An alternative proof of Brafman's generating function for Legendre polynomials

Informal note by Tom Koornwinder, `T.H.Koornwinder@uva.nl`*, 26 July 2013*

Brafman [1, (13)] gave a generating function for Legendre polynomials:

$$\sum_{n=0}^{\infty} \frac{(s)_n(1-s)_n}{n!\,n!}\,P_n(x)\,z^n = {}_2F_1\left(\begin{matrix} s,1-s \\ 1 \end{matrix};\frac{1-\rho-z}{2}\right)\,{}_2F_1\left(\begin{matrix} s,1-s \\ 1 \end{matrix};\frac{1-\rho+z}{2}\right) \qquad (-1\le x\le 1,\ |z|<1), \tag{1}$$

where

$$\rho := (1-2xz+z^2)^{1/2}.$$

He obtained it as a special case of a generating formula for Jacobi polynomials which followed from an expansion of an Appell F_4 function in terms of Jacobi polynomials.

Here I will provide an alternative proof of (1). First observe that by [2, 3.6(3)],

$$P_\nu(z) = {}_2F_1\left(\begin{matrix} 1+\nu,-\nu \\ 1 \end{matrix};\tfrac12-\tfrac12 z\right),$$

we have on the right-hand side of (1) a product $P_{-s}(\rho+z)P_{-s}(\rho-z)$ of Legendre functions. We can rewrite (1) as

$$\sum_{n=0}^{\infty} c_n\,P_n(x)\,z^n = P_{-s}(\rho+z)P_{-s}(\rho-z), \tag{2}$$

where (putting $x=1$) the c_n are such that

$$\sum_{n=0}^{\infty} c_n\,z^n = P_{-s}(1-2z). \tag{3}$$

In order to prove (2), plug in the integral representation [3, (10.10(42)],

$$P_n(x) = \pi^{-1}\int_0^\pi \big(x+i(1-x^2)^{\frac12}\cos\phi\big)^n\,d\phi.$$

Thus, by (3), the left-hand side of (2) equals

$$\pi^{-1}\int_0^\pi P_{-s}\big(1-2z(x+i(1-x^2)^{\frac12}\cos\phi)\big)\,d\phi.$$

Now we see that this equals the right-hand side of (2) by the product formula for Legendre functions,

$$P_\nu(z)P_\nu(w) = \pi^{-1}\int_0^\pi P_\nu\big(zw+\sqrt{(1-z^2)(1-w^2)}\,\cos\phi\big)\,d\phi,$$

which follows by integration from the addition formula [2, 3.11(1)] for Legendre functions, and which is valid as long as $(1-z^2)(1-w^2)$ has positive real part and the arguments of the Legendre functions stay away from $(-\infty,-1]$.

It is tempting to try a similar proof for Brafman's addition formula for Jacobi polynomials [1, (12)] by using the product formula [4, (4.1)]. However, we would need there an integral representing $R_\nu^{(\alpha,\beta)}(x)R_\nu^{(\beta,\alpha)}(y)$ rather than $R_\nu^{(\alpha,\beta)}(x)R_\nu^{(\alpha,\beta)}(y)$.

Note added December 21, 2018 In 2013 Wadim Zudilin discussed the generating function (1) with me in connection with his paper [5] joint with Wan. This gave rise to the present note.

References

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[5] J. Wan and W. Zudilin, *Generating functions of Legendre polynomials: a tribute to Fred Brafman*, J. Approx. Theory 164 (2012), 488–503.