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# Two-Variable Analogues of the Classical Orthogonal Polynomials

*Tom Koornwinder*

Contents

## 1. Introduction

By now the theory of Jacobi polynomials is rather well settled cf. Askey [5] and Gasper [24]. The surprising richness of this theory is closely tied up with the group theoretic interpretation of Jacobi polynomials for certain values of the parameters. Hardly any comparable results have been obtained for orthogonal polynomials in two or more variables, probably because people did not study the right classes of polynomials. Most emphasis was laid on certain biorthogonal systems, while some remarkable orthogonal systems of polynomials in two variables, scattered over the literature, did not get as much attention as they deserved.

Analogues in severable variables of the Jacobi polynomials seem to be highly nontrivial generalizations of the one-variable case. Therefore, it is a good approach to restrict oneself first to the case of two variables. In this paper a number of distinct classes of orthogonal polynomials in two variables will be introduced for which many properties hold which are analogous to properties of Jacobi polynomials. The polynomials belonging to these orthogonal systems are eigenfunctions of two algebraically independent partial differential operators. In the Chebyshev cases





(i.e. if all parameters are equal to $\pm\frac{1}{2}$) these polynomials can be interpreted as quotients of two eigenfunctions of the Laplacian on a two-dimensional torus or sphere, which satisfy symmetry relations with respect to certain reflections. For two of the classes which will be considered and for certain values of the parameters the polynomials can be interpreted as spherical functions on certain compact symmetric Riemannian spaces of rank 2.

This paper is an extended version of the survey given in the author's thesis [47, sections 8, 9, 10, 11]. It also includes some new material, see in particular sections 4.4 and 4.5. Most results are given without proofs.

The Proceedings of this Advanced Seminar include three other papers dealing with orthogonal polynomials in several variables, cf. James [35] and Karlin and McGregor [40], [41]. The present paper has some relationship with [40], cf. section 3.7.3, and an important relationship with the zonal polynomials and the generalized Jacobi polynomials considered by James, cf. sections 4.4 and 4.5.

Notation. Throughout this paper $c$ denotes a constant factor, which is usually nonzero. The symbol $\partial_{x_1 x_2 \ldots x_k}$ denotes the partial derivative $\frac{\partial^k}{\partial x_1 \partial x_2 \ldots \partial x_k}$.

## 2. Jacobi polynomials

The classical orthogonal polynomials in one variable are the Jacobi polynomials, the Laguerre and the Hermite polynomials. Since Laguerre and Hermite polynomials are limit cases of the Jacobi polynomials, we shall restrict ourselves to Jacobi polynomials and their two-variable analogues, i.e., to the case of a bounded orthogonality region.

Two-variable analogues of the Jacobi polynomials may be related to Jacobi polynomials in several different ways:

(a) It may be possible to express them in terms of Jacobi polynomials.

(b) They may occur in certain formulas for Jacobi polynomials.

(c) Their properties will be analogous to the properties of Jacobi polynomials.

(d) The methods of proving these properties may be analogous to the proofs in the Jacobi case.

We shall briefly consider those properties of the Jacobi polynomials which we want to generalize. For standard results about Jacobi polynomials the reader is referred to Szegő [68, Chap. 4] and Erdélyi [17, Chap. 10].

2.1. Simple analytic properties

Let $\alpha, \beta > -1$. Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $n = 0, 1, 2, \ldots$, are orthogonal polynomials with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ on the interval $(-1, 1)$. They are normalized such that $P_n^{(\alpha,\beta)}(1) = (\alpha+1)_n/n!$. We shall often use the notation $R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$.

The pair of differential recurrence relations

(2.1) $$\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(n+\alpha+\beta+1)\, P_{n-1}^{(\alpha+1,\beta+1)}(x) ,$$

(2.2) $$(1-x)^{-\alpha}(1+x)^{-\beta} \frac{d}{dx}\left[(1-x)^{\alpha+1}(1+x)^{\beta+1} P_{n-1}^{(\alpha+1,\beta+1)}(x)\right] = -2n\, P_n^{(\alpha,\beta)}(x)$$

can immediately be derived from the definition. There are three important corollaries of (2.1) and (2.2). Combination of (2.1) and (2.2) gives the second order differential equation

(2.3) $$\left[(1-x^2)\frac{d^2}{dx^2} + (\beta-\alpha-(\alpha+\beta+2)x)\frac{d}{dx}\right] P_n^{(\alpha,\beta)}(x) = -n(n+\alpha+\beta+1)\, P_n^{(\alpha,\beta)}(x) .$$

Iteration of (2.2) leads to the Rodrigues formula

(2.4) $$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n\, n!}(1-x)^{-\alpha}(1+x)^{-\beta} \frac{d^n}{dx^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right] .$$





Finally, repeated application of (2.1) together with the value of $P_n^{(\alpha,\beta)}(1)$ gives the power series expansion

(2.5) $$R_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \frac{(-n)_k(n+\alpha+\beta+1)_k}{(\alpha+1)_k\; k!}\left(\frac{1-x}{2}\right)^k =$$

$$= {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; \tfrac{1}{2}(1-x)) .$$

Orthogonal polynomials $p_n(x)$ with respect to a weight function $w(x)$ are called classical if they satisfy one of the following three equivalent conditions:

(a) The polynomials $p_n(x)$ are eigenfunctions of a second order linear differential operator.

(b) The system of polynomials $dp_{n+1}(x)/dx$, $n = 0, 1, 2, \ldots$, is an orthogonal system.

(c) There is a polynomial $\rho(x)$ such that $p_n(x)$ is given by the Rodrigues formula

$$p_n(x) = c \cdot (w(x))^{-1} \frac{d^n}{dx^n}\left[(\rho(x))^n\, w(x)\right] .$$

For Jacobi polynomials these three properties are contained in formulas (2.3), (2.1) and (2.4), respectively.

2.2. Chebyshev polynomials

Consider the unit circle parametrized by the angle $\theta$, let $\sigma = \pm 1$, and let the functions $f_n^\sigma(\theta)$, $n = 0, 1, 2, \ldots$, be the successive eigenfunctions of $d^2/d\theta^2$ which satisfy the symmetry relation

$$f_n^\sigma(-\theta) = \sigma f_n^\sigma(\theta) .$$

Then $f_n^1(\theta) = \cos n\theta$, $f_n^{-1}(\theta) = \sin(n+1)\theta$, and $f_n^\sigma(\theta)/f_0^\sigma(\theta)$ is a polynomial of degree $n$ in $\cos\theta$. In this way we obtain the special Jacobi polynomials

(2.6) $$T_n(\cos\theta) = R_n^{(-\frac{1}{2},-\frac{1}{2})}(\cos\theta) = \cos n\theta ,$$

(2.7) $$U_n(\cos\theta) = (n+1)\, R_n^{(\frac{1}{2},\frac{1}{2})}(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} ,$$

which are called Chebyshev polynomials of the first and second kind, respectively.

In a similar way we can consider eigenfunctions $f_n^{\sigma,\tau}(\theta)$ $(\sigma,\tau=\pm 1,$ $n = 0,1,2,\ldots)$ of $d^2/d\theta^2$ which satisfy symmetry relations

$$f_n^{\sigma,\tau}(-\theta) = \sigma f_n^{\sigma,\tau}(\theta),\quad f_n^{\sigma,\tau}(\pi-\theta) = \tau f_n^{\sigma,\tau}(\theta)$$

with respect to the reflections in $\theta = 0$ and $\theta = \pi/2$. Then $f_n^{\sigma,\tau}(\theta)/f_0^{\sigma,\tau}(\theta)$ is a polynomial in $\cos 2\theta$. Thus we obtain (2.6) and (2.7) with $\theta$ replaced by $2\theta$, and, furthermore, the special Jacobi polynomials

(2.8) $$(2n+1)\, R_n^{(\frac{1}{2},-\frac{1}{2})}(\cos 2\theta) = \frac{\sin(2n+1)\theta}{\sin\theta} ,$$

(2.9) $$R_n^{(-\frac{1}{2},\frac{1}{2})}(\cos 2\theta) = \frac{\cos(2n+1)\theta}{\cos\theta} .$$

This interpretation of Jacobi polynomials of order $(\pm\frac{1}{2},\pm\frac{1}{2})$ is a motivation for the study of Jacobi polynomials of general order $(\alpha,\beta)$. Many simple formulas for Jacobi polynomials of general order $(\alpha,\beta)$ (for instance the differential equation (2.3)) are "analytic continuations" of the cases $\alpha,\beta = \pm\frac{1}{2}$. So we may first derive these formulas in the Chebyshev cases, then predict the formulas in the general case, and finally give a formal proof.

## 2.3. Jacobi polynomials as spherical functions and deeper analytic properties

First we give some definitions and results about homogeneous spaces and spherical functions. For further details we refer to





Helgason [26, Chap. 10] and Coifman and Weiss [10, Chap. 1 and 2].

Let $G$ be a compact group acting transitively on a compact Hausdorff space $M$, fix $e \in M$ and let $K$ be the subgroup of $G$ which leaves $e$ fixed. Then $M = G/K$ is called a homogeneous space of $G$. Let a function $f$ on $M$ be called zonal if $f$ is invariant with respect to $K$. It is possible to decompose $L^2(M)$ as the orthogonal direct sum of finite dimensional subspaces which are invariant and irreducible with respect to $G$. The functions belonging to these irreducible subspaces are continuous on $M$. Each $G$-invariant subspace of $L^2(M)$ of nonzero dimension contains zonal functions which are not identically zero. The class of zonal $L^1$-functions on $M$ can be considered as a subalgebra of the convolution algebra of $L^1$-functions on $G$.

Theorem 2.1. The following four statements are equivalent:

(a) In each subspace of $L^2(M)$ which is irreducible with respect to $G$, the class of zonal functions has dimension 1.

(b) The decomposition of $L^2(M)$ into irreducible subspaces with respect to $G$ is unique.

(c) The representation of $G$ in $L^2(M)$ contains each irreducible representation of $G$ at most once.

(d) The convolution algebra of zonal $L^1$-functions on $M$ is commutative.

Definition 2.2. Let $M = G/K$ have the equivalent properties of Theorem 2.1.

(a) A function $f$ on $M$ is called a harmonic if $f$ belongs to an irreducible subspace of $L^2(M)$ with respect to $G$.

(b) A function $f$ on $M$ is called a spherical function if (i) $f$ is a harmonic, (ii) $f$ is zonal, (iii) $f(e) = 1$.

Unless otherwise stated, the homogeneous spaces considered in this paper will satisfy the equivalent properties of Theorem 2.1.

Sometimes it is possible to parametrize the set of K-orbits $Kx$ in $M$ such that the spherical functions expressed in terms of these parameters become well-known special functions. On the other hand, new special functions may be obtained in this way. If special functions can be interpreted as spherical functions then certain deeper analytic results immediately follow from the group theoretic interpretation. Among these results are an integral representation, a product formula, a positive convolution structure and, sometimes, an addition formula. A related, but more elementary result is the inequality

(2.10) $$|f(x)| \leq f(e) = 1, \quad x \in M,$$

for spherical functions $f$ on $M$.

For certain discrete values of $(\alpha, \beta)$ (see Figure 1) Jacobi polynomials $R_n^{(\alpha,\beta)}(x)$ can be interpreted as spherical functions on two-point homogeneous spaces or, equivalently, symmetric spaces of rank 1, cf. Helgason [27] and Gangolli [21]. The most elementary case is the unit circle $S^1 = O(2)/O(1)$ with spherical functions $\cos n\theta = R_n^{(-\frac{1}{2}, -\frac{1}{2})}(\cos\theta)$. Gegenbauer polynomials $R_n^{(q/2-3/2, q/2-3/2)}(x)$, $q = 3, 4, \ldots$, can be interpreted as spherical functions on the sphere $S^{q-1} = O(q)/O(q-1)$. The harmonics on this homogeneous space are the well-known spherical harmonics, cf. for instance Müller [59]. The other cases are the real projective space $SO(q)/O(q-1)$ ($\alpha = q/2-3/2$, $\beta = -\frac{1}{2}$), the complex projective space $SU(q)/U(q-1)$ ($\alpha = q-2$, $\beta = 0$), the quaternionic projective space $Sp(q)/Sp(q-1) \times Sp(1)$ ($\alpha = 2q-3$, $\beta = 1$) and the Cayley projective plane ($\alpha = 7$, $\beta = 3$). Jacobi polynomials of order $(\frac{1}{2}, \frac{1}{2})$ can also be interpreted as the characters on the group $SU(2)$.

Among the deeper analytic properties of Jacobi polynomials are the inequality

(2.11) $$|R_n^{(\alpha,\beta)}(x)| \leq 1, \quad |x| \leq 1, \quad \alpha \geq \beta > -1, \quad \alpha \geq -\tfrac{1}{2},$$

(cf. Szegö [68, §7.32]), the Laplace type integral representation, the





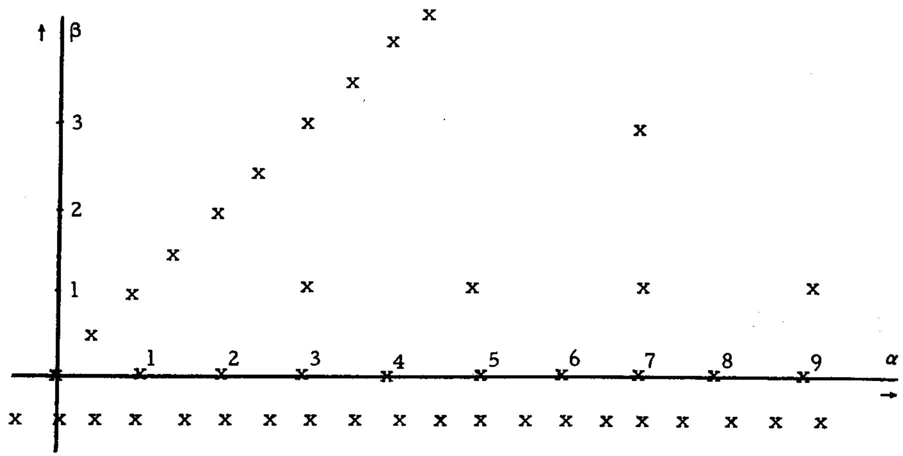


Figure 1

product formula and the addition formula for Jacobi polynomials (cf. Koornwinder [42]), and a positive convolution structure for Jacobi series (cf. Gasper [22], [23]). For the values of $(\alpha, \beta)$ given in Figure 1 these results follow from the group theoretic interpretation. If it is known that a certain result for Jacobi polynomials holds for special values of $(\alpha, \beta)$ then it is easier to obtain such a result in the general case, either by analytic manipulation of the known cases or by first predicting the general result as an "analytic continuation" of the known cases and next giving some new analytic proof.

