Some simple applications and variants of the q-binomial formula

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1. Introduction

Fix $q \in \mathbb{C}$, say $0 < q < 1$. The q-binomial formula states that for variables x, y satisfying the relation $xy = qyx$ and for n a nonnegative integer we have

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^k \, x^{n-k}. \tag{1.1}$$

This formula is due to Schützenberger [4], see also Koornwinder [2] and the references given there. In q-analysis one often encounters identities involving commuting variables which have a form similar to (1.1) in the sense that the right-hand side is a sum from $k = 0$ to n having $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ as a coefficient. It is natural to suspect an operational interpretation of such a formula which involves q-commuting operators x and y, such that the formula follows from identity (1.1) for these operators. For two simple cases this turns out to be the case, as we will illustrate below. This concerns the iterated q-Leibniz rule (the n^{th} q-derivative of a product) and the rewriting of the n^{th} q-derivative of f at x as a linear combination of values $f(xq^k)$ $(k = 0, 1, \ldots, n)$. I thank Hjalmar Rosengren for a reference to Folke Ryde's thesis from 1921 concerning this last formula.

Two other formulas involving q-binomial coefficients concern the expansion of a function f at x resp. $q^n x$ in terms of the k^{th} q-derivatives of f at $q^{n-k}x$ resp. x $(k = 0, 1, \ldots, n)$. The abstract versions of these two formulas are variants of the q-binomial formula, but not precisely equal to it.

Let us introduce some notation. For a function f defined on a subset of $\mathbb{C}\backslash\{0\}$ which is invariant under the map $x \mapsto qx$ define its q-derivative ∂f by

$$(\partial f)(x) := \frac{f(x) - f(qx)}{(1-q)x}\,. \tag{1.2}$$

Also define operators Q, X and X^{-1} by

$$(Qf)(x) := f(qx), \quad (Xf)(x) := x\,f(x), \quad (X^{-1}f)(x) := x^{-1}\,f(x). \tag{1.3}$$

Note that

$$\partial Q = q\,Q\partial, \quad QX = q\,XQ, \quad X^{-1}Q = q\,QX^{-1}. \tag{1.4}$$

2. The iterated q-Leibniz rule

This formula says that

$$\big(\partial^n(fg)\big)(t)=\sum_{k=0}^n \begin{bmatrix} n\\ k\end{bmatrix}_q (\partial^{n-k}f)(q^kt)\,(\partial^k g)(t). \tag{2.1}$$

Proof Use the notation ∂_1, ∂_2 for partial q-derivatives acting on a function in two commuting variables, and similarly for Q_1, Q_2.

Suppose $f(t)=F(t,t)$ for some function F defined on a subset of $\mathbb{C}^2$ which is invariant under the maps $(x,y)\mapsto(qx,y)$ and $(x,y)\mapsto(x,qy)$. Then

$$\begin{aligned}(\partial f)(t)&=\frac{F(t,t)-F(qt,qt)}{(1-q)t}\\ &=\frac{F(t,t)-F(qt,t)}{(1-q)t}+\frac{F(qt,t)-F(qt,qt)}{(1-q)t}=\big((\partial_1+Q_1\partial_2)F\big)(t,t).\end{aligned}$$

Iteration of this result and combination with (1.1) yield:

$$(\partial^n f)(t)=\big((\partial_1+Q_1\partial_2)^nF\big)(t,t)=\sum_{k=0}^n \begin{bmatrix} n\\ k\end{bmatrix}_q \big((Q_1\partial_2)^k\,\partial_1^{n-k}F\big)(t,t). \tag{2.2}$$

Now replace $f(t)$ by $f(t)g(t)$ and replace $F(x,y)$ by $f(x)g(y)$. Then $\big((Q_1\partial_2)^k\,\partial_1^{n-k}F\big)(x,y)$ is replaced by $(\partial^{n-k}f)(q^kx)\,(\partial^kg)(y)$. Hence these substitutions in formula (2.2) yield formula (2.1). $\square$

3. Expansion of the iterated q-derivative

This formula, which was first obtained in 1921 by Ryde [3], is as follows:

$$(\partial^n f)(x)=(1-q)^{-n}\,x^{-n}\sum_{k=0}^n \begin{bmatrix} n\\ k\end{bmatrix}_q (-1)^k\,q^{-k(n-k)}\,q^{-\frac12 k(k-1)}\,f(q^kx). \tag{3.1}$$

Proof From (1.4) we have $X^{-1}\,(-X^{-1}Q)=q\,(-X^{-1}Q)\,X^{-1}$. Combination with formula (1.1) yields:

$$\begin{aligned}(1-q)^n\,\partial^n=(X^{-1}-X^{-1}Q)^n&=\sum_{k=0}^n \begin{bmatrix} n\\ k\end{bmatrix}_q (-X^{-1}Q)^k\,(X^{-1})^{n-k}\\ &=\sum_{k=0}^n \begin{bmatrix} n\\ k\end{bmatrix}_q (-1)^k\,q^{-k(n-k)}\,q^{-\frac12 k(k-1)}\,X^{-n}\,Q^k.\end{aligned}$$

When we apply this last identity to f and evaluate at x then we obtain formula (3.1). $\square$

4. A variant of the q-binomial formula

Proposition 4.1 Let x, y, z be variables such that $xy = qyx$ and $x + yz = 1$. Then

$$1 = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^k x^{n-k} z^k \qquad (n \in \mathbb{Z}_{\geq 0}) \tag{4.1}$$

and

$$x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k \, q^{\frac{1}{2}k(k-1)} \, y^k z^k \qquad (n \in \mathbb{Z}_{\geq 0}). \tag{4.2}$$

Proof Both identities are obtained by induction with respect to n by using that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q. \qquad \square$$

As an application of this Proposition take $x := Q$, $y := X$ and $z := (1-q)\,\partial$ in (4.1) or (4.2) (where X, Y, ∂ are defined by (1.2) and (1.3)) and let both sides act on a function f. Then

$$f(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (1-q)^k \, x^k \, (\partial^k f)(q^{n-k} x) \tag{4.3}$$

and

$$f(q^n x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k \, q^{\frac{1}{2}k(k-1)} \, (1-q)^k \, x^k \, (\partial^k f)(x). \tag{4.4}$$

Formula (4.3) was earlier observed in Carnovale & Koornwinder [1, (8.3)].

References

[1] G. Carnovale & T. H. Koornwinder, *A q-analogue of convolution on the line*, preprint, 1999, downloadable from `math.CA/9909025`.

[2] T. H. Koornwinder, *Special functions and q-commuting variables*, in *Special Functions, q-Series and Related Topics*, M. E. H. Ismail, D. R. Masson & M. Rahman (eds.), Fields Institute Communications 14, American Mathematical Society, 1997, pp. 131–166.

[3] F. Ryde, *A contribution to the theory of linear homogeneous geometric difference equations (q-difference equations)*, Dissertation, Lund, 1921.

[4] M. P. Schützenberger, *Une interprétation de certaines solutions de l'équation fonctionnelle:* $F(x+y) = F(x)F(y)$, C. R. Acad. Sci. Paris 236 (1953), 352–353.

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