SIAM J. MATH. ANAL.
Vol. 5, No. 1, February 1974

JACOBI POLYNOMIALS, II. AN ANALYTIC PROOF OF THE PRODUCT FORMULA*

TOM KOORNWINDER†

Abstract. An analytic proof is given for the author's product formula for Jacobi polynomials and a new integral representation is obtained for the product $J_\alpha(x)J_\beta(y)$ of two Bessel functions. Similarly, a product formula for Jacobi polynomials due to Dijksma and the author is derived in an analytic way. The proofs are based on Bateman's work on special solutions of the biaxially symmetric potential equation. The paper concludes with new proofs for Gasper's evaluation of the convolution kernel for Jacobi series and for Watson's evaluation of the integral

$$\int_0^\infty J_\alpha(\lambda x)J_\beta(\lambda y)J_\beta(\lambda z)\lambda^{1-\alpha}\,d\lambda.$$

1. Introduction. In recent papers [13], [14], [15] the author derived the addition formula for Jacobi polynomials by group theoretic methods. It was pointed out in [13] that the product formula and the Laplace type integral representation for Jacobi polynomials immediately follow from the addition formula. The way of obtaining these results illustrated the power of the group theoretic approach to special functions. However, it was felt unsatisfying that no analytic proofs were available for the addition formula and its corollaries.

Next, an elementary analytic proof of the Laplace type integral representation was given by Askey [1]. Our main result in the present paper is an analytic derivation of the product formula. It is based on important but rather unknown results of Bateman [3], [4] concerning special solutions of the biaxially symmetric potential equation. The present paper is a continuation of Askey's paper [1]. We would like to thank Askey for communicating us the results contained in [1] and Gasper for calling our attention to [3].

Immediately after this work was done both Gasper and the author extended the results to an analytic proof of the addition formula. They used different methods and will publish their proofs separately in subsequent papers.

Section 2 of this paper contains a review of Bateman's work on the biaxially symmetric potential equation [3], [4]. Admitting transformations of the variables, Bateman obtained solutions of this equation by separating the variables in three different ways. We prove that, in a certain sense, these three possibilities are the only ones. Bateman's special solutions involve Bessel functions, Jacobi polynomials and nth powers. They can be expressed in terms of each other by means of a number of identities, one of which is the bilinear sum obtained in [1].

By using these identities the product formula for Jacobi polynomials and a new product formula for Bessel functions can be derived from the Laplace type integral representation for Jacobi polynomials. This is done in § 3. Section 4 discusses the analogous results connected with an integral representation for Jacobi polynomials due to Braaksma and Meulenbeld [5] and a new proof is given of a product formula due to Dijksma and the author [7].

* Received by the editors September 6, 1972.
† Mathematisch Centrum, Amsterdam, the Netherlands.

Gasper [10], [11] settled the positivity of the convolution structure for Jacobi series. His explicit expression for the convolution kernel is derived from our product formula in § 5. Some formulas from Watson [17], which Gasper applied in his proof in [10], here arise in a natural way. Thus, a deeper understanding of Gasper's proof is achieved.

2. The biaxially symmetric potential equation. The partial differential equation

$$\left(\frac{\partial^2}{\partial u^2} + \frac{2\beta + 1}{u}\frac{\partial}{\partial u} + \frac{\partial^2}{\partial v^2} + \frac{2\alpha + 1}{v}\frac{\partial}{\partial v}\right)F(u, v) = 0 \tag{2.1}$$

arises naturally from the potential equation in two different ways.

First, if α and β are nonnegative integers and if $(x_1, x_2, x_3, x_4) = (u\cos\phi, u\sin\phi, v\cos\chi, v\sin\chi)$, then the equation

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2}\right)(u^\beta v^\alpha e^{i(\beta\phi + \alpha\chi)}F(u, v)) = 0 \tag{2.2}$$

is equivalent to (2.1) (cf. Bateman [4, p. 389]).

Second, if $2\alpha + 1$ and $2\beta + 1$ are nonnegative integers and if

$$u = \sqrt{x_1^2 + x_2^2 + \cdots + x_{2\beta+2}^2}$$

and

$$v = \sqrt{y_1^2 + y_2^2 + \cdots + y_{2\alpha+2}^2},$$

then the equation

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_{2\beta+2}^2} + \frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_{2\alpha+2}^2}\right)F(u, v) = 0 \tag{2.3}$$

is equivalent to (2.1).

Therefore, (2.1) is called the biaxially symmetric potential equation. Special solutions of this equation were studied by Bateman in [3] and in [4, pp. 389–394]. We will summarize some of Bateman's results in this section.

