On a monotonticity property of the normalized incomplete gamma function

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The *incomplete gamma functions* $\gamma(a,x)$, $\Gamma(a,x)$ and the normalized incomplete gamma function $P(a,x)$ are defined for $\operatorname{Re} a > 0$, $x > 0$ by

$$\gamma(a,x) := \int_0^x e^{-t} t^{a-1}\, dt, \qquad \Gamma(a,x) := \int_x^\infty e^{-t} t^{a-1}\, dt,$$
$$P(a,x) := \frac{\gamma(a,x)}{\Gamma(a)} = \frac{\gamma(a,x)}{\gamma(a,x)+\Gamma(a,x)}\,, \tag{1}$$

see [1, §6.5]. For $x > 0$ the function $a \mapsto P(a,x)\colon (0,\infty) \to (0,1)$ is monotonically decreasing. This can be established by a slight variation of the proof given in Tricomi [2, §9]. From (1) we have

$$\frac{\partial P(a,x)}{\partial a} = -\frac{1}{\Gamma^2(a)} \left\{ \gamma(a,x)\, \frac{\partial \Gamma(a,x)}{\partial a} - \Gamma(a,x)\, \frac{\partial \gamma(a,x)}{\partial a} \right\}.$$

The quantity in curly braces can be written as

$$\int_x^\infty e^{-t} t^{a-1} \log t\, dt \times \int_0^x e^{-u} u^{a-1}\, du - \int_x^\infty e^{-t} t^{a-1}\, dt \times \int_0^x e^{-u} u^{a-1} \log u\, du$$
$$= \int_x^\infty \Big(\int_0^x e^{-(u+t)} (ut)^{a-1} \log(t/u)\, du \Big)\, dt > 0$$

since $t/u \ge 1$ in the integration domain. Hence $\partial P(a,x)/\partial a < 0$ for $a > 0$ and $x > 0$.

As a consequence, the function $a \mapsto \Gamma(a,x)/\Gamma(a)\colon (0,\infty) \to (0,1)$ is monotonically increasing for $x > 0$.

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions*, National Bureau of Standards, 1964.

[2] F. G. Tricomi, *Sulla funzione gamma incompleta*, Ann. Mat. Pura Appl. (4) 31 (1950), 263–279.