.

References

[1] G. Gasper and M. Rahman, *Basic hypergeometric series*, Encyclopedia of Mathematics and its Applications 35, Cambridge University Press, 1990.

[2] D. Zeilberger, *The method of creative telescoping*, J. Symb. Comput. 11 (1991), 195-204.