The differential operator in (2.1) has two singular lines $u = 0$ and $v = 0$. It is natural to consider solutions of (2.1) in the upper right quarter-plane. Equation (2.1) admits solutions by separation of variables. Regular solutions of this type are

$$F(u, v) = u^{-\beta}J_\beta(\lambda u)v^{-\alpha}I_\alpha(\lambda v), \tag{2.4}$$

where the functions J_β and I_α are Bessel functions.

Let $\mathscr{D}_1$ be a simply connected domain in the (s, t)-plane and let $\mathscr{D}_2 = \{(u, v) | u > 0, v > 0\}$. Suppose that the mapping $(s, t) \to (u, v)$ is a conformal mapping of $\mathscr{D}_1$ onto $\mathscr{D}_2$. It means that $u(s, t)$ and $v(s, t)$ satisfy the Cauchy–Riemann equations

$$u_s = v_t \quad \text{and} \quad u_t = -v_s \tag{2.5}$$

and that

$$\Delta(s, t) \equiv u_s v_t - u_t v_s \neq 0 \quad \text{on } \mathscr{D}_1. \tag{2.6}$$

After this transformation, equation (2.1) becomes

$$\frac{1}{\Delta(s,t)}\left[\frac{\partial^2}{\partial s^2} + \left((2\beta+1)\frac{u_s}{u} + (2\alpha+1)\frac{v_s}{v}\right)\frac{\partial}{\partial s} + \frac{\partial^2}{\partial t^2} + \left((2\beta+1)\frac{u_t}{u} + (2\alpha+1)\frac{v_t}{v}\right)\frac{\partial}{\partial t}\right]F(u(s,t),v(s,t)) = 0. \tag{2.7}$$

It is not difficult to prove that for a fixed conformal mapping $(s,t) \to (u,v)$ as introduced above the following three statements are equivalent.

(A) For all values of α and β, equation (2.7) admits separation of variables.

(B) Both the functions $u(s,t)$ and $v(s,t)$ are the products of a function of s and a function of t.

(C) The mapping $(s,t) \to (u,v)$ is given by one of the three complex analytic functions

$$u + iv = s + it, \quad u + iv = e^{s+it} \quad \text{or} \quad u + iv = \cos(s + it),$$

up to translations, dilatations and rotations over an angle $k(\pi/2)$ of the (s,t)-plane and up to dilatations of the (u,v)-plane.

We did not succeed in proving or disproving the equivalence of (B) with the following statement (A′).

(A′) There is a value of α and β $(-\frac{1}{2} \neq \alpha \neq \beta \neq -\frac{1}{2})$ for which equation (2.7) admits separation of variables.

However, the equivalence of the statements (A) and (C) suggests that one should especially consider the three forms of equation (2.1) connected by the transformations

$$u + iv = e^{x+iy} = \cos(\xi + i\eta). \tag{2.8}$$

The pictures in Fig. 1 show the domains which are thus mapped onto each other.

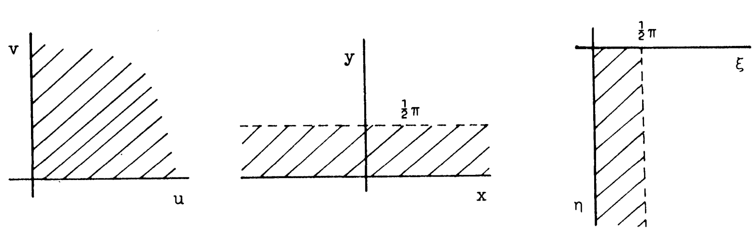

FIG. 1

The first identity in (2.8) is equivalent to

$$u = e^x \cos y, \quad v = e^x \sin y \tag{2.9}$$

and equation (2.7) becomes

$$\left[\frac{\partial^2}{\partial x^2} + 2(\alpha+\beta+1)\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + ((2\alpha+1)\operatorname{cotg} y - (2\beta+1)\operatorname{tg} y)\frac{\partial}{\partial y}\right] \cdot F(e^x \cos y, e^x \sin y) = 0 \tag{2.10}$$

with the special regular solutions

$$F(e^x \cos y, e^x \sin y) = e^{2nx} P_n^{(\alpha,\beta)}(\cos 2y) \tag{2.11}$$

(cf. Bateman [4, p. 389]). Here the function $P_n^{(\alpha,\beta)}$ denotes a Jacobi polynomial.