In section 2.2 we pointed out that the analytic structure of elementary formulas for Jacobi polynomials can already be predicted from the Chebyshev cases. This is no longer true for the deeper results, since these formulas may become degenerate if $\alpha = \beta$ or $\beta = \frac{1}{2}$. For instance, the formula expressing the product $R_n^{(\alpha,\beta)}(x)\, R_n^{(\alpha,\beta)}(y)$ as an integral of a Jacobi polynomial of the same degree and order, is a sum of two terms if $\alpha = \beta = -\frac{1}{2}$, a single integral if $\alpha = \beta$ or $\beta = -\frac{1}{2}$ and a double integral if $\alpha > \beta > -\frac{1}{2}$. Hence, $SU(3)/U(2)$ is the least complicated homogeneous space on which the product formula is nondegenerate.

## 3. Methods of constructing two-variable analogues of the Jacobi polynomials

### 3.1. General orthogonal polynomials in two variables

For the few things which are known about general orthogonal polynomials in two variables the reader is referred to Jackson [32], Erdélyi [17, Chap. 12] and Bertran [8]. Some further results have been obtained in connection with numerical cubature problems, cf. Stroud [67].

Let $\Omega$ be an open subset of $\mathbb{R}^2$ and let $w(x, y)$ be a nonnegative $L^1$-function on $\Omega$. For convenience we shall always suppose that the region $\Omega$ is bounded and that the function $w(x, y)$ is strictly positive and continuous on $\Omega$. The set $\mathscr{V}_n$, $n = 0, 1, \ldots,$ of orthogonal polynomials of degree $n$ with respect to the weight function $w(x, y)$ on the region $\Omega$ is defined as the $(n+1)$-dimensional set of all polynomials $p(x, y)$ of degree $n$ with complex coefficients such that

$$\iint_\Omega p(x, y)\, q(x, y)\, w(x, y)\, dx\,dy = 0$$

if $q(x, y)$ is a polynomial of degree less than $n$.

For a special region $\Omega$ and a special weight function $w(x, y)$ it may be possible to find an explicit basis of polynomials $p_{n,k}(x, y)$, $k = 0, 1, \ldots, n$, for $\mathscr{V}_n$. In particular, it is of interest to find an orthogonal basis for $\mathscr{V}_n$. There are infinitely many possibilities to choose an orthogonal basis for $\mathscr{V}_n$. One evident method for constructing such an orthogonal basis is to define $p_{n,k}(x, y)$ as the polynomial with highest term $c \cdot x^{n-k} y^k$ obtained by applying the Gram-Schmidt orthogonalization process to the sequence of monomials

$$1, x, y, x^2, xy, y^2, x^3, x^2y, \ldots, x^n, x^{n-1}y, \ldots, x^{n-k}y^k. \tag{3.1}$$

However, this is not a very canonical method, since the transformation $(x, y) \to (y, x)$ generally does not map the thus obtained orthogonal basis $\{p_{n,0}(x, y), p_{n,1}(x, y), \ldots, p_{n,n}(x, y)\}$ of $\mathscr{V}_n$ onto itself, even if $\Omega$ and $w(x, y)$ are invariant under this transformation.





Sometimes it is possible to preserve the symmetry in $x$ and $y$ by choosing a nonorthogonal basis $p_{n,k}(x, y)$, $k = 0, 1, \ldots, n$, for $\mathscr{V}_n$, and another nonorthogonal basis $q_{n,k}(x, y)$, $k = 0, 1, \ldots, n$, for $\mathscr{V}_n$, such that $p_{n,k}(x, y)$ is orthogonal to $q_{n,\ell}(x, y)$ if $k \neq \ell$. Then the systems $\{p_{n,k}(x, y)\}$ and $\{q_{n,k}(x, y)\}$ together are called a biorthogonal system (cf. Erdélyi [17, Chap. 12] and Schleusner [65]).

It is the author's opinion that if a general theory of sufficient depth is possible for orthogonal polynomials in two variables then it probably will be given for the classes $\mathscr{V}_n$ rather than for some basis of $\mathscr{V}_n$. However, for certain special regions and weight functions there are quite interesting properties for polynomials belonging to some explicit basis of $\mathscr{V}_n$.

Let us return to the polynomials $p_{n,k}(x, y)$ which are obtained by orthogonalizing the sequence of monomials (3.1). Let us arrange the pairs of integers $(n, k)$, $n \geq k \geq 0$, by lexicographic ordering, i.e. $(m, \ell) < (n, k)$ if either $m < n$ or $m = n$ and $\ell < k$. Then $p_{n,k}(x, y)$ has a power series expansion

$$p_{n,k}(x, y) = \sum_{(m,\ell) \leq (n,k)} c(m, \ell; n, k)\, x^{m-\ell} y^\ell, \tag{3.2}$$

and there are recurrence relations

$$x p_{n,k}(x, y) = \sum_{(m,\ell)=(n-1,k)}^{(n+1,k)} a(m, \ell; n, k)\, p_{m,\ell}(x, y) \tag{3.3}$$

and

$$y p_{n,k}(x, y) = \sum_{(m,\ell)=(n-1,k-1)}^{(n+1,k+1)} b(m, \ell; n, k)\, p_{m,\ell}(x, y). \tag{3.4}$$

Note that the number of terms in the recurrence relations becomes arbitrarily large if $n \to \infty$. This is in striking contrast with the one-variable case, where the recurrence relation contains at most three terms, independent of $n$.

Next consider the trivial example of a direct product of two systems of orthogonal polynomials in one variable. Let $\{p_n(x)\}$ and $\{q_n(y)\}$ be orthogonal systems with respect to the weight functions $w_1(x)$ and $w_2(y)$, respectively. Define

$$p_{n,k}(x,y) = p_{n-k}(x)\, q_k(y) \quad . \tag{3.5}$$

Then $p_{n,k}(x,y)$ can be obtained by orthogonalizing the sequence (3.1) with respect to the weight function $w_1(x)\, w_2(y)$. Now the formulas (3.2), (3.3) and (3.4) can be simplified, since $c(m,\ell;n,k)$ can only be non-zero if $m-\ell \leq n-k$ and $\ell \leq k$, $a(m,\ell;n,k)$ vanishes except for $(m,\ell) = (n+1,k)$, $(n,k)$ or $(n-1,k)$, and $b(m,\ell;n,k)$ vanishes except for $(m,\ell) = (n+1,k+1)$, $(n,k)$ or $(n-1,k-1)$.

We shall meet less trivial examples of orthogonal polynomials $p_{n,k}(x,y)$ for which certain coefficients in (3.1), (3.2) and (3.3) vanish, such that the number of nonvanishing coefficients in the recurrence relations remains bounded for $n \to \infty$. It is worthwhile to consider system having these properties, even if the symmetry in $x$ and $y$ is destroyed.

In some examples which will be considered in this paper, there is a partial ordering $\prec$ of the pairs of integers $(n,k)$, $n \geq k \geq 0$, such th[at] $(m,\ell) \prec (n,k)$ implies that $(m,\ell) \leq (n,k)$ and such that

$$p_{n,k}(x,y) = \sum_{(m,\ell)\prec(n,k)} c(m,\ell;n,k)\, x^{m-\ell}\, y^{\ell} \quad . \tag{3.6}$$

The following useful lemma gives some consequences of this property.

<u>Lemma 3.1.</u> Let $\phi_1, \phi_2, \phi_3, \ldots$ be a linearly independent sequence of polynomials in $x$ and $y$ and let $p_n$ be a nonzero polynomial with "highest" term $c \cdot \phi_n$ which is obtained by orthogonalizing the sequen[ce] $\phi_1, \phi_2, \ldots, \phi_n$ with respect to a certain weight function on a certain re[-]gion. Let $\prec$ be a partial ordering of the set $\mathbb{N}$ of natural numbers su[ch] that $i \prec j$ implies $i \leq j$ and such that for each $n \in \mathbb{N}$ $p_n$ is a linear combination of polynomials $\phi_m$, $m \prec n$. Then:

(a) For each $n$, $\phi_n$ is a linear combination of polynomials $p_m$, $m \prec n$.

(b) If $q$ is a linear combination of polynomials $\phi_m$, $m \prec n$, and if $q$ is orthogonal to $\phi_m$ for $m \prec n$ and $m \neq n$, then $q = c \cdot p_n$.

(c) If not $m \succ n$ then $p_n$ is orthogonal to $\phi_m$.

The easy proof is left to the reader.





## 3.2. <u>The definition of classical orthogonal polynomials in two variables</u>

It seems natural to look for some two-variable analogue of the one-variable criterium for classical orthogonal polynomials that the polynomials must be eigenfunctions of a second order differential operator. Krall and Sheffer [53] and independently Engelis [16] considered the case that the classes $\mathscr{V}_n$, $n = 0,1,2,\ldots$, of orthogonal polynomials of degree $n$ with respect to a certain weight function $w(x,y)$ on a certain region $\Omega$ are eigenspaces of a second order linear partial differential operator. They classified all partial differential operators having this property. The only bounded regions occurring in their classification are the unit disk with weight function $(1-x^2-y^2)^{\alpha}$ and the triangular region $\{(x,y) \mid 0<y<x<1\}$ with weight function $(1-x)^{\alpha}(x-y)^{\beta}y^{\gamma}$.

In the present paper we shall consider examples of orthogonal systems $\{p_{n,k}(x,y)\}$, such that the polynomials $p_{n,k}(x,y)$ are the joint eigenfunctions of two commuting partial differential operators $D_1$ and $D_2$, where $D_1$ has order two and $D_2$ may have any arbitrary order, and where $D_1$ and $D_2$ are algebraically independent, i.e., if $Q$ is a polynomial in two variables and if $Q(D_1, D_2)$ is the zero operator then $Q$ is the zero polynomial. If the eigenvalue of $p_{n,k}(x,y)$ with respect to $D_1$ only depends on the degree $n$ then we are back in the situation studied by Krall and Sheffer [53]. In that case the operator $D_2$ provides us a canonical method to choose an orthogonal basis for $\mathscr{V}_n$, i.e. by taking the eigenfunctions of $D_2$.

It does not yet seem to be the time to make a final decision, which systems should be called two-variable analogues of the classical orthogonal polynomials. Rather than trying to classify all orthogonal systems which are eigenfunctions of differential operators, we shall emphasize the methods by which such systems can be constructed. These methods, which are suggested by the results for Jacobi polynomials stated in section 2, are the following:

(a) Consider orthogonal polynomials in two variables which can be expressed in terms of Jacobi polynomials in some elementary way.

(b) Consider orthogonal polynomials in two variables which are analogous to Chebyshev polynomials, i.e., which can be expressed as elementary trigonometric functions in two variables or as spherical harmonics on the sphere $S^2$ satisfying symmetry relations with respect to certain reflections.

(c) Consider orthogonal polynomials in two variables which can be interpreted as spherical functions on homogeneous spaces of rank 2. Informally stated, a homogeneous space $M = G/K$ has rank r if G and K are Lie groups and the set of K-orbits on M is a manifold of dimension r (except possibly for a set of measure zero).

(d) Construct new orthogonal polynomials in two variables by performing quadratic transformations on known ones.

(e) Construct new orthogonal polynomials in two variables from known ones by doing "analytic continuation" with respect to some parameter.

## 3.3. Examples of two-variable analogues of the Jacobi polynomials

Below we introduce seven different classes of orthogonal polynomials in two variables.

Class I. For $\alpha > -1$, $z = x + iy$, $\bar{z} = x - iy$, the polynomials

$$(3.7)\qquad {}_1P^{\alpha}_{m,n}(z,\bar{z}) = \begin{cases} P_n^{(\alpha,\, m-n)}(2z\bar{z}-1)\, z^{m-n} & \text{if} \quad m \geq n\,, \\ P_m^{(\alpha,\, n-m)}(2z\bar{z}-1)\, \bar{z}^{n-m} & \text{if} \quad m < n\,, \end{cases}$$





are orthogonal with respect to the weight function $(1-x^2-y^2)^{\alpha}$ on the unit disk.

Class II. For $\alpha > -1$ the polynomials

$$(3.8)\qquad {}_2P^{\alpha}_{n,k}(x,y) = P_{n-k}^{(\alpha+k+\frac{1}{2},\, \alpha+k+\frac{1}{2})}(x)\,(1-x^2)^{\frac{1}{2}k} \cdot P_k^{(\alpha,\alpha)}((1-x^2)^{-\frac{1}{2}}y), \qquad n \geq k \geq 0\,,$$

are also orthogonal with respect to the weight function $(1-x^2-y^2)^{\alpha}$ on the unit disk.

Class III. For $\alpha, \beta > -1$ the polynomials

$$(3.9)\qquad {}_3P^{\alpha,\beta}_{n,k}(x,y) = P_{n-k}^{(\alpha,\, \beta+k+\frac{1}{2})}(2x-1)x^{\frac{1}{2}k}\, P_k^{(\beta,\beta)}(x^{-\frac{1}{2}}y), \qquad n \geq k \geq 0\,,$$

are orthogonal with respect to the weight function $(1-x)^{\alpha}(x-y^2)^{\beta}$ on the region $\{(x,y) \mid y^2 < x < 1\}$, which is bounded by a straight line and a parabola.

Class IV. For $\alpha, \beta, \gamma > -1$ the polynomials

$$(3.10)\qquad {}_4P^{\alpha,\beta,\gamma}_{n,k}(x,y) = P_{n-k}^{(\alpha,\, \beta+\gamma+2k+1)}(2x-1)x^k \cdot P_k^{(\beta,\gamma)}(2x^{-1}y-1), \quad n \geq k \geq 0\,,$$

are orthogonal with respect to the weight function $(1-x)^{\alpha}(x-y)^{\beta}y^{\gamma}$ on the triangular region $\{(x,y) \mid 0 < y < x < 1\}$.

Class V. For $\alpha, \beta, \gamma, \delta > -1$ the polynomials

$$(3.11)\qquad {}_5P^{\alpha,\beta,\gamma,\delta}_{n,k}(x,y) = P_{n-k}^{(\alpha,\beta)}(x)\, P_k^{(\gamma,\delta)}(y), \qquad n \geq k \geq 0\,,$$

are orthogonal with respect to the weight function $(1-x)^{\alpha}(1+x)^{\beta}(1-y)^{\gamma}(1+y)^{\delta}$ on the square $\{(x,y) \mid -1 < x < 1, \ -1 < y < 1\}$.

