Comments for my 1984 paper *Orthogonal polynomials with weight function* $(1-x)^\alpha(1+x)^\beta + M\delta(x+1) + N\delta(x-1)$

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These are comments for the paper

T. H. Koornwinder,
Orthogonal polynomials with weight function $(1-x)^\alpha(1+x)^\beta + M\delta(x+1) + N\delta(x-1)$,
Canad. Math. Bull. 27 (1984), 205–214.

The orthogonal polynomials $P_n^{\alpha,\beta,M,N}(x)$, defined by (2.1), are semiclassical by their property

$$(1-x^2)^2\,\frac{\mathrm{d}}{\mathrm{d}x}P_n^{\alpha,\beta,M,N}(x) = \sum_{j=-3}^{3} c_{n,j}\,P_{n+j}^{\alpha,\beta,M,N}(x) \tag{K1}$$

for certain foefficients $c_{n,j}$.

Proof of (K1). Recall that the polynomials $P_n^{\alpha,\beta,M,N}(x)$ are orthogonal with respect to a measure $\mu_{\alpha,\beta,M,N}$ on $[-1,1]$ which is defined by

$$\int_{-1}^{1} f(x)\,\mathrm{d}\mu_{\alpha,\beta,M,N}(x) = C_{\alpha,\beta}\int_{-1}^{1} f(x)(1-x)^\alpha(1+x)^\beta\,\mathrm{d}x + Mf(-1) + Nf(1) \quad (f\in C([-1,1]),$$

where

$$C_{\alpha,\beta} = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\,.$$

Now let $n\ge 4$ and let $q(x)$ be a polynomial of degree $\le n-4$. Then

$$\begin{aligned}
&\int_{-1}^{1}(1-x^2)^2\,q(x)\,\frac{\mathrm{d}}{\mathrm{d}x}P_n^{\alpha,\beta,M,N}(x)\,\mathrm{d}\mu_{\alpha,\beta,M,N}(x)\\
&= C_{\alpha,\beta}\int_{-1}^{1}(1-x)^{\alpha+2}(1+x)^{\beta+2}\,q(x)\,\frac{\mathrm{d}}{\mathrm{d}x}P_n^{\alpha,\beta,M,N}(x)\,\mathrm{d}x\\
&= -C_{\alpha,\beta}\int_{-1}^{1}P_n^{\alpha,\beta,M,N}(x)\,\frac{\mathrm{d}}{\mathrm{d}x}\Big((1-x)^{\alpha+2}(1+x)^{\beta+2}\,q(x)\Big)\,\mathrm{d}x\\
&= -\int_{-1}^{1}P_n^{\alpha,\beta,M,N}(x)\Big(\big((\alpha+2)(1+x)+(\beta+2)(1-x)\big)(1-x^2)q(x)\\
&\qquad\qquad + (1-x^2)^2\,q'(x)\Big)\mathrm{d}\mu_{\alpha,\beta,M,N}(x) = 0.
\end{aligned}$$

□

In the same way we can prove that

$$(1-x^2)(1+x)\,\frac{\mathrm{d}}{\mathrm{d}x}P_n^{\alpha,\beta,M,0}(x)=\sum_{j=-2}^{2}c_{n,j}P_{n+j}^{\alpha,\beta,M,0}(x), \tag{K2}$$

and that the orthogonal polynomials $L_n^{\alpha,N}(x)$, defined by (4.8), satisfy

$$x^2\,\frac{\mathrm{d}}{\mathrm{d}x}L_n^{\alpha,N}(x)=\sum_{j=-2}^{1}c_{n,j}L_{n+j}^{\alpha,N}(x). \tag{K3}$$

I thank Kenier Castillo [K1] for pointing out to me that the polynomials $P_n^{\alpha,\beta,M,N}(x)$ are semiclassical, and that this is already implied by Maroni's result [K2, Theorem 3.1]. Note also that Kwon & Park [K3] explicitly mention that the polynomials $P_n^{\alpha,\beta,M,N}(x)$ are semiclassical.

References

[K1] K. Castillo and D. Mbouna, *Epilegomena to the study of semiclassical orthogonal polynomials*, `arXiv:2307.10331`.

[K2] P. Maroni, *Sur la suite de polynômes orthogonaux associée à la forme* $u=\delta_c+\lambda(x-c)^{-1}L$, Period. Math. Hungar. 21 (1990), 223–248.

[K3] K. H. Kwon and S. B. Park, *Two-point masses perturbation of regular moment functionals*, Indag. Math. (N.S.) 8 (1997), 79–93.