The mapping $(\xi, \eta) \to (u, v)$ in (2.8) can be written as

$$u = \cos \xi \operatorname{ch} \eta, \quad v = -\sin \xi \operatorname{sh} \eta \tag{2.12}$$

and after this transformation equation (2.1) takes the form

$$\begin{aligned} &\left[\frac{\partial^2}{\partial \xi^2} + ((2\alpha + 1)\operatorname{cotg} \xi - (2\beta + 1)\operatorname{tg} \xi)\frac{\partial}{\partial \xi}\right. \\ &\left. + \frac{\partial^2}{\partial \eta^2} + ((2\alpha + 1)\operatorname{cth} \eta + (2\beta + 1)\operatorname{th} \eta)\frac{\partial}{\partial \eta}\right] \\ &\cdot F(\cos \xi \operatorname{ch} \eta, -\sin \xi \operatorname{sh} \eta) = 0 \end{aligned} \tag{2.13}$$

with the special regular solutions

$$F(\cos \xi \operatorname{ch} \eta, -\sin \xi \operatorname{sh} \eta) = P_n^{(\alpha,\beta)}(\cos 2\xi) P_n^{(\alpha,\beta)}(\operatorname{ch} 2\eta) \tag{2.14}$$

(cf. Bateman [4, pp. 392–393]).

Bateman [3], [4] has derived some identities which relate the special solutions (2.4), (2.11) and (2.14) of equation (2.1) to each other. We need two of these identities.

Solutions of type (2.4) and (2.11) are related to each other by

$$u^{-\beta} J_\beta(u) v^{-\alpha} I_\alpha(v) = \sum_{n=0}^{\infty} a_n (u^2 + v^2)^n \frac{P_n^{(\alpha,\beta)}((u^2 - v^2)/(u^2 + v^2))}{P_n^{(\alpha,\beta)}(1)}, \tag{2.15}$$

where the coefficients a_n are defined by

$$\frac{1}{2^\alpha \Gamma(\alpha + 1)} u^{-\beta} J_\beta(u) = \sum_{n=0}^{\infty} a_n u^{2n} \tag{2.16}$$

(formula (2.15) with $v = 0$). For a detailed proof, see Bateman [3, pp. 113, 114]. Formula (2.15) is a generating function for Jacobi polynomials, which is also mentioned in Erdélyi [8, vol. III, § 19.9(12)].

The substitution

$$s = \cos 2\xi, \quad t = \operatorname{ch} 2\eta \tag{2.17}$$

combined with the substitutions (2.9) and (2.12) gives

$$e^{2x} = s + t, \quad \cos 2y = \frac{1 + st}{s + t}. \tag{2.18}$$

In terms of the variables s and t, the solutions of type (2.11) and (2.14) can be related to each other by the identity

$$\frac{P_n^{(\alpha,\beta)}(s)}{P_n^{(\alpha,\beta)}(1)} \frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(\alpha,\beta)}(1)} = \sum_{k=0}^{n} b_{k,n} (s + t)^k \frac{P_k^{(\alpha,\beta)}((1 + st)/(s + t))}{P_k^{(\alpha,\beta)}(1)}, \tag{2.19}$$

where $b_{k,n}$ is defined by (2.19) when $t = 1$, i.e.,

$$\frac{P_n^{(\alpha,\beta)}(s)}{P_n^{(\alpha,\beta)}(1)} = \sum_{k=0}^{n} b_{k,n}(s+1)^k. \tag{2.20}$$

Formula (2.19) is proved in Bateman [4, pp. 392, 393] by using the fact that both sides of (2.19) are solutions of the same partial differential equation (2.13) (after the transformation (2.17)). The converse identity (formula (4.1) in Askey [1]) was first obtained in [3, pp. 122, 123]. For another result of Bateman, which expresses the solution (2.4) in terms of the solutions (2.14), the reader is referred to [3, p. 115] or [17, p. 370].

The preceding results might be extended by considering other special solutions of (2.1). For instance, one may take n complex in the solutions (2.11) and (2.14). In this way Flensted-Jensen and the author [9] generalized (2.19) for complex values of n. Another possibility is to replace one or both of the factors in (2.4), (2.11), (2.14) by a second solution of the (ordinary) differential equation.

It should be pointed out that Appell's hypergeometric function

$$F_4(\gamma, \delta; 1+\alpha, 1+\beta; -v^2, u^2),$$

defined in [8, vol. I, § 5.7.1], is also a solution of (2.1). This can be verified by termwise differentiating the power series of the function F_4. The methods of this section may be applied in order to prove the generating function for Jacobi polynomials mentioned in [8, vol. III, § 19.10 (26)] and the Poisson kernel for Jacobi polynomials (see Bailey [2, p. 102, example 19]).