Class VI. Let $\alpha, \beta, \gamma > -1$, $\alpha + \gamma + 3/2 > 0$, $\beta + \gamma + 3/2 > 0$. Let ${}_6P^{\alpha,\beta,\gamma}_{n,k}(u,v)$, $n \geq k \geq 0$, be a polynomial with highest term $c \cdot u^{n-k}v^{k}$ obtained by orthogonalization of the sequence $1, u, v, u^2, uv, \ldots$ with respect to the weight function

$$(1-u+v)^{\alpha}\,(1+u+v)^{\beta}\,(u^2-4v)^{\gamma}$$

on the region $\{(u,v) \mid |u| < v+1, \ u^2-4v > 0\}$, which is bounded by two straight lines and a parabola touching these lines. In particular we have

$$(3.12) \qquad {}_6P^{\alpha,\beta,-\frac{1}{2}}_{n,k}(x+y, xy) =$$

$$= c \cdot [P^{(\alpha,\beta)}_n(x)\, P^{(\alpha,\beta)}_k(y) + P^{(\alpha,\beta)}_k(x)\, P^{(\alpha,\beta)}_n(y)]$$

and

$$(3.13) \qquad {}_6P^{\alpha,\beta,\frac{1}{2}}_{n,k}(x+y, xy) =$$

$$= c \cdot (x-y)^{-1}[P^{(\alpha,\beta)}_{n+1}(x)\, P^{(\alpha,\beta)}_k(y) - P^{(\alpha,\beta)}_k(x)\, P^{(\alpha,\beta)}_{n+1}(y)] \; .$$

Class VII. Let $\alpha > -5/6$, $z = x + iy$, $\bar{z} = x - iy$, $m, n \geq 0$. Let ${}_7P^{\alpha}_{m,n}(z,\bar{z}) = c \cdot z^m \bar{z}^n$ + polynomial in $z, \bar{z}$ of degree less than $m+n$ such that ${}_7P^{\alpha}_{m,n}(z,\bar{z})$ is orthogonal to all polynomials $q(z,\bar{z})$ of degree less than $m+n$ with respect to the weight function

$$[-(x^2+y^2+9)^2 + 8(x^3-3xy^2) + 108]^{\alpha}$$

on the region bounded by the three-cusped deltoid (or Steiner's hypocycloid)

$$-(x^2+y^2+9)^2 + 8(x^3-3xy^2) + 108 = 0 \; .$$





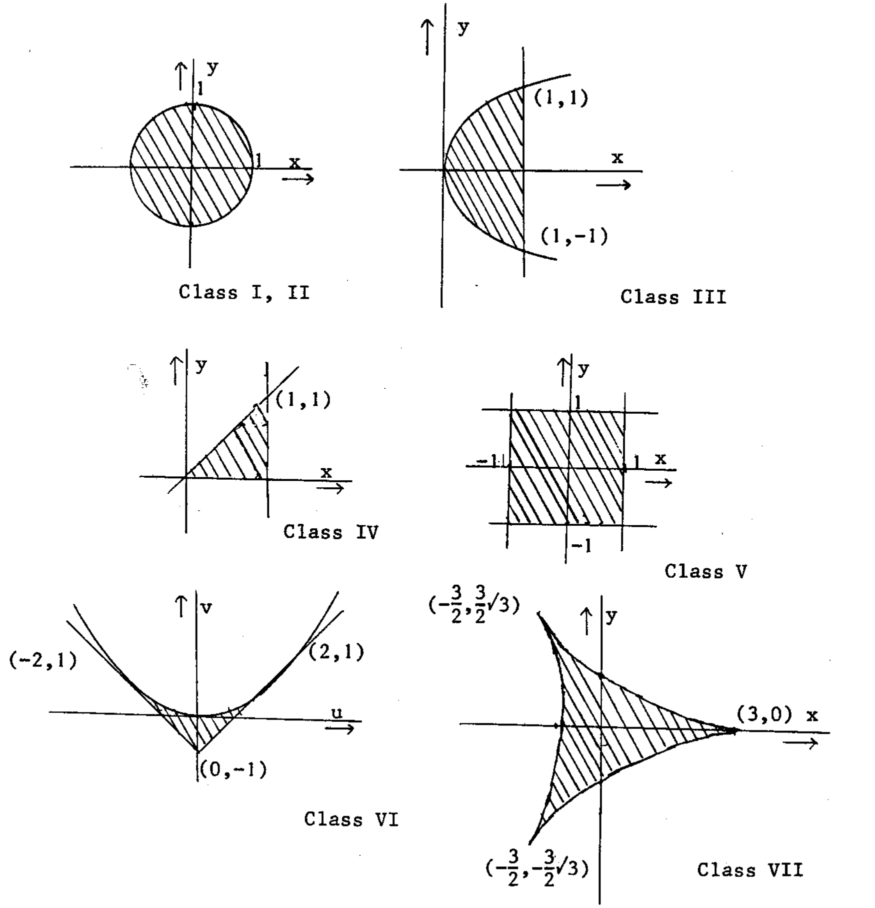


Figure 2

It can be proved that the polynomials ${}_7P^{\alpha}_{n-k,k}(z,\bar{z})$, $k = 0, 1, \ldots, n$, form an orthogonal basis for the class $\mathcal{H}_n$ with respect to this weight function and region. For $\alpha = \pm\frac{1}{2}$ we have an explicit expression. Let

$$e^{\pm}_{m,n}(\sigma,\tau) = e^{i(m\sigma+n\tau)} \pm e^{i((m+n)\sigma-n\tau)} + e^{i(-(m+n)\sigma+m\tau)} \pm e^{i(-n\sigma-m\tau)} + e^{i(n\sigma-(m+n)\tau)} \pm e^{i(-m\sigma+(m+n)\tau)} \tag{3.14}$$

and

$$z = e^{i\sigma} + e^{-i\tau} + e^{i(-\sigma+\tau)} . \tag{3.15}$$

Then

$${}_7P^{-\frac{1}{2}}_{m,n}(z,\bar{z}) = c \cdot e^{+}_{m,n}(\sigma,\tau) \tag{3.16}$$

and

$${}_7P^{\frac{1}{2}}_{m,n}(z,\bar{z}) = c \cdot e^{-}_{m+1,n+1}(\sigma,\tau)/e^{-}_{1,1}(\sigma,\tau) . \tag{3.17}$$

In Figure 2 we give pictures of the orthogonality regions for these seven classes.

At this stage it is useful to make a remark about the notation chos in this paper. The author believes that the theory of special orthogonal polynomials in several variables is not yet developed far enough in orde to make a final decision about the notation. The notation introduced above is therefore only intended for the present paper. We shall write $p_{n,k}$ instead of $P_{n,k}$ if the coefficient of the highest term of the ortho onal polynomial equals 1. The notation $R_{n,k}$ instead of $P_{n,k}$ will b used if the orthogonal polynomial equals 1 at a vertex of the orthogona ity region. The left under index denoting the class will be deleted if nc confusion is possible.





In the classes I and VII the polynomials have the form $c \cdot z^m \bar{z}^n +$ + a polynomial of degree less than $m + n$. This is an important difference with the polynomials of the other classes, which are obtained by orthogonalizing the sequence $1, x, y, x^2, xy, \ldots$. The polynomials of the classes I-V have a simple explicit expression in terms of Jacobi polynomials. However, in the classes VI and VII such elementary expressions only exist for certain special values of the parameters. It can therefore be expected that the study of these last two classes is much more difficult than the study of the other classes. On the other hand, the theory of the classes I-IV is by no means a trivial corollary of the theory of Jacobi polynomials. It is helpful to consider the classes VI and VII as more difficult analogues of the classes IV and I, respectively.

### 3.4. Some references and applications

Orthogonal polynomials on the disk of class I were introduced by Zernike and Brinkman [74]. Zernike [73] used the case $\alpha = 0$ for the study of diffraction problems. For further applications in optics see the references in Myrick [62], cf. also Marr [58]. If $\alpha = q - 2$, $q = 2, 3, 4, \ldots$, then these polynomials are the spherical functions on the sphere $S^{2q-1}$ considered as homogeneous space $U(q)/U(q-1)$, cf. Vilenkin and Šapiro [70], [64], Ikeda and Kayama [30], [31], Koornwinder [43], Boyd [9], Folland [19], [20], Dunkl [13], and Annabi [2]. From this group theoretic interpretation there follows an addition formula for the polynomials of class I (cf. Šapiro [64]), which implies the addition formulas for Jacobi polynomials and for Laguerre polynomials (cf. Koornwinder [44] and [50], respectively). A related result is a positive convolution structure for polynomials of class I, cf. Annabi and Trimèche [3], Trimèche [69] and Kanjin [37].

Orthogonal polynomials on the disk of class II were introduced by Didon [12] in the case $\alpha = 0$ and by Koschmieder [51] in the general case. A generating function is given by Koschmieder [52]. Using this class of polynomials Koornwinder [49] obtained a new proof of the addition formula for Gegenbauer polynomials. Hermite and Didon introduced a well-known biorthogonal system of polynomials on the unit disk with respect to the weight function $(1-x^2-y^2)^\alpha$, cf. Erdélyi [17, sections 12.5 and 12.6].

Orthogonal polynomials of class III are implicitly contained in Agahanov [1]. The addition formula for Jacobi polynomials can be considered as an orthogonal expansion in terms of the polynomials of class III, cf. Koornwinder [48, §3].

Orthogonal polynomials of class IV on a triangular region were introduced by Proriol [63]. They were applied to the problem of solving the Schrödinger equation for the Helium atom, cf. Munschy and Pluvinage [61], [60]. The same class was independently obtained by Karlin and McGregor [39] in view of applications to genetics. Appell's polynomials on the triangle (cf. Erdélyi [17, §12.4]) provide a nonorthogonal basis for the class $\mathscr{K}_n$ with respect to the weight function and region considered in class IV. In the case of order $(\alpha, \beta, 0)$ Appell extended this basis to a biorthogonal system. Engelis [15] and independently Fackerell and Littler [18] obtained a biorthogonal system in the case of general order.

The polynomials of class VI were introduced by Koornwinder [45]. The motivation was that for special values of the parameters and in terms of suitable coordinates these polynomials are eigenfunctions of the Laplace-Beltrami operator on certain compact symmetric spaces of rank 2, cf. section 5. A further analysis was given by Sprinkhuizen [66], see also section 4.3. For $\gamma = 0$ and $n = k$ these are the generalized Jacobi polynomials of $2 \times 2$ matrix argument introduced by Herz [29], cf. section 4.4. For the interpretation of polynomials of class VI as spherical functions the reader is referred to section 5.

The author [46] also introduced the polynomials of class VII, motivated in a similar way as in the case of class VI. These polynomials were independently considered by Eier and Lidl [14], [54] for $n = 0$. The symmetric (or antisymmetric) polynomials generated by the monomial $x_1^{n_1} x_2^{n_2} x_3^{n_3}$ can be expressed in terms of the elementary symmetric polynomials in three variables by using the polynomials of class VII of order $\pm\frac{1}{2}$. Zonal polynomials of $3 \times 3$ matrix argument (cf. James [35]) can be expressed in terms of polynomials of class VII of order $0$. For all these results we refer to section 4.5. Section 5 contains references about the interpretation by means of spherical functions.





### 3.5. Two-variable analogues of the Chebyshev polynomials

The best way to bring more systematics in the examples of section 3.3 is to look at the Chebyshev cases, i.e. when the parameters are equal to $+\frac{1}{2}$ or $-\frac{1}{2}$. For these cases there exist interpretations analogous to the interpretation of Chebyshev polynomials given in section 2.2. It turns out that the classes I-IV are related to symmetries on the sphere $S^2$ and the classes V-VII to symmetries on certain two-dimensional tori. The results below are just observations which can be verified for our examples. However, these observations suggest the existence of a general theory.

On the sphere $S^2$ we use spherical coordinates $\theta$ $(0 \le \theta \le \pi)$ and $\phi$ (mod $2\pi$) such that the mapping $(\theta, \phi) \to (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ gives the natural embedding of $S^2$ as a subspace of $\mathbb{R}^3$, cf. Fig. 3.

A two-dimensional torus is obtained by choosing two linearly independent vectors $e_1$ and $e_2$ in $\mathbb{R}^2$ and by identifying points in $\mathbb{R}^2$ whose difference is $ke_1 + \ell e_2$, $k$, $\ell$ integers. All such tori are topologically equivalent with $S^1 \times S^1$, but they are different as Riemannian spaces. We shall consider two particular tori for which the group of all isometries is sufficiently rich for our purposes. The first one is the square torus denoted by $T^2$, where $e_1 = (2\pi, 0)$, $e_2 = (0, 2\pi)$ and each point in $\mathbb{R}^2$ has one and only one representative in the square region $\{(s,t) \mid -\pi < s \le \pi,\ -\pi < t \le \pi\}$, cf. Figure 4. The other one is the hexagonal torus denoted by $H$, where $e_1 = (\pi, \pi\sqrt{3})$, $e_2 = (\pi, -\pi\sqrt{3})$ and each point in $\mathbb{R}^2$ has one and only one representative in the hexagonal region

$$\{(s,t) \mid -\pi < s \le \pi,\ -\pi < \tfrac{1}{2}s + \tfrac{1}{2}\sqrt{3}\,t \le \pi,\ -\pi < \tfrac{1}{2}s - \tfrac{1}{2}\sqrt{3}\,t \le \pi\},$$

cf. Figure 5.

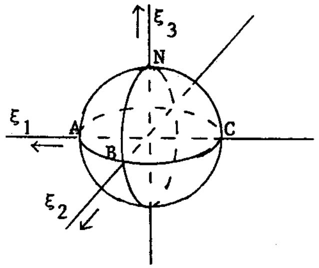


Figure 3

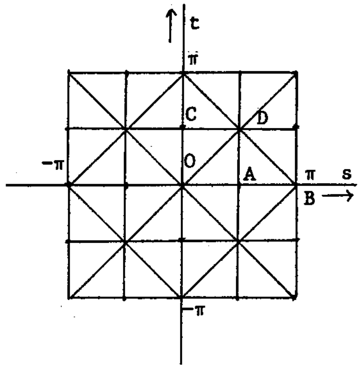


Figure 4

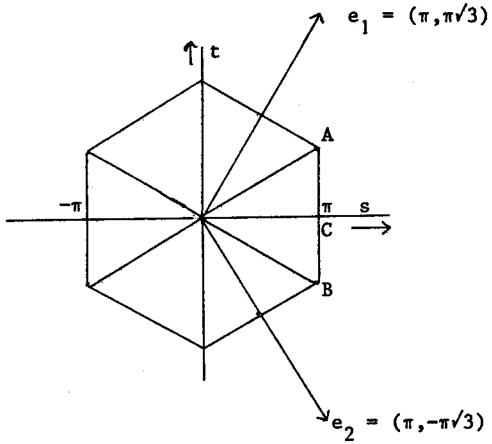


Figure 5





Corresponding with each of the seven classes introduced in section 3.3 we shall choose a Riemannian manifold $M$ (one of the spaces $S^2$, $T^2$ or $H$), a group $G$ of isometries acting transitively on $M$ (not necessarily the group of all isometries), a polygonal region $R$ in $M$ which is bounded by geodesics such that the reflections in these geodesics generate a discrete subgroup $\Gamma$ of $G$ having $R$ as a fundamental region, and finally a $C^\infty$-mapping $F$ from $M$ onto the orthogonality region $\Omega$ of the polynomials such that $F(T\xi) = F(\xi)$ for all $T \in \Gamma$ and $F$ restricted to $R$ is a diffeomorphism. By using the mapping $F$ any orthogonal system of polynomials belonging to one of the seven classes can be considered as an orthogonal system of functions on $R$ with respect to a certain measure $\rho(\xi)\,d\xi$, where $d\xi$ is the measure on $M$ induced by the Riemannian metric ($d\xi = \sin\theta\, d\theta\, d\phi$ on $S^2$, $d\xi = ds\,dt$ on $T^2$ and $H$).

The choices of $M$, $G$, $R$ and $F$ corresponding to the different classes, and the resulting weight function $\rho$ are listed in Table 1. The Table contains references to the Figures 3, 4 and 5.

