q-SPECIAL FUNCTIONS, A TUTORIAL

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Summary

This tutorial provides the necessary prerequisites on q-special functions for understanding the lectures by Koornwinder and Koelink at this conference (see the summaries in this Volume).

Fix a *base* q, for convenience $0 < q < 1$. A *q-shifted factorial* $(a;q)_k$ is a product of k factors $1 - aq^j$ $(j = 0, 1, \ldots, k-1)$. The limit for $k \to \infty$ is a meaningful infinite product denoted by $(a;q)_\infty$. A *q-hypergeometric series* is a sum $\sum_{k=0}^\infty c_k$ such that $c_0 = 1$ and c_{k+1}/c_k is rational in q^k. Such a series is denoted by ${}_r\phi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z)$, which stands for a power series in z with coefficients given by quotients involving a.o. q-shifted factorials $(a_i;q)_k$ and $(b_j;q)_k$. After some rescaling this tends to a hypergeometric series ${}_rF_s(a_1, \ldots a_r; b_1, \ldots, b_s; z)$ as $q \uparrow 1$. The *q-binomial series* ${}_1\phi_0(a;;q,z)$ can be evaluated as the quotient of two infinite q-products. By specialization or limit transition one gets an evaluation of ${}_1\phi_0(0;;q,z)$ and ${}_0\phi_0(;;q,z)$, which are *q-exponential functions*.

A *q-integral* $\int_0^1 f(t)\, d_qt$ is defined as the sum over k from 0 to ∞ of $f(q^k)\,(q^k - q^{k+1})$. The evaluation formula for the q-binomial series can equivalently be written as a q-analogue of the integral representation for the beta function.

The ${}_2\phi_1$ q-hypergeometric series was introduced by Heine as a q-analogue of the Gaussian hypergeometric series ${}_2F_1$. Analogous to Euler's integral representation it has a q-integral representation. There are also q-analogues of the various transformation formulas and the summation formula (at $z = 1$) for the ${}_2F_1$.

Little q-Jacobi polynomials are orthogonal polynomials on the interval $[0,1]$ with respect to the q-beta measure. In particular, in the little q-Legendre case there is orthogonality measure d_qx on $[0,1]$. These polynomials are expressible as terminating ${}_2\phi_1$'s. They correspond to Jacobi polynomials of argument $1-2x$ (i.e., living on [0,1]). Corresponding to Jacobi polynomials living on an arbitrary bounded interval we have *big q-Jacobi polynomials* which are expressible as ${}_3\phi_2$'s. Little and big q-Jacobi polynomials have many properties similar to those of the classical orthogonal polynomials (Jacobi, Laguerre and Hermite). For instance, they are eigenfunctions of a second order q-difference operator.

The most general class of orthogonal polynomials which is yet considered as 'classical' is formed by the *Askey-Wilson polynomials*. This is a four-parameter

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family of orthogonal polynomials on the interval $[-1, 1]$ with respect to a continuous weight function. The polynomials are expressible as ${}_4\phi_3$'s and contain all other 'classical' orthogonal polynomials as special cases or limit cases.

For further reading see Gasper & Rahman [2] on q-hypergeometric series and Askey & Wilson [1] on Askey-Wilson polynomials.

References

1. R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. **54** (1985), no. 319.
2. G. Gasper and M. Rahman, *Basic hypergeometric series*, Cambridge University Press, 1990.

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