Comment on the paper "Asymptotic behavior of matrix coefficients of admissible representations" by W. Casselman & D. Miličić

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Casselman & Miličić [3, Example 3.7] compute the τ-radial component $\Pi_r(C)$ of the Casimir element C for $SL(2,\mathbb{R})$. After an identification of the subgroup A with $\mathbb{R}_+^*$ this becomes the differential operator

$$\Pi_r(C) = \left(z\frac{d}{dz}\right)^2 - \frac{1+z^2}{1-z^2}\,z\,\frac{d}{dz} - (n^2+m^2)\,\frac{z^2}{(1-z^2)^2} - nm\,\frac{z(1+z^2)}{(1-z^2)^2}\,.$$

I want to observe here that this can be transformed into a hypergeometric differential operator (see also Bargmann [2, §9,10] and see a check of the computation in my accompanying Mathematica notebook):

$$\frac{\Pi_r(C)\Big(\big((z+z^{-1}-2)/4\big)^{(m+n)/4}\big((z+z^{-1}+2)/4\big)^{(m-n)/4} f\big((2-z-z^{-1})/4\big)\Big)}{\big((z+z^{-1}-2)/4\big)^{(m+n)/4}\big((z+z^{-1}+2)/4\big)^{(m-n)/4}}$$
$$= -w(1-w)f''(w) - \big(\tfrac12(m+n)+1-(m+2)w\big)f'(w) + \tfrac14 m(m+2)f(w)\Big|_{w=(2-z-z^{-1})/4}\,.$$

Thus, a possible solution of the differential equation

$$\Pi_r(C)h(z) = -\tfrac14(\lambda^2+1)h(z),$$

expressed in terms of hypergeometric functions (see [4, Ch. 2], [1, Ch. 2]), is

$$h(z) = \big((z+z^{-1}-2)/4\big)^{(m+n)/4}\big((z+z^{-1}+2)/4\big)^{(m-n)/4} \times {}_2F_1\left(\begin{matrix}\frac12(m+1+i\lambda),\frac12(m+1-i\lambda)\\ \frac12(m+n)+1\end{matrix};\tfrac14(2-z-z^{-1})\right).$$

In terms of Jacobi functions (see [5], [7], [6, (2.28)]) this becomes

$$h(e^{2t}) = (\sinh t)^{(m+n)/2}(\cosh t)^{(m+n)/2}\,\phi_\lambda^{((m+n)/2,(m-n)/2)}(-\sinh^2 t).$$

In the Dissertation [8] by Vincent van der Noort the differential equation satsified by $g(w) := h(e^{2w})$ and with $\lambda = i\zeta$ should be his formula (2.29). However, in the transformation from (2.27) to (2.29) by replacing w by e^{2w} an error was made. Formula (2.29) in [8] becomes correct if all powers of e^w are replaced by powers of e^{2w}.

References

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[5] T. Koornwinder, *A new proof of a Paley-Wiener type theorem for the Jacobi transform*, Ark. Mat. 13 (1975), 145–159.

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