After having made these choices the following results can be obtained by inspection.

(a) The manifold $M$ as a homogeneous space of the group $G$ satisfies the equivalent statements of Theorem 2.1. Hence the harmonics on $M$ with respect to $G$ are well-defined by Definition 2.2.

(b) The two coordinates of the mapping $F$ are harmonics which are invariant with respect to $\Gamma$.

(c) Let $\sigma$ be a one-dimensional representation of the group $\Gamma$. Then $\sigma(T) = \pm 1$ for each $T \in \Gamma$. Call a function $f$ on $M$ of type $\sigma$ if $f(T\xi) = \sigma(T)\, f(\xi)$ for each $T \in \Gamma$.

(d) If an orthogonal system of polynomials as given in Table 1 has parameters $\pm\frac{1}{2}$ then the corresponding weight function $\rho$ on $R$ is the square of a harmonic on $M$ of certain type $\sigma$ which is strictly positive on $R$. If all parameters are equal to $-\frac{1}{2}$ then $\rho$ is the constant function on $R$ and if all parameters are equal to $+\frac{1}{2}$ then $\rho(\xi)$ is the square

| system of polynomials | manifold M | (generators of) G | region R | mapping F: $M \to \bar{\Omega}$ | weight function $\rho$ with respect to invariant measure on M |
|---|---|---|---|---|---|
| ${}_1P^{\alpha}_{m,n}(z,\bar{z})$ | $S^2$ | O(3) | $0 < \theta < \pi/2$ | $z = \sin\theta\, e^{i\phi}$ <br> $\bar{z} = \sin\theta\, e^{-i\phi}$ | $(\cos\theta)^{2\alpha+1}$ |
| ${}_2P^{\alpha}_{n,k}(x,y)$ | $S^2$ | O(3) | $0 < \phi < \pi$ | $x = \cos\theta$, <br> $y = \sin\theta\cos\phi$ | $(\sin\theta\sin\phi)^{2\alpha+1}$ |
| ${}_3P^{\alpha,\beta}_{n,k}(x,y)$ | $S^2$ | O(3) | ACN | $x = \sin^2\theta$, <br> $y = \sin\theta\cos\phi$ | $(\cos\theta)^{2\alpha+1}(\sin\theta\sin\phi)^{2\beta+1}$ |
| ${}_4P^{\alpha,\beta,\gamma}_{n,k}(x,y)$ | $S^2$ | O(3) | ABN | $x = \sin^2\theta$, <br> $y = \sin^2\theta\cos^2\phi$ | $(\cos\theta)^{2\alpha+1}(\sin\theta\sin\phi)^{2\beta+1}$ <br> $\cdot\ (\sin\theta\cos\phi)^{2\gamma+1}$ . |
| ${}_5P^{\alpha,\beta,\gamma,\delta}_{n,k}(x,y)$ | $T^2$ | O(2) × O(2) | OADC | $x = \cos 2s$, <br> $y = \cos 2t$ | $(\sin s)^{2\alpha+1}(\cos s)^{2\beta+1}$ <br> $\cdot\ (\sin t)^{2\gamma+1}(\cos t)^{2\delta+1}$ . |
| ${}_6P^{\alpha,\alpha,\gamma}_{n,k}(u,v)$ | $T^2$ | O(2) × O(2), <br> $(s,t) \to (t,s)$ | OBE | $u = \cos s + \cos t$ <br> $v = \cos s\ \cos t$ | $(\sin s\sin t)^{2\alpha+1}$ <br> $\cdot\ (\sin(\frac{s+t}{2})\sin(\frac{s-t}{2}))^{2\gamma+1}$ . |
| ${}_6P^{\alpha,\beta,\gamma}_{n,k}(u,v)$ | $T^2$ | O(2) × O(2) <br> $(s,t) \to (t,s)$ | OAD | $u = \cos 2s + \cos 2t$ <br> $v = \cos 2s\ \cos 2t$ | $(\sin s\sin t)^{2\alpha+1}$ <br> $\cdot (\cos s\cos t)^{2\beta+1}$ <br> $\cdot\ (\sin(s+t)\sin(s-t))^{2\gamma+1}$ . |
| ${}_7P^{\alpha}_{m,n}(z,\bar{z})$ | H | SO(2) × SO(2), reflections in OA and OB | OAB | $z = e^{i(s+t/\sqrt{3})} +$ <br> $+ e^{i(-s+t/\sqrt{3})} +$ <br> $+ e^{-2it/\sqrt{3}}$ | $(\sin s)^{2\alpha+1}$ <br> $\cdot\ (\sin(\frac{1}{2}s + \frac{1}{2}\sqrt{3}\,t))^{2\alpha+1}$ <br> $\cdot\ (\sin(\frac{1}{2}s - \frac{1}{2}\sqrt{3}\,t))^{2\alpha+1}$ . |

Table 1





of a harmonic which equals the Jacobian $\partial F(\xi)/\partial\xi$ and which changes sign with respect to any reflection in a boundary of R .

(e) There is a one-to-one correspondence between the one-dimensional representations $\sigma$ of $\Gamma$ and the Chebyshev cases of the class of orthogonal polynomials under consideration. In fact, for fixed values $\pm\frac{1}{2}$ of the parameters, the polynomials considered as functions of $\xi$ can be written as a quotient of two harmonics of type $\sigma$, where the denominator equals $(\rho(\xi))^{\frac{1}{2}}$ .

Let us illustrate these rather abstract statements by means of a few examples. The harmonics on $S^2 = O(3)/O(2)$ are the well-known spherical harmonics of degree $n = 0, 1, 2, \ldots$ . If the class of spherical harmonics of degree n is decomposed into irreducible subclasses with respect so O(2) then we obtain the functions

(3.18) $$P^{(k,k)}_{n-k}(\cos\theta)\,(\sin\theta)^k\,(A\cos k\phi + B\sin k\phi)\ ,$$

$k = 0, 1, 2, \ldots, n$, where A and B are constants. A further decomposition into irreducible subclasses with respect to S O(2) gives the functions

(3.19) $$P^{(k,k)}_{n-k}(\cos\theta)(\sin\theta)^k\, e^{\pm ik\phi}, \qquad k = 0, 1, \ldots, n\ .$$

Examples of Chebyshev cases for the classes I-IV are, for instance:

(3.20) $${}_1P^{-\frac{1}{2}}_{m,n}(\sin\theta\, e^{i\phi},\ \sin\theta\, e^{-i\phi}) =$$

$$= c \cdot P^{(m-n,\,m-n)}_{2n}(\cos\theta)\,(\sin\theta)^{m-n}\, e^{i(m-n)\phi}\ ,$$

$m \geq n$ ,

(3.21) $${}_1P^{\frac{1}{2}}_{m,n}(\sin\theta\, e^{i\phi},\ \sin\theta\, e^{-i\phi}) = \frac{P^{(m-n,\,m-n)}_{2n+1}(\cos\theta)(\sin\theta)^{m-n}\, e^{i(m-n)\phi}}{\cos\theta}\ ,$$

$m \geq n$ ,

(3.22) $$ {}_2P^{\frac{1}{2}}_{n,k}(\cos\theta,\ \sin\theta\cos\phi) = c\cdot\frac{P^{(k+1,\,k+1)}_{n-k}(\cos\theta)(\sin\theta)^{k+1}\sin(k+1)\phi}{\sin\theta\,\sin\phi}\ , $$

(3.23) $$ {}_3P^{\frac{1}{2},\frac{1}{2}}_{n,k}(\sin^2\theta,\ \sin\theta\cos\phi) = c\cdot\frac{P^{(k+1,\,k+1)}_{2n-2k+1}(\cos\theta)(\sin\theta)^{k+1}\sin(k+1)\phi}{\cos\theta\,\sin\theta\,\sin\phi}\ , $$

(3.24) $$ {}_4P^{-\frac{1}{2},-\frac{1}{2},\frac{1}{2}}_{n,k}(\sin^2\theta,\ \sin^2\theta\cos^2\phi) = c\cdot\frac{P^{(2k+1,\,2k+1)}_{2n-2k}(\cos\theta)(\sin\theta)^{2k+1}\cos(2k+1)\phi}{\sin\theta\,\cos\phi}\ . $$

The harmonics on the square torus $T^2$ with respect to the group generated by $O(2)\times O(2)$ and the reflection $(s,t)\to(t,s)$ are the linear combinations of $\begin{Bmatrix}\cos ns\\ \sin ns\end{Bmatrix}\cdot\begin{Bmatrix}\cos kt\\ \sin kt\end{Bmatrix}$ and $\begin{Bmatrix}\cos ks\\ \sin ks\end{Bmatrix}\cdot\begin{Bmatrix}\cos nt\\ \sin nt\end{Bmatrix}$, where $(n,k)$, $n\geq k\geq 0$, is fixed. We have, for instance

(3.25) $$ {}_6P^{-\frac{1}{2},-\frac{1}{2},\frac{1}{2}}_{n,k}(\cos s+\cos t,\ \cos s\cos t) = c\cdot\frac{\cos(n+1)s\cos kt - \cos ks\cos(n+1)t}{\cos t-\cos s}\ , $$

(3.26) $$ {}_6P^{\frac{1}{2},-\frac{1}{2},\frac{1}{2}}_{n,k}(\cos 2s+\cos 2t,\ \cos 2s\cos 2t) = c\cdot\frac{\sin(2n+3)s\,\sin(2k+1)t - \sin(2k+1)s\,\sin(2n+3)t}{\sin 3s\,\sin t - \sin s\,\sin 3t}\ . $$

Finally let us consider the hexagonal torus $H$ as a homogeneous space of the group generated by $SO(2)\times SO(2)$ and the reflections in $OA$ and $OB$. In terms of new coordinates



$$ \sigma = s + t/\sqrt{3},\quad \tau = s - t/\sqrt{3} $$

the harmonics on $H$ are the linear combinations of

$$ e^{i(m\sigma+n\tau)},\ e^{i((m+n)\sigma-n\tau)},\ e^{i(-(m+n)\sigma+m\tau)}, $$

$$ e^{i(-n\sigma-m\tau)},\ e^{i(n\sigma-(m+n)\tau)}\ \text{and}\ e^{i(-m\sigma+(m+n)\tau)}, $$

where $m$ and $n$ are integers $\geq 0$. Now the formulas (3.14), (3.15), (3.16) and (3.17) give an interpretation of the Chebyshev cases of class VII.

Observe that the groups $\Gamma$ are abelian in the cases I-V but non-abelian in the cases VI and VII. If the classes of harmonics on $M$ are decomposed into irreducible subclasses with respect to nonabelian $\Gamma$ then some of these subclasses do not correspond with one-dimensional representations $\sigma$ of $\Gamma$. The harmonics corresponding with higher dimensional irreducible representations of $\Gamma$ do not fit in our picture. It is not clear how to express such harmonics as orthogonal polynomials.

In the cases V-VII each one-dimensional representation $\sigma$ of $\Gamma$ occurs at most once in each class of harmonics on $M$. This is no longer true for the cases I-IV. There we first have to split up the spherical harmonics with respect to $O(2)$ or $SO(2)$ in order to get uniqueness with respect to the one-dimensional representations $\sigma$ of $\Gamma$.

The reader may observe some missing cases in Table 1. If $M$ is the hexagonal torus $H$, if $G$ is the group of all isometries of $H$ (i.e. the reflection in $OC$ included) and if $R$ is the region $OCA$ (cf. Figure 5), then probably the Chebyshev cases of another interesting class of orthogonal polynomials can be obtained. On the sphere $S^2$ we may take for the region $R$ a spherical triangle with angles $\{\pi/2, \pi/3, \pi/3\}$ or $\{\pi/2, \pi/3, \pi/4\}$ or $\{\pi/2, \pi/3, \pi/5\}$. It is not clear at all whether the approach of this subsection applied to these regions will give new classes of orthogonal polynomials.

3.6. Partial differential operators for which the orthogonal polynomials are eigenfunctions

It follows easily from the results of section 3.5 that for the Chebyshev cases of the classes I-VII the polynomials are eigenfunctions of certain partial differential operators. In this subsection it will be discussed how these differential operators can be generalized for other values of the parameters.

The harmonics on the sphere $S^2$ are eigenfunctions of the Laplace-Beltrami operator

$$\partial_{\theta\theta} + \cot\theta\ \partial_\theta + (\sin\theta)^{-2}\partial_{\phi\phi}\ . \tag{3.27}$$

The harmonics on $S^2$ belonging to irreducible subclasses with respect to SO(2) are also eigenfunctions of $\partial_\phi$ (cf. (3.19)), and the harmonics belonging to irreducible subclasses with respect to O(2) are also eigenfunctions of $\partial_{\phi\phi}$ (cf. (3.18)).

The harmonics on the tori $T^2$ and $H$ are eigenfunctions of the Laplace operator $\partial_{ss} + \partial_{tt}$. In case V the harmonics on $T^2$ with respect to the symmetry group $O(2)\times O(2)$ are also eigenfunctions of $\partial_{ss}$. In case VI the harmonics on $T^2$ with respect to the group of all isometries of $T^2$ are also eigenfunctions of $\partial_{sstt}$. In case VII the harmonics on $H$ with respect to the group generated by $SO(2)\times SO(2)$ and the reflections in OA and OB are also eigenfunctions of the operator

$$\partial_t(\sqrt{3}\ \partial_s + \partial_t)(-\sqrt{3}\ \partial_s + \partial_t)\ . \tag{3.28}$$

It follows that in each case of Table 1 the harmonics are eigenfunctions of two algebraically independent partial differential operators, one of which is the Laplace-Beltrami operator on $M$.

Fix one of the classes I-VII and let $D$ be a partial differential operator for which the harmonics are eigenfunctions. Fix the parameters of the orthogonal system of polynomials equal to $\pm\frac{1}{2}$. By using result





(e) of section 3.5 it follows that each polynomial of the system considered as a function of $\xi$ is an eigenfunction of the operator

$$(\rho(\xi))^{-\frac{1}{2}}\, D \circ (\rho(\xi))^{\frac{1}{2}}\ .$$

Since $(\rho(\xi))^{\frac{1}{2}}$ is a harmonic (cf. result (d) in section 3.5), the polynomials of the system are also eigenfunctions of the operator

$$(\rho(\xi))^{-\frac{1}{2}}\, D \circ (\rho(\xi))^{\frac{1}{2}} - (\rho(\xi))^{-\frac{1}{2}}\,(D(\rho(\xi))^{\frac{1}{2}})\ , \tag{3.29}$$

which is a partial differential operator without zero order part.

Let us next consider an orthogonal system of polynomials of arbitrary order belonging to one of the classes I-VII and let $\rho$ be the corresponding weight function on the region $R$ as given in Table 1. Then, if the partial differential operator $D$ has order 1 or 2, the polynomials of the system considered as functions of $\xi$ are still eigenfunctions of the operator (3.29). This can be proved by first observing that the operator (3.29) is selfadjoint on the region $R$ with respect to the measure $\rho(\xi)d\xi$. Next, for each individual class one has to rewrite the operator (3.29) in terms of coordinates $x, y$ (or $z, \bar{z}$) and one has to verify that this operator maps $x^{n-k}y^k$ to a polynomial with "highest" term $\lambda_{n,k}x^{n-k}y^k$ (or $z^m\bar{z}^n$ to a polynomial with highest term $\lambda_{m,n}\ z^m\bar{z}^n$). Then $\lambda_{n,k}$ is the eigenvalue of the eigenfunction $P_{n,k}$. For the classes VI and VII (with $D = \partial_{ss} + \partial_{tt}$) the detailed proof has been given in Koornwinder [45, §4] and [46, §5], respectively. Note that the operator (3.29) is the unique analytic continuation of the Chebyshev cases such that it depends linearly on the parameters.

In the cases VI and VII there are partial differential operators of order 4 and 3, respectively, for which the harmonics are eigenfunctions. In these cases it is much more difficult to find the generalization of (3.29) for arbitrary values of the parameters. It can be proved that ${}_6P^{\alpha,\beta,\gamma}_{n,k}$ $(\cos 2s + \cos 2t,\ \cos 2s\cos 2t)$ is an eigenfunction of the fourth order operator

(3.30) $$\rho^{-1}(\partial_s \circ \rho_2 \partial_t + \partial_t \circ \rho_2 \partial_s) \circ$$
$$\circ\ \rho_1 \rho_2^{-1}(\partial_s \circ \rho_2 \partial_t + \partial_t \circ \rho_2 \partial_s) ,$$

where $\rho_1(s,t) = (\sin s \sin t)^{2\alpha+1}(\cos s \cos t)^{2\beta+1}$, $\rho_2(s,t) = (\sin(s+t) \cdot \sin(s-t))^{2\gamma+1}$ and $\rho = \rho_1\rho_2$, cf. Koornwinder [45, (5.15)]. It can also be proved that ${}_7P^{\alpha}_{m,n}(F(s,t))$ is an eigenfunction of the third order operator

(3.31) $$X_1X_2X_3 + (\alpha+\tfrac{1}{2})\,\rho^{-1}[(X_1\rho)X_2X_3 + (X_2\rho)X_3X_1 + (X_3\rho)X_1X_2] +$$
$$+ (\alpha+\tfrac{1}{2})^2\,\rho^{-1}[(X_1X_2\rho)X_3 + (X_2X_3\rho)X_1 + (X_3X_1\rho)X_2] ,$$

where $F(s,t)$ and $\rho(s,t)$ are given in Table 1 and

$$X_1 = i(\tfrac{3}{2}\,\partial_s - \tfrac{1}{2}\sqrt{3}\,\partial_t),\quad X_2 = -i(\tfrac{3}{2}\,\partial_s + \tfrac{1}{2}\sqrt{3}\,\partial_t) ,$$
$$X_3 = i\sqrt{3}\,\partial_t ,$$

cf. Koornwinder [46, §6].

Summarizing our results we have for each of the classes I-VII and for each choice of the parameters two partial differential operators (say $D_1$ and $D_2$) for which the polynomials of the system are eigenfunctions. The operator $D_1$ has order 2 in all cases and the operator $D_2$ has order 1, 2, 2, 2, 2, 4 or 3, respectively. It can be proved that the operators $D_1$ and $D_2$ commute, that they are algebraically independent and that they generate the algebra of all differential operators which have the polynomials of the system as eigenfunctions (cf. Koornwinder [45, §6] and [46, §7] for the classes VI and VII, respectively). It can also be proved that two distinct polynomials belonging to the same orthogonal system cannot have the same pair of eigenvalues with respect to $D_1$ and $D_2$.





### 3.7. General methods of constructing orthogonal polynomials in two variables from orthogonal polynomials in one variable

There are known some rather general classes of orthogonal polynomials in two variables which can be expressed in terms of orthogonal polynomials in one variable in some elementary way. Many of the special classes considered in section 3.3 are included in these general classes. We shall discuss three general classes. For each of these classes certain coefficients in the expansions (3.2), (3.3) and (3.4) will vanish.

#### 3.7.1. Rotation invariant weight functions

Let $w(x)$ be a weight function on the interval $(0,1)$. For each integer $k \geq 0$ consider polynomials $p_n^k(x)$ of degree $n$ which are orthogonal with respect to the weight function $x^k\,w(x)$ on the interval $(0,1)$. Define for integers $m,n \geq 0$ and for $z = x+iy$, $\bar{z} = x - iy$, $x,y \in \mathbb{R}$,

(3.32) $$p_{m,n}(z,\bar{z}) = \begin{cases} p_n^{m-n}(z\bar{z})\,z^{m-n} & \text{if } m \geq n , \\ p_m^{n-m}(z\bar{z})\,\bar{z}^{\,n-m} & \text{if } m < n . \end{cases}$$

Then

(3.33) $$\iint_{x^2+y^2<1} p_{m,n}(x+iy, x-iy)\ \overline{p_{k,\ell}(x+iy, x-iy)}\ \cdot$$
$$\cdot\ w(x^2+y^2)\,dx\,dy = 0 \quad \text{if } (m,n) \neq (k,\ell) ,$$

and $p_{m,n}(z,\bar{z}) - c \cdot z^m \bar{z}^n$ is a polynomial of degree less than $m+n$. Hence, the polynomials $p_{n-k,k}(z,\bar{z})$, $k = 0,1,\ldots,n$, form an orthogonal basis of the class $\mathscr{V}_n$ with respect to the rotation invariant weight function $w(x^2+y^2)$ on the unit disk. This method of constructing orthogonal polynomials in two variables is due to Maldonado [57]. Clearly, the polynomials of class I (cf. section 3.3) have the form (3.32). Since

(3.34) $$p_{m,n}(z,\bar z) = \overline{p_{n,m}(\bar z, z)} ,$$

(3.35) $$p_{m,n}(e^{i\phi}z, \overline{e^{i\phi}z}) = e^{i(m-n)\phi}\, p_{m,n}(z,\bar z) ,$$

the orthogonal basis $\{p_{n,0}, p_{n-1,1}, \ldots, p_{0,n}\}$ of $\mathscr{V}_n$ is essentially invariant with respect to orthogonal transformations of the $(x,y)$-plane.

The power series expansion of $p_{m,n}(z,\bar z)$ only contains terms $c \cdot z^{m-j}\,\bar z^{n-j}$, $j = 0,1,\ldots,\min(m,n)$. The recurrence relations express $z\,p_{m,n}(z,\bar z)$ as a linear combination of $p_{m+1,n}$ and $p_{m,n-1}$, and $\bar z\,p_{m,n}(z,\bar z)$ as a linear combination of $p_{m,n+1}$ and $p_{m-1,n}$.

### 3.7.2. Weight functions of the form $w_1(x)\, w_2((\rho(x))^{-1}y)$

Let $w_1(x)$ be a weight function on the interval $(a,b)$. Let $\rho(x)$ be a positive function on $(a,b)$ which is either a polynomial of degree $r$ $(r = 0,1,2,\ldots)$ or the square root of a polynomial of degree $2r$ $(r = \frac12, 1, \frac32, \ldots)$. For each integer $k \geq 0$ let the polynomials $p_n^k(x)$, $n = 0,1,2,\ldots$, be orthogonal with respect to the weight function $(\rho(x))^{2k+1}\, w_1(x)$ on $(a,b)$. Let $w_2(y)$ be a weight function on the interval $(c,d)$. If $\rho(x)$ is not a polynomial then suppose that $c = -d$ and that $w_2(y)$ is an even function on $(-d,d)$. Let the polynomials $q_n(y)$ be orthogonal with respect to the weight function $w_2(y)$ on $(c,d)$. Define

(3.36) $$p_{n,k}(x,y) = p_{n-k}^k(x)\,(\rho(x))^k\, q_k\!\left(\frac{y}{\rho(x)}\right), \quad n \geq k \geq 0 .$$

Then $p_{n,k}(x,y)$ is a linear combination of monomials $x^{m-\ell}\,y^{\ell}$ such that $\ell \leq k$ and $m + (r-1)\ell \leq n + (r-1)k$, and for $(n,k) \neq (m,\ell)$ we have

(3.37) $$\int_{x=a}^{b} \int_{y=c\,\rho(x)}^{d\,\rho(x)} p_{n,k}(x,y)\, p_{m,\ell}(x,y) \,.$$

$$\cdot\, w_1(x)\, w_2((\rho(x))^{-1}y)\, dx\,dy = 0 .$$





If $\rho(x) \equiv 1$ then (3.36) reduces to (3.5). If $r = 0, \frac12$ or $1$ then the polynomial $p_{n,k}(x,y)$ has degree $n$ and it can be obtained by orthogonalizing the sequence $1, x, y, x^2, xy, \ldots,$ but for $r > 1$ this is no longer true.

In the special cases that $\rho(x) = (1-x^2)^{\frac12}$, $(a,b) = (-1,1)$, $(c,d) = (-1,1)$, $r = 1$, or $\rho(x) = x^{\frac12}$, $(a,b) = (0,1)$, $(c,d) = (-1,1)$, $r = \frac12$, or $\rho(x) = x$, $(a,b) = (0,1)$, $(c,d) = (0,1)$, $r = 1$, this method of constructing orthogonal polynomials in two variables has been described by Agahanov [1]. Further specialization of the weight functions leads to the classes II, III and IV, respectively (cf. section 3.3).

Let us make some further remarks about the case $r = 1$. Introduce a partial ordering $<$ such that $(m,\ell) < (n,k)$ if $m \leq n$ and $\ell \leq k$. For this partial ordering and for $p_{n,k}(x,y)$ defined by (3.36) with $r = 1$, the power series expansion (3.6) is valid, cf. Figure 6. By using (for instance) Lemma 3.1 it can be proved that $x\,p_{n,k}(x,y)$ is a linear combination of the three polynomials $p_{n-1,k}$, $p_{n,k}$, $p_{n+1,k}$ (cf. Figure 7) and that $y\,p_{n,k}(x,y)$ is a linear combination of the nine polynomials $p_{m,\ell}$ such that $(n-1,k-1) < (m,\ell) < (n+1,k+1)$ (cf. Figure 8).

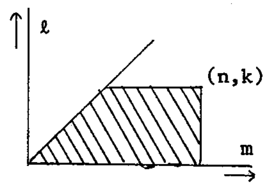


Figure 6

Figure 7

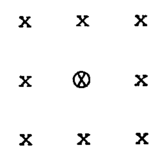

Figure 8

### 3.7.3. Symmetric and antisymmetric products

Let the polynomials $p_n(x)$ be orthogonal with respect to the weight function $w(x)$ on some interval. Then both the system of symmetric polynomials

$$(3.38)\qquad p_n(x)\,p_k(y) + p_k(x)\,p_n(y), \qquad n \ge k \ge 0\ ,$$

and the system of antisymmetric polynomials

$$(3.39)\qquad p_{n+1}(x)\,p_k(y) - p_k(x)\,p_{n+1}(y), \qquad n \ge k \ge 0\ ,$$

are orthogonal systems with respect to the weight function $w(x)\,w(y)$, $y < x$. The system (3.39) is a special case of the systems considered by Karlin and McGregor [38], [40].

Let $u = x + y$, $v = xy$ and define

$$(3.40)\qquad P^{-\frac12}_{n,k}(u,v) = p_n(x)\,p_k(y) + p_k(x)\,p_n(y), \quad n \ge k \ge 0\ ,$$

$$(3.41)\qquad P^{\frac12}_{n,k}(u,v) = \frac{p_{n+1}(x)\,p_k(y) - p_k(x)\,p_{n+1}(y)}{x-y}\ , \quad n \ge k \ge 0\ ,$$

$$(3.42)\qquad W(u,v) = w(x)\,w(y), \quad x > y\ .$$

Note that $P^{\frac12}_{n,n}(x+y,\,xy)$ denotes the Christoffel-Darboux kernel (up to a constant factor) for the orthogonal polynomials $p_k(x)$, $k = 0, 1, \ldots, n$.

Since any symmetric polynomial in $x$ and $y$ is a polynomial in the elementary symmetric polynomials $u = x + y$, $v = xy$, it follows that the systems $\{P^{-\frac12}_{n,k}(u,v)\}$ and $\{P^{\frac12}_{n,k}(u,v)\}$ are orthogonal systems with respect to the weight function $(u^2 - 4v)^{-\frac12}\,W(u,v)$ and $(u^2 - 4v)^{\frac12}\,W(u,v)$, respectively.

Note that $P^{-\frac12}_{n,k}(x+y,\,xy)$ is a linear combination of terms $(x^m y^\ell + x^\ell y^m)$, $m \ge \ell \ge 0$, $m \le n$, $\ell \le k$, and that $P^{\frac12}_{n,k}(x+y,\,xy)$ is a linear combination of terms $(x-y)^{-1}(x^{m+1} y^\ell - x^\ell y^{m+1})$, $m \ge \ell \ge 0$, $m \le n$, $\ell \le k$. Let us define the polynomials $Z^{\pm\frac12}_{n,k}(u,v)$, $n \ge k \ge 0$, by

$$(3.43)\qquad Z^{-\frac12}_{n,k}(x+y,\,xy) = \frac{x^n y^k + x^k y^n}{1+\delta_{n,k}}\ ,$$





$$(3.44)\qquad Z^{\frac12}_{n,k}(x+y,\,xy) = \frac{x^{n+1} y^k - x^k y^{n+1}}{x-y}\ .$$

Since $\frac12(t^n + t^{-n}) = T_n(\frac12(t+t^{-1}))$ and $\dfrac{t^{n+1} - t^{-n-1}}{t - t^{-1}} = U_n(\frac12(t+t^{-1}))$, cf. (2.6) and (2.7), it follows that

$$(3.45)\qquad Z^{-\frac12}_{n,k}(u,v) = \frac{2}{1+\delta_{n,k}}\; v^{\frac12(n+k)}\; T_{n-k}(\tfrac12 v^{-\frac12} u)$$

and

$$(3.46)\qquad Z^{\frac12}_{n,k}(u,v) = v^{\frac12(n+k)}\; U_{n-k}(\tfrac12 v^{-\frac12} u)\ .$$

Hence, $Z^{-\frac12}_{n,k}(u,v)$ and $Z^{\frac12}_{n,k}(u,v)$ are both linear combinations of monomials $u^{n-k-2i} v^{k+i}$, $i = 0, 1, \ldots, [\frac12(n-k)]$, cf. Figure 9, and $P^{-\frac12}_{n,k}(u,v)$ and $P^{\frac12}_{n,k}(u,v)$ have expansions of the form

$$(3.47)\qquad P^{\pm\frac12}_{n,k}(u,v) = \sum_{\ell=0}^{k}\ \sum_{m=\ell}^{n}\ c^{\pm\frac12}_{m,\ell;n,k}\ Z^{\pm\frac12}_{m,\ell}(u,v)\ ,$$

cf. Figure 10. It follows that both $P^{-\frac12}_{n,k}(u,v)$ and $P^{\frac12}_{n,k}(u,v)$ are linear combinations of monomials $u^{m-\ell} v^\ell$, $m \le n$, $m + \ell \le n+k$, cf. Figure 11.

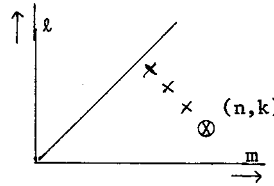


Figure 9

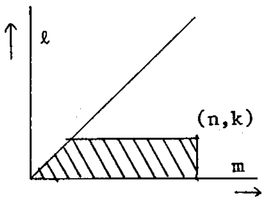


Figure 10

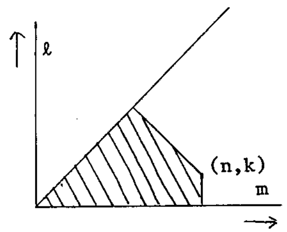


Figure 11

The above results imply that the polynomials $P^{-\frac12}_{n,k}(u,v)$ and $P^{\frac12}_{n,k}(u,v)$ can be obtained by orthogonalizing the sequence $1, u, v, u^2, uv, \ldots$ .

For special choices of the weight function $w(x)$ we obtain the polynomials of class VI of order $(\alpha, \beta, \pm\frac{1}{2})$, cf. section 3.3.

Regarding the recurrence relations it can easily be proved that $u\,P_{n,k}(u,v)$ is a linear combination of the five polynomials $P_{n+1,k}$, $P_{n,k+1}$, $P_{n,k}$, $P_{n,k-1}$, $P_{n-1,k}$, cf. Figure 12, and that $v\,P_{n,k}(u,v)$ is a linear combination of the nine polynomials $P_{m,\ell}$ such $n-1 \le m \le n+1$, $k-1 \le \ell \le k+1$, cf. Figure 13.

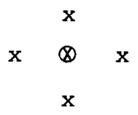

Figure 12

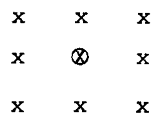

Figure 13

## 4. Differential recurrence relations and other analytic properties

In this section we shall discuss some analytic properties of the classes I, IV, VI and VII. An important tool for the analysis is certain differential recurrence relations analogous to (2.1) and (2.2), which have been obtained for most of the classes I-VII. Instead of one pair of ordinary differential recurrence relations we now have two pairs of partial differential recurrence relations such that all values of $(n, k)$ (or of $(m, n)$ in the case of Class I) are connected with each other by the recurrence relations. The partial differential operators may have order more than one and they may depend on the parameters.

By using the differential recurrence relations we obtain properties analogous to the properties characterizing classical orthogonal polynomials in one variable (cf. section 2.1):

(a) There exist partial differential operators which map the orthogonal system onto another orthogonal system of polynomials in two variables.

(b) Iteration of differential recurrence relations gives a Rodrigues type formula.

(c) Composition of two differential operators belonging to the same pair of differential recurrence relations leads to partial differential equations as considered in section 3.6.





On applying the differential recurrence relations which raise the degree, it is possible to prove certain properties of the orthogonal polynomials by complete induction with respect to the degree, cf. Sprinkhuizen [66].

In subsections 4.1, 4.2 and 4.3 we shall give the differential recurrence relations for the classes I, IV and VI, respectively. The results for class III are quite analogous to those of class IV. For class II probably no satisfactory results exist. In the case of class VII the research has not yet been done. Finally, in subsections 4.4 and 4.5 we shall discuss the relation of the polynomials of class VI and VII with the polynomials discussed by James [35].

### 4.1. Polynomials on the disk

For the polynomials $P^{\alpha}_{m,n}(z,\bar{z})$ of class I there is a pair of first order differential recurrence relations

$$(4.1) \qquad \partial_z\, P^{\alpha}_{m,n}(z,\bar{z}) = c \cdot P^{\alpha+1}_{m-1,n}(z,\bar{z}) ,$$

$$(4.2) \qquad (1-z\bar{z})^{-\alpha}\, \partial_{\bar{z}}\, [(1-z\bar{z})^{\alpha+1}\, P^{\alpha+1}_{m-1,n}(z,\bar{z})] = c \cdot P^{\alpha}_{m,n}(z,\bar{z}) ,$$

and a similar pair connecting $P^{\alpha}_{m,n}$ and $P^{\alpha+1}_{m,n-1}$, which can be obtained from (4.1) and (4.2) by complex conjugation. Iteration of (4.2) and its complex conjugate gives the Rodrigues type formula

$$(4.3) \qquad P^{\alpha}_{m,n}(z,\bar{z}) = c \cdot (1-z\bar{z})^{-\alpha}\, (\partial_z)^n\, (\partial_{\bar{z}})^m\, [(1-z\bar{z})^{\alpha+m+n}] .$$

It is of interest to compare (4.3) with the Rodrigues type formula

$$U^{2\alpha+1}_{m,n}(x,y) = c \cdot (1-x^2-y^2)^{-\alpha}\, (\partial_x)^m\, (\partial_y)^n\, [(1-x^2-y^2)^{\alpha+m+n}]$$

(cf. Erdélyi [17, 12.6(11)]) for polynomials $U^{2\alpha+1}_{m,n}(x,y)$ belonging to a biorthogonal system of polynomials on the unit disk with respect to the weight function $(1-x^2-y^2)^\alpha$.

4.2. Polynomials on the triangle

Consider the polynomials $P^{\alpha,\beta,\gamma}_{n,k}(x,y)$ of class IV and let

$$w_{\alpha,\beta,\gamma}(x,y) = (1-x)^\alpha (x-y)^\beta y^\gamma, \qquad 0 < y < x < 1 ,$$

denote the weight function. There is a pair of first order differential recurrence relations given by

$$(4.4) \qquad \partial_y P^{\alpha,\beta,\gamma}_{n,k} = c \cdot P^{\alpha,\beta+1,\gamma+1}_{n-1,k-1} ,$$

$$(4.5) \qquad (w_{\alpha,\beta,\gamma})^{-1} \partial_y (w_{\alpha,\beta+1,\gamma+1} P^{\alpha,\beta+1,\gamma+1}_{n-1,k-1}) = c \cdot P^{\alpha,\beta,\gamma}_{n,k} .$$

Define the second order partial differential operator $\underline{E}^{\beta,\gamma}$ by

$$(4.6) \qquad \underline{E}^{\beta,\gamma} = x\,\partial_{xx} + 2y\,\partial_{xy} + y\,\partial_{yy} + (\beta+\gamma+2)\,\partial_x + (\gamma+1)\,\partial_y .$$

Let $D^*$ denote the formal adjoint of a partial differential operator $D$. A calculation shows that $(\underline{E}^{\beta,\gamma})^* = \underline{E}^{-\beta,-\gamma}$. Now we have the pair of second order differential recurrence relations

$$(4.7) \qquad \underline{E}^{\beta,\gamma} P^{\alpha,\beta,\gamma}_{n,k} = c \cdot P^{\alpha+2,\beta,\gamma}_{n-1,k} ,$$

$$(4.8) \qquad (w_{\alpha,\beta,\gamma})^{-1} \underline{E}^{-\beta,-\gamma}(w_{\alpha+2,\beta,\gamma} P^{\alpha+2,\beta,\gamma}_{n-1,k}) = c \cdot P^{\alpha,\beta,\gamma}_{n,k} .$$





These formulas can be proved by using the second order differential recurrence relations for Jacobi polynomials which are derived in Koornwinder [48, (2.4) and (2.5)].

Iteration of (4.8) gives the Rodrigues type formula

$$(4.9) \qquad P^{\alpha,\beta,\gamma}_{n,k} = c \cdot (w_{\alpha,\beta,\gamma})^{-1} (\underline{E}^{-\beta,-\gamma})^{n-k} (\partial_y)^k \cdot [w_{\alpha+2n-2k,\beta+k,\gamma+k}] .$$

It is of interest to compare (4.9) with the Rodrigues type formula for Appell's polynomials on the triangle, cf. Erdélyi [17, 12.4(4)].

4.3. Polynomials on a region bounded by two straight lines and a parabola

Consider polynomials $P^{\alpha,\beta,\gamma}_{n,k}(u,v)$ of class VI. Let

$$w_{\alpha,\beta,\gamma}(u,v) = (1-u+v)^\alpha (1+u+v)^\beta (u^2-4v)^\gamma$$

denote the weight function. The differential recurrence relations involve the two second order partial differential operators

$$(4.10) \qquad \underline{D}^\gamma = \partial_{uu} + u\,\partial_{uv} + v\,\partial_{vv} + (\gamma + \tfrac{3}{2})\,\partial_v ,$$

$$(4.11) \qquad \underline{E}^{\alpha,\beta} = u\,\partial_{uu} + 2(v+1)\,\partial_{uv} + u\,\partial_{vv} + (\alpha+\beta+2)\,\partial_u + (\beta-\alpha)\,\partial_v .$$

A calculation shows that $(\underline{D}^\gamma)^* = \underline{D}^{-\gamma}$ and $(\underline{E}^{\alpha,\beta})^* = \underline{E}^{-\alpha,-\beta}$.

The first pair of partial differential recurrence relations is given by

$$(4.12) \qquad \underline{D}^\gamma P^{\alpha,\beta,\gamma}_{n,k} = c \cdot P^{\alpha+1,\beta+1,\gamma}_{n-1,k-1} ,$$

$$(4.13) \qquad (w_{\alpha,\beta,\gamma})^{-1} \underline{D}^{-\gamma}(w_{\alpha+1,\beta+1,\gamma} P^{\alpha+1,\beta+1,\gamma}_{n-1,k-1}) = c \cdot P^{\alpha,\beta,\gamma}_{n,k} ,$$

cf. Koornwinder [45, §5]. If $\gamma = \pm\frac{1}{2}$ then these formulas immediately follow from (2.1), (2.2), (3.12) and (3.13).

The second pair of differential recurrence relations is given by

$$E_-^{\alpha,\beta}\, P_{n,k}^{\alpha,\beta,\gamma} = c \cdot P_{n-1,k}^{\alpha,\beta,\gamma+1}, \tag{4.14}$$

$$(w_{\alpha,\beta,\gamma})^{-1} E_-^{-\alpha,-\beta} (w_{\alpha,\beta,\gamma+1}\, P_{n-1,k}^{\alpha,\beta,\gamma+1}) = c \cdot P_{n,k}^{\alpha,\beta,\gamma}, \tag{4.15}$$

cf. Sprinkhuizen [66, §4]. If $\gamma = -\frac{1}{2}$ then these formulas are corollaries of (2.3), (3.12) and (3.13).

In the remainder of this subsection we summarize some interesting consequences of (4.12), (4.13), (4.14) and (4.15), which were obtained by Sprinkhuizen [66].

Iteration of (4.13) and (4.15) gives the Rodrigues type formula

$$P_{n,k}^{\alpha,\beta,\gamma} = c \cdot (w_{\alpha,\beta,\gamma})^{-1} (D_-^{-\gamma})^k (E_-^{-\alpha-k,-\beta-k})^{n-k} \cdot [w_{\alpha+k,\beta+k,\gamma+n-k}] . \tag{4.16}$$

Let $(m,\ell) < (n,k)$ if $m \le n$ and $m+\ell \le n+k$. This defines a partial ordering.

Theorem 4.1. $P_{n,k}^{\zeta,\ell,\gamma}(u,v)$ is a linear combination of monomials $u^{m-\ell} v^{\ell}$ such that $(m,\ell) < (n,k)$ (cf. Figure 11).

For $\gamma = \pm\frac{1}{2}$ this result was already obtained in section 3.7.3. In the case of general $\gamma$ the theorem is evident if $n = k$. Complete induction with respect to $n-k$ and use of (4.15) gives the general theorem.

Denote the second order operator (3.29) and the fourth order operator (3.30), expressed in terms of $u$ and $v$, by $D_1^{\alpha,\beta,\gamma}$ and $D_2^{\alpha,\beta,\gamma}$, respectively (see Koornwinder [45] for the explicit expressions). Then the polynomials $P_{n,k}^{\alpha,\beta,\gamma}(u,v)$ are eigenfunctions of both differential operators with eigenvalues





$$\lambda_{n,k}^{\alpha,\beta,\gamma} = -n(n+\alpha+\beta+2\gamma+2) - k(k+\alpha+\beta+1) \tag{4.17}$$

and

$$\mu_{n,k}^{\alpha,\beta,\gamma} = k(k+\alpha+\beta+1)(n+\gamma+\tfrac{1}{2})(n+\alpha+\beta+\gamma+\tfrac{3}{2}), \tag{4.18}$$

respectively. If $(n,k) \neq (m,\ell)$ then $(\lambda_{n,k}^{\alpha,\beta,\gamma}, \mu_{n,k}^{\alpha,\beta,\gamma}) \neq (\lambda_{m,\ell}^{\alpha,\beta,\gamma}, \mu_{m,\ell}^{\alpha,\beta,\gamma})$. However, if we only consider $\lambda^{\alpha,\beta,\gamma}$ then degeneracies may occur. For instance, $\lambda_{2,2}^{3/2,-1/2,0} = \lambda_{3,0}^{3/2,-1/2,0}$. But if $(m,\ell) \neq (n,k)$ and $(m,\ell) < (n,k)$ then $\lambda_{m,\ell}^{\alpha,\beta,\gamma} > \lambda_{n,k}^{\alpha,\beta,\gamma}$, i.e., $\lambda_{n,k}^{\alpha,\beta,\gamma}$ is a monotonic function of $(n,k)$ with respect to the partial ordering. This result together with Theorem 4.1 implies:

Theorem 4.2. If a function $f(u,v)$ is an eigenfunction of $D_1^{\alpha,\beta,\gamma}$ and if

$$f(u,v) = \sum_{(m,\ell) < (n,k)} c_{m,\ell}\, u^{m-\ell} v^{\ell}$$

for certain coefficients $c_{m,\ell}$ with $c_{n,k} \neq 0$, then $f(u,v) = c \cdot P_{n,k}^{\alpha}(u,v)$.

By applying Theorem 4.1 there follow quadratic transformation formulas

$$\frac{P_{n+k,n-k}^{\alpha,\alpha,\gamma}(u,v)}{P_{n+k,n-k}^{\alpha,\alpha,\gamma}(2,1)} = \frac{P_{n,k}^{\gamma,-\frac{1}{2},\alpha}(2v, u^2-2v-1)}{P_{n,k}^{\gamma,-\frac{1}{2},\alpha}(2,1)}, \tag{4.19}$$

$$\frac{P_{n+k+1,n-k}^{\alpha,\alpha,\gamma}(u,v)}{P_{n+k+1,n-k}^{\alpha,\alpha,\gamma}(2,1)} = \frac{u\, P_{n,k}^{\gamma,\frac{1}{2},\alpha}(2v, u^2-2v-1)}{2\, P_{n,k}^{\gamma,\frac{1}{2},\alpha}(2,1)} . \tag{4.20}$$

Application of Theorem 4.1 and Lemma 3.1 gives the recurrence relations

$$u\, P_{n,k}^{\alpha,\beta,\gamma}(u,v) = \sum_{(n-1,k) < (m,\ell) < (n+1,k)} a(m,\ell;n,k)\, P_{m,\ell}^{\alpha,\beta,\gamma}(u,v) \tag{4.21}$$

and

$$(4.22)\qquad v\,P^{\alpha,\beta,\gamma}_{n,k}(u,v) = \sum_{(n-1,\,k-1)<(m,\ell)<(n+1,\,k+1)} b(m,\ell;n,k)\, P^{\alpha,\beta,\gamma}_{m,\ell}(u,v)\,.$$

These recurrence relations involve nine terms (cf. Figure 14) and fifteen terms (cf. Figure 15), respectively. However, a calculation shows that four of the nine coefficients in (4.21) vanish and that six of the fifteen coefficients in (4.22) vanish, as is indicated in Figures 14 and 15. Hence, the recurrence relations have the same structure as in the cases $\gamma = \pm\frac{1}{2}$, cf. Figures 12 and 13 in section 3.7.3.

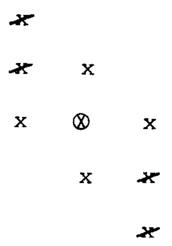

Figure 14

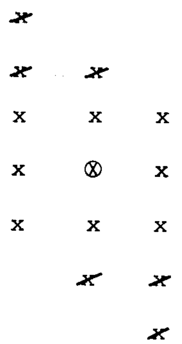

Figure 15

4.4. Expansions in terms of James type zonal polynomials

It seems difficult to derive some explicit power series expansion analogous to (2.5) for the polynomials of class VI. However, some results can be obtained for expansions in terms of the so-called James type zonal polynomials, which are explicitly known in the case of two variables. In particular, the polynomials ${}_6P^{\alpha,\beta,0}_{n,n}(u,v)$ can be identified with certain hypergeometric functions of $2\times 2$ matrix argument. The results announced in this subsection will be published elsewhere in more details.





It is convenient to introduce new variables $\xi = 1-\frac{1}{2}u$, $\eta = \frac{1}{4}(1-u+v)$, and to define

$$(4.23)\qquad R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta) = \frac{{}_6P^{\alpha,\beta,\gamma}_{n,k}(2-2\xi,\,1-2\xi+4\eta)}{{}_6P^{\alpha,\beta,\gamma}_{n,k}(2,1)}\,.$$

Then the polynomials $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$ are polynomials with highest term $c\cdot\xi^{n-k}\eta^{k}$ obtained by orthogonalization of the sequence $1,\xi,\eta,\xi^2,\xi\eta,\eta^2,\xi^3,\xi^2\eta,\ldots$ with respect to the weight function $\eta^{\alpha}(1-\xi+\eta)^{\beta}(\xi^2-4\eta)^{\gamma}$ on the region $\{(\xi,\eta)\mid \eta>0,\ 1-\xi+\eta>0,\ \xi^2-4\eta>0\}$, cf. Figure 16. They are normalized such that $R^{\alpha,\beta,\gamma}_{n,k}(0,0)=1$.

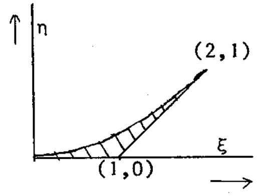


Figure 16

It follows from Theorem 4.1 that $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$ is a linear combination of monomials $\xi^{m-\ell}\eta^{\ell}$ such that $(m,\ell)<(n,k)$. The recurrence relations for $\xi\, R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$ and $\eta\, R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$ have the structure of Figures 12 and 13, respectively.

For $\gamma=\pm\frac{1}{2}$ we have

$$(4.24)\qquad R^{\alpha,\beta,-\frac{1}{2}}_{n,k}(x+y,\,xy) =$$

$$= \tfrac{1}{2}[R^{(\alpha,\beta)}_{n}(1-2x)\,R^{(\alpha,\beta)}_{k}(1-2y) + R^{(\alpha,\beta)}_{k}(1-2x)\,R^{(\alpha,\beta)}_{n}(1-2y)]$$

$$= \sum_{\ell=0}^{k}\sum_{m=\ell}^{n} \tfrac{1}{2}\left[\frac{(-n)_m(-k)_\ell(n+\alpha+\beta+1)_m(k+\alpha+\beta+1)_\ell}{(\alpha+1)_m(\alpha+1)_\ell\, m!\,\ell!} + \frac{(-k)_m(-n)_\ell(k+\alpha+\beta+1)_m(n+\alpha+\beta+1)_\ell}{(\alpha+1)_m(\alpha+1)_\ell\, m!\,\ell!}\right]\cdot\frac{x^m y^\ell + x^\ell y^m}{1+\delta_{m,\ell}}$$

and

$$(4.25)\qquad R_{n,k}^{\alpha,\beta,\frac{1}{2}}(x+y,\ xy) =$$

$$= -\frac{(\alpha+1)(R_{n+1}^{(\alpha,\beta)}(1-2x)\,R_k^{(\alpha,\beta)}(1-2y) - R_k^{(\alpha,\beta)}(1-2x)\,R_{n+1}^{(\alpha,\beta)}(1-2y))}{(n-k+1)\,(n+k+\alpha+\beta+2)\,(x-y)}$$

$$= \sum_{\ell=0}^{k}\ \sum_{m=\ell}^{n}\left[\frac{(-n-1)_{m+1}(-k)_\ell\,(n+\alpha+\beta+2)_{m+1}(k+\alpha+\beta+1)_\ell}{(\alpha+1)_{m+1}\,(\alpha+1)_\ell\,(m+1)!\ \ell\,!}\right.$$

$$\left.-\frac{(-k)_{m+1}(-n-1)_\ell\ (k+\alpha+\beta+1)_{m+1}\,(n+\alpha+\beta+2)_\ell}{(\alpha+1)_{m+1}\,(\alpha+1)_\ell\,(m+1)!\ \ell\,!}\right].$$

$$\cdot\ \frac{\alpha+1}{(-n+k-1)(n+k+\alpha+\beta+2)}\ \cdot\ \frac{x^{m+1}y^\ell - x^\ell y^{m+1}}{x-y}\ .$$

Hence, in view of (3.43) and (3.44), we have explicit expansions of $R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)$ in terms of $Z_{m,\ell}^{\gamma}(\xi,\eta)$, $\gamma = \pm\frac{1}{2}$ .

As a generalization of (3.45) and (3.46) let us define

$$(4.26)\qquad Z_{n,k}^{\gamma}(\xi,\eta) = \frac{2^{2n-2k}(n-k)!}{(2\gamma+n-k+1)_{n-k}}\ \eta^{\frac{1}{2}(n+k)}\ P_{n-k}^{(\gamma,\gamma)}(\tfrac{1}{2}\eta^{-\frac{1}{2}}\xi)$$

$$= \sum_{i=0}^{[\frac{1}{2}(n-k)]}\frac{(-n+k)_{2i}}{(-n+k-\gamma+\frac{1}{2})_i\ i!}\ \xi^{n-k-2i}\ \eta^{k+i},\quad n \geq k \geq 0\ .$$

Note that the coefficient of $\xi^{n-k}\eta^{k}$ equals 1 .

Then the coefficients in the expansion

$$(4.27)\qquad R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta) = \sum_{(m,\ell)<(n,k)} c_{m,\ell;n,k}^{\alpha,\beta,\gamma}\ Z_{m,\ell}^{\gamma}(\xi,\eta)$$

are uniquely determined. It is our purpose to find explicit expressions for these coefficients. The expansion (4.27) is motivated by formulas (4.24) and (4.25) and also by the following facts.





Comparing formula (4.16) with a similar Rodrigues type formula in Herz [29, (6.4')] we can express $R_{n,n}^{\alpha,\beta,0}(x+y,\ xy)$ as a hypergeometric function of matrix argument:

$$(4.28)\qquad R_{n,n}^{\alpha,\beta,0}(x+y,\ xy) =$$

$$= {}_2F_1\left(-n,\ n+\alpha+\beta+\frac{3}{2}\ ;\ \alpha+\frac{3}{2};\ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right)\ .$$

Herz [29] defined hypergeometric functions of matrix argument by means of generalized Laplace transforms and their inverses. However, Constantine [11, (25)] obtained the series expansion

$$(4.29)\qquad {}_2F_1(a,b;c;X) = \sum_{m=0}^{\infty}\ \sum_{\ell=0}^{m}\frac{(a)_{m,\ell}\,(b)_{m,\ell}}{(c)_{m,\ell}\,(m+\ell)!}\ C_{m,\ell}(X)\ ,$$

where $(a)_{m,\ell} = (a)_m(a-\frac{1}{2})_\ell$ and the functions $C_{m,\ell}(X)$ are the so-called zonal polynomials depending on the eigenvalues of the $2\times 2$ symmetric matrix $X$ . In fact, the theory was given for hypergeometric functions depending on the eigenvalues of $m\times m$ symmetric matrices. The zonal polynomials were introduced by James [33]. The polynomial $C_{m,\ell}(X)$ is the spherical function on $GL(2,\mathbb{R})/O(2)$ belonging to the irreducible representation $\{2m,\ 2\ell\}$ of $GL(2,\mathbb{R})$ . James [34, (7.9)] pointed out that

$$C_{n,k}\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = C_{n,k}(I_2)(xy)^{\frac{1}{2}(n+k)}\ P_{n-k}\left(\frac{x+y}{2(xy)^{\frac{1}{2}}}\right)\ .$$

It follows that

$$(4.30)\qquad C_{n,k}\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = \frac{(n+k)!(\frac{3}{2})_{n-k}}{(n-k)!\,k!\,(\frac{3}{2})_n}\ Z_{n,k}^{0}(x+y,\ xy)\ .$$

Combination of (4.28), (4.29) and (4.30) gives

(4.31) $$R^{\alpha,\beta,0}_{n,n}(\xi,\eta) = \sum_{\ell=0}^{k}\sum_{m=\ell}^{n} \frac{(-n)_m(-n-\frac{1}{2})_\ell(n+\alpha+\beta+\frac{3}{2})_m(n+\alpha+\beta+1)_\ell(\frac{3}{2})_{m-\ell}}{(\alpha+\frac{3}{2})_m(\alpha+1)_\ell(\frac{3}{2})_m\,\ell!\,(m-\ell)!} \cdot Z^0_{m,\ell}(\xi,\eta),$$

which gives the coefficients in (4.27) if $\gamma = 0$ and $n = k$.

Let us now consider the general case of (4.27). The operator $D^\gamma_-$ (cf (4.10)), expressed in terms of the coordinates $\xi$ and $\eta$, acts on $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$ and $Z^\gamma_{n,k}(\xi,\eta)$ in a similar way. We have

(4.32) $$D^\gamma_- R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta) = \frac{k(k+\alpha+\beta+1)(n+\gamma+\frac{1}{2})(n+\alpha+\beta+\gamma+\frac{3}{2})}{4(\alpha+1)(\alpha+\gamma+\frac{3}{2})} R^{\alpha+1,\beta+1,\gamma}_{n-1,k-1}(\xi,\eta),$$

(4.33) $$D^\gamma_- Z^\gamma_{n,k}(\xi,\eta) = \frac{1}{4}k(n+\gamma+\frac{1}{2})Z^\gamma_{n-1,k-1}(\xi,\eta).$$

If $k = 0$ then (4.32) and (4.33) have to be interpreted such that the right hand sides become zero. By complete induction with respect to $k$ formulas (4.32) and (4.33) imply that the coefficients in (4.27) can only be nonzero if $\ell \le k$ and $m \le n$, cf. Figure 10. Using this result and Lemma 3.1 we can now immediately prove that the recurrence relation for $\eta R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$ has the structure given by Figure 13.

It also follows from (4.32), (4.33) and (4.27) that

(4.34) $$c^{\alpha,\beta,\gamma}_{m,\ell;n,k} = \frac{k(k+\alpha+\beta+1)(n+\gamma+\frac{1}{2})(n+\alpha+\beta+\gamma+\frac{3}{2})}{(\alpha+1)(\alpha+\gamma+\frac{3}{2})\,\ell\,(m+\gamma+\frac{1}{2})} \cdot c^{\alpha+1,\beta+1,\gamma}_{m-1,\ell-1;n-1,k-1}.$$





Hence, it is sufficient to calculate the coefficients $c^{\alpha,\beta,\gamma}_{m,\ell;n,k}$ if $\ell = 0$. These are also the coefficients in the power series of the boundary value $R^{\alpha,\beta,\gamma}_{n,k}(\xi,0)$:

(4.35) $$R^{\alpha,\beta,\gamma}_{n,k}(\xi,0) = \sum_{m=0}^{n} c^{\alpha,\beta,\gamma}_{m,0;n,k}\,\xi^m.$$

Using the action of the differential operator $E^{\alpha,\beta}_-$ (cf. (4.11)) on $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$ and $Z^\gamma_{n,k}(\xi,\eta)$ we can obtain a four-term recurrence relation expressing $c^{\alpha,\beta,\gamma}_{m+1,0;n,k}$ as a linear combination of $c^{\alpha,\beta,\gamma}_{m,0;n,k}$, $c^{\alpha+1,\beta+1,\gamma}_{m-1,0;n-1,k-1}$ and $c^{\alpha,\beta,\gamma+1}_{m-1,0;n-1,k}$. If either $n = k$ or $k = 0$ then this reduces to a three-term recurrence relation and then we are able to calculate the coefficients. The results are:

(4.36) $$c^{\alpha,\beta,\gamma}_{m,\ell;n,n} = \frac{(-n)_m(-n-\gamma-\frac{1}{2})_\ell(n+\alpha+\beta+\gamma+\frac{3}{2})_m(n+\alpha+\beta+1)_\ell(\gamma+\frac{3}{2})_{m-\ell}}{(\alpha+\gamma+\frac{3}{2})_m(\alpha+1)_\ell(\gamma+\frac{3}{2})_m\,\ell!\,(m-\ell)!},$$

(4.37) $$c^{\alpha,\beta,\gamma}_{m,0;n,0} = \frac{(-n)_m(n+\alpha+\beta+2\gamma+2)_m(\gamma+\frac{1}{2})_m}{(\alpha+\gamma+\frac{3}{2})_m(2\gamma+1)_m\,m!}.$$

Formula (4.36) was first proved by Sprinkhuizen (unpublished). For $\gamma = 0$ formula (4.36) is the Herz-Constantine-James result (4.31). For $\gamma = \pm\frac{1}{2}$ formulas (4.36) and (4.37) imply (4.24) and (4.25) ($n = k$ or $k = 0$). Finally formulas (4.36) and (4.37) give nice explicit boundary values:

(4.38) $$R^{\alpha,\beta,\gamma}_{n,n}(\xi,0) = R^{(\alpha+\gamma+\frac{1}{2},\beta)}_n(1-2\xi),$$

(4.39) $$R^{\alpha,\beta,\gamma}_{n,0}(\xi,0) = {}_3F_2\left(\begin{matrix}-n,\ n+\alpha+\beta+2\gamma+2,\ \gamma+\frac{1}{2}\\ \alpha+\gamma+\frac{3}{2},\ 2\gamma+1\end{matrix}\,\middle|\,\xi\right).$$

## 4.5. Polynomials on the deltoid

Consider the polynomials $P^\alpha_{m,n}(z,\bar{z})$ of class VII. They have power series expansion

(4.40) $$P^{\alpha}_{m,n}(z,\bar z) = c_{m,n}\, z^m \bar z^n + \sum_{k+\ell<m+n} c_{k,\ell}\, z^k \bar z^\ell .$$

Because of the symmetries of the orthogonality region it follows that $c_{k,\ell} = 0$ if $k - \ell \neq m - n \pmod 3$.

Let us denote the second order operator (3.29) and the third order operator (3.31), expressed in terms of $z$ and $\bar z$, by $D_1^\alpha$ and $D_2^\alpha$, respectively (cf. Koornwinder [46] for the explicit expressions). Both differential operators have the polynomials $P^\alpha_{m,n}(z,\bar z)$ as eigenfunctions. The eigenvalues are

(4.41) $$\lambda^\alpha_{m,n} = -\frac{4}{3}(m^2 + mn + n^2 + 3(\alpha+\frac{1}{2})(m+n))$$

and

(4.42) $$\mu^\alpha_{m,n} = (m-n)(2m+n+3\alpha+\frac{3}{2})(m+2n+3\alpha+\frac{3}{2}) ,$$

respectively. If $(k,\ell) \neq (m,n)$ then $(\lambda^\alpha_{k,\ell}, \mu^\alpha_{k,\ell}) \neq (\lambda^\alpha_{m,n}, \mu^\alpha_{m,n})$. The two differential equations for $P^\alpha_{m,n}(z,\bar z)$ give recurrence relations for the coefficients in (4.40). From these recurrence relations it can be derived that $c_{k,\ell} \neq 0$ only if $k + 2\ell \leq m+2n$ and $2k+\ell \leq 2m+n$, cf. Figure 17.

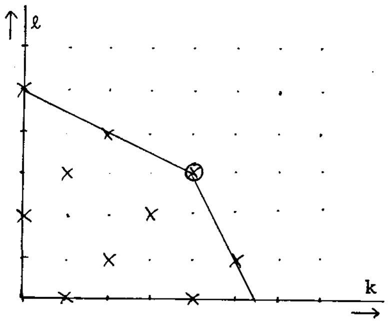


Figure 17





Observe that $\lambda^\alpha_{k,\ell} > \lambda^\alpha_{m,n}$ if $(k,\ell) \neq (m,n)$, $k + 2\ell \leq m + 2n$, $2k+\ell \leq 2m+n$. This implies:

Theorem 4.3. Let $P(z,\bar z)$ be a linear combination of monomials $z^k \bar z^\ell$ such that $2k+\ell \leq 2m+n$, $k+2\ell \leq m+2n$, $k-\ell = m-n \pmod 3$ and the coefficient of $z^m \bar z^n$ is nonzero. If $P(z,\bar z)$ is also an eigenfunction of $D_1^\alpha$ then $P(z,\bar z) = c \cdot P^\alpha_{m,n}(z,\bar z)$.

Lidl [54, §3.c.β] defines generalized Gegenbauer polynomials $C^a_k(x,y)$ by the generating function

(4.43) $$(1 - xz + yz^2 - z^3)^{-a} = \sum_{k=0}^\infty C^a_k(x,y)\, z^k .$$

By using Theorem 4.3 it can be proved that $C^a_k(x,y)$ is a constant multiple of ${}_7P^{a-\frac{1}{2}}_{k,0}(x,y)$.

Let us define

(4.44) $$Z^\alpha_{n_1,n_2,n_3}(\xi,\eta,\zeta) = c \cdot \zeta^{(n_1+n_2+n_3)/3}\, {}_7P^\alpha_{n_1-n_2,\,n_2-n_3}(\zeta^{-\frac{1}{3}}\xi,\ \zeta^{-\frac{2}{3}}\eta) ,$$

where $n_1 \geq n_2 \geq n_3 \geq 0$ and the coefficient of $\xi^{n_1-n_2}\eta^{n_2-n_3}\zeta^{n_3}$ equals 1. These functions are polynomials in $\xi, \eta, \zeta$ and they can be considered as three-variable analogues of the polynomials defined by (4.26). If $\alpha = -\frac{1}{2}, \frac{1}{2}$ or $0$ and if

$$\xi = x_1+x_2+x_3,\quad \eta = x_1x_2 + x_2x_3 + x_3x_1,\quad \zeta = x_1x_2x_3$$

are the elementary symmetric polynomials in $x_1, x_2, x_3$, then the polynomials defined by (4.44) have special interpretations.

It follows from (3.14), (3.15), (3.16) and (3.17) that

(4.45) $$Z^{-\frac{1}{2}}_{n_1,n_2,n_3}(\xi,\eta,\zeta) = c \cdot \sum x_{i_1}^{n_1} x_{i_2}^{n_2} x_{i_3}^{n_3},$$

where the sum is taken over all permutations $(i_1, i_2, i_3)$ of $(1, 2, 3)$, and that

$$(4.46)\qquad Z^{\frac{1}{2}}_{n_1, n_2, n_3}(\xi, \eta, \zeta) = $$

$$= \left| \begin{matrix} x_1^{n_1+2} & x_2^{n_1+2} & x_3^{n_1+2} \\ x_1^{n_2+1} & x_2^{n_2+1} & x_3^{n_2+1} \\ x_1^{n_3} & x_2^{n_3} & x_3^{n_3} \end{matrix} \right| \Bigg/ \left| \begin{matrix} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{matrix} \right| .$$

The polynomials satisfying (4.45) and (4.46) in the special case that $n_2 = n_3 = 0$, have been studied by Eier and Lidl [14] and Lidl [54], respectively.

The polynomial $Z^{\alpha}_{n_1, n_2, n_3}(\xi, \eta, \zeta)$ can be characterized up to a constant factor as a symmetric polynomial in $x_1, x_2, x_3$, which is a linear combination of monomials $x_1^{k_1} x_2^{k_2} x_3^{k_3}$ such that $k_1 + k_2 + k_3 = n_1 + n_2 + n_3$, $k_1 \le n_1$, $k_3 \ge n_3$ and the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3}$ is nonzero, and which is an eigenfunction of a certain second order differential operator. By using this characterization and by using the results in James [34], the case $\alpha = 0$ of these polynomials can be identified with the zonal polynomials of $3 \times 3$ matrix argument considered by James.

It seems probable that, analogous to the results of section 4.5, the three-variable analogues of the polynomials of class VI will have nice expansions in terms of the polynomials $Z_{n_1, n_2, n_3}(\xi, \eta, \zeta)$.

## 5. Orthogonal polynomials in two variables as spherical functions

In this final section we briefly discuss the known cases in which polynomials of the classes I-VII can be interpreted as spherical functions. The importance of such a group theoretic interpretation should be clear from section 2.3.





The polynomials, ${}_1P^{q-2}_{m,n}(z, \bar{z})$, $q = 2, 3, 4, \ldots$, are the spherical functions on the sphere $S^{2q-1}$ considered as the homogeneous space $U(q)/U(q-1)$, cf. the references given in section 3.4. A related result is the inequality

$$(5.1)\qquad |{}_1P^{\alpha}_{m,n}(z, \bar{z})| \le P^{\alpha}_{m,n}(1, 1), \quad |z| \le 1 ,$$

which holds for $\alpha \ge 0$. The author proved (unpublished) analogous inequalities

$$(5.2)\qquad |{}_3P^{\alpha,\beta}_{n,k}(x, y)| \le {}_3P^{\alpha,\beta}_{n,k}(1, 1), \quad y^2 \le x \le 1, \quad \alpha \ge |\beta + \tfrac{1}{2}| ,$$

$\alpha \geq \beta + \frac{1}{2} \geq 0$

and

$$(5.3)\qquad |{}_4P^{\alpha,\beta,\gamma}_{n,k}(x, y)| \le {}_4P^{\alpha,\beta,\gamma}_{n,k}(1, 1), \quad 0 \le y \le x \le 1 ,$$

$$\alpha \ge |\beta + \gamma + 1|, \quad \beta \ge \max(\gamma, -\tfrac{1}{2}) .$$

A spherical function interpretation and a positive convolution structure are unknown for the classes III and IV, but formulas (5.2) and (5.3) suggest that further research in this direction may be worthwhile.

In the case of the classes VI and VII and for certain values of the parameters the second order differential operator (3.29) (with $D = \partial_{ss} + \partial_{tt}$ and $\rho(s, t)$ as given in Table 1) is the radial part of the Laplace-Beltrami operator on certain compact symmetric Riemannian spaces of rank 2, cf. Harish-Chandra [25, §7], or Helgason [28, (3.3)] together with the volume element ratio given in Helgason [26, Chap. 10, §1.5]. The spherical functions on these symmetric spaces are eigenfunctions of the Laplace-Beltrami operator. Since the Laplace-Beltrami operator may have degenerate eigenvalues, this does not prove that the polynomials of class VI and VII can be interpreted as spherical functions. Still it strongly motivated the author to introduce the classes VI and VII.

In the case of class VI and of order $(\alpha, \beta, \gamma)$ the restricted root vectors of the corresponding symmetric space have a Dynkin diagram $0 \Longrightarrow 0$ and a vector diagram as given by Figure 18, where the restricted root vectors $\lambda_1$, $2\lambda_1$ and $\lambda_2$ have multiplicities $2\alpha - 2\beta$, $2\beta + 1$ and $2\gamma + 1$, respectively.

In the case of class VII and of order $\alpha$ the restricted root vectors have Dynkin diagram $0 - 0$ and multiplicity $2\alpha + 1$. The vector diagram for this case is given by Figure 19.

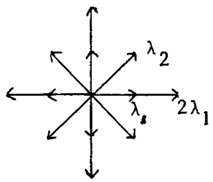

Figure 18

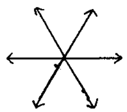
Figure 19

A list of all compact symmetric spaces of rank 2 is included in Helgason [26, p. 354, Table II]. The corresponding Dynkin diagrams and multiplicities of the restricted root vectors are given by Araki [4, pp. 32, 33] and by Loos [55, pp. 119, 146]. For our classes VI and VII we give the results in Tables 2 and 3, respectively.

In these tables the tori $T^2$ and H are considered as homogeneous spaces of the groups given in Table 1. The groups SO(5) and SU(3) are considered as homogeneous spaces of $SO(5) \times SO(5)$ and $SU(3) \times SU(3)$, respectively.

On a symmetric space M of rank r the class of all invariant differential operators is a commutative algebra with r generators, one of which is the Laplace-Beltrami operator (cf. Helgason [26, Chap. 10, §2]). The spherical functions on M can be characterized as the zonal functions which are eigenfunctions of all invariant differential operators on





| homogeneous space | $\alpha$ | $\beta$ | $\gamma$ |
|---|---|---|---|
| square torus $T^2$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |
| SO(5) | $\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ |
| $O(q)/SO(2) \times O(q-2)$ | 0 | 0 | $\frac{1}{2}q - \frac{5}{2}$ |
| $O(q)/O(2) \times O(q-2)$ | $\frac{1}{2}q - \frac{5}{2}$ | $-\frac{1}{2}$ | 0 |
| $U(q)/U(2) \times U(q-2)$ | $q - 4$ | 0 | $\frac{1}{2}$ |
| $Sp(q)/Sp(2) \times Sp(q-2)$ | $2q - 7$ | 1 | $\frac{3}{2}$ |
| SO(10)/U(5) | 2 | 0 | $\frac{3}{2}$ |
| E III | 4 | 0 | $\frac{5}{2}$ |

Table 2

| homogeneous space | $\alpha$ |
|---|---|
| hexagonal torus H | $-\frac{1}{2}$ |
| SU(3) | $\frac{1}{2}$ |
| SU(3)/SO(3) | 0 |
| SU(6)/Sp(3) | $\frac{3}{2}$ |
| E IV | $\frac{7}{2}$ |

Table 3

M. Except for the Laplace-Beltrami operator, the radial parts of the invariant differential operators are not known in general. However, it follows from our results that for each of the spaces M listed in Table 2 (or Table 3) the following two statements are equivalent:

(a) The polynomials ${}_6P^{\alpha,\beta,\gamma}_{m,n}(u,v)$ (or ${}_7P^{\alpha}_{m,n}(z,\bar{z})$) as functions of s and t are the spherical functions on M considered as functions of the radial coordinates s and t .

(b) The differential operator (3.30) (or (3.31)) is the radial part of an invariant differential operator on M .

In several special cases it has been proved that statements (a) and (b) are true. For the two tori $T^2$ and H it is evident. In the group cases SO(5) and SU(3) the spherical functions are the characters on the group. Then Weyl's character formula (cf. Weyl [71, (37)], [72, (29)]) together with formulas (3.26) and (3.17) proves statement (a), and statement (b) follows from Berezin [6]. For complex Grassmann manifolds $U(q)/U(2)\times U(q-2)$ the spherical functions and the invariant differential operators have been given by Berezin and Karpelevic [7].

In the case of real Grassmann manifolds $O(q)/O(2)\times O(q-2)$ Maass [56] first derived the invariant differential operators and then obtained the spherical functions as orthogonal polynomials. Another approach was followed by James and Constantine [36]. They obtained the spherical functions by using zonal polynomials of $2\times 2$ matrix argument. More generally, it follows from their results that our polynomials of class VI and of order $(\frac{1}{2}(q-p-3), \frac{1}{2}(p-3), 0)$ can be interpreted as functions on O(q), right invariant with respect to $O(2)\times O(q-2)$, left invariant with respect to $O(p)\times O(q-p)$, and belonging to some irreducible representation of O(q) . This also gives a group theoretic explanation for the results of section 4.4 if $\gamma = 0$ and $\alpha$ and $\beta$ are integers or half integers. Recently, the author found yet another approach to the analysis on Grassmann manifolds of rank two. In this approach the harmonics on the Grassmann manifold are obtained as restrictions of certain doubly homogeneous polynomials which satisfy certain orthogonality properties. These results will be soon available in preprint form.



Deeper analytic properties of polynomials of class VI and VII may be first derived in the cases, where a group theoretic interpretation is known, and next be generalized to other values of the parameters. A study of the most simple case, where all parameters are equal to $-\frac{1}{2}$, already gives an indication of the difficulties which can be expected in the general case.

## Note Added in Proof

Recently I. Sprinkhuizen together with the author proved that

$$R^{\alpha,\beta,\gamma}_{n,k}(\xi,0) = \sum_{m=k}^{n} \frac{(\gamma+\frac{1}{2})_{m-k}(\gamma+\frac{1}{2})_{n-m}(n-k)!}{(m-k)!\,(n-m)!\,(2\gamma+1)_{n-k}} \cdot$$

$$\cdot \frac{(n+k+\alpha+\beta+2\gamma+2)_{m-k}(m+k+\alpha+\beta+2)_{n-m}}{(m+k+\alpha+\beta+\gamma+\frac{3}{2})_{m-k}(2m+\alpha+\beta+\gamma+\frac{5}{2})_{n-m}} R_m^{(\alpha+\gamma+\frac{1}{2},\beta)}(1-2\xi) ,$$

where $R^{\alpha,\beta,\gamma}_{n,k}(\xi,0)$ is defined by (4.23).

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