It would also be of interest to express the solutions (2.11) and (2.14) in terms of the solutions (2.4) by means of definite integrals over λ.

Finally, we mention the work of Henrici [12], who used equation (2.1) in order to prove the addition formula for Gegenbauer functions.

3. The product formulas for Jacobi polynomials and for Bessel functions. The Laplace type integral representation for Jacobi polynomials is

$$R_n^{(\alpha,\beta)}(x) = \int_{r=0}^{1}\int_{\phi=0}^{\pi}\left(\frac{1+x}{2} - \frac{1-x}{2}r^2 + i\sqrt{1-x^2}\,r\cos\phi\right)^n dm_{\alpha,\beta}(r,\phi), \qquad \alpha > \beta > -\tfrac{1}{2}, \tag{3.1}$$

where

$$dm_{\alpha,\beta}(r,\phi) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}(1-r^2)^{\alpha-\beta-1}r^{2\beta+1}(\sin\phi)^{2\beta}\,dr\,d\phi. \tag{3.2}$$

Following Gasper [10] we use the notation

$$R_n^{(\alpha,\beta)}(x) \equiv \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}. \tag{3.3}$$

The measure (3.2) is normalized by

$$\int_0^1\int_0^\pi dm_{\alpha,\beta}(r,\phi) = 1. \tag{3.4}$$

Formula (3.1) was first proved by the author [13] from the addition formula. Next, an elementary analytic proof of (3.1) was obtained by Askey [1, § 3]. The derivations given below were suggested by the way Askey proved the converse of (2.19) (see [1, § 4]).

It follows from (3.1) that

$$(x+y)^n R_n^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right) = \int_0^1 \int_0^\pi [\tfrac{1}{2}(1+x)(1+y) + \tfrac{1}{2}(1-x)(1-y)r^2 + \sqrt{1-x^2}\sqrt{1-y^2}\,r\cos\phi]^n \, dm_{\alpha,\beta}(r,\phi) \tag{3.5}$$

and

$$(x^2+y^2)^n R_n^{(\alpha,\beta)}\left(\frac{x^2-y^2}{x^2+y^2}\right) = \int_0^1 \int_0^\pi (x^2 - y^2r^2 + 2ixyr\cos\phi)^n \, dm_{\alpha,\beta}(r,\phi). \tag{3.6}$$

Combination of formulas (2.19), (2.20) and (3.5) gives the product formula

$$R_n^{(\alpha,\beta)}(x) R_n^{(\alpha,\beta)}(y) = \int_0^1 \int_0^\pi R_n^{(\alpha,\beta)}[\tfrac{1}{2}(1+x)(1+y) + \tfrac{1}{2}(1-x)(1-y)r^2 + \sqrt{1-x^2}\sqrt{1-y^2}\,r\cos\phi - 1] \, dm_{\alpha,\beta}(r,\phi), \qquad \alpha > \beta > -\tfrac{1}{2}. \tag{3.7}$$

In his original proof the author [13] derived (3.7) from the addition formula by integration.

In a similar way, it follows from the formulas (2.15), (2.16) and (3.6) that

$$\begin{aligned} x^{-\beta} J_\beta(x) y^{-\alpha} I_\alpha(y) &= \sum_{n=0}^\infty a_n \int_0^1 \int_0^\pi (x^2 - y^2r^2 + 2ixyr\cos\phi)^n \, dm_{\alpha,\beta}(r,\phi) \\ &= \int_0^1 \int_0^\pi \sum_{n=0}^\infty a_n (x^2 - y^2r^2 + 2ixyr\cos\phi)^n \, dm_{\alpha,\beta}(r,\phi) \\ &= \frac{1}{2^\alpha \Gamma(\alpha+1)} \int_0^1 \int_0^\pi \frac{J_\beta((x^2 - y^2r^2 + 2ixyr\cos\phi)^{1/2})}{(x^2 - y^2r^2 + 2ixyr\cos\phi)^{(1/2)\beta}} \, dm_{\alpha,\beta}(r,\phi). \end{aligned}$$

The interchanging of summation and integration is allowed because the infinite sum converges uniformly in r and ϕ. By using that

$$y^{-\alpha} I_\alpha(y) = (iy)^{-\alpha} J_\alpha(iy)$$

and by analytic continuation it follows that

$$x^{-\beta} J_\beta(x) y^{-\alpha} J_\alpha(y) = \frac{1}{2^\alpha \Gamma(\alpha+1)} \cdot \int_0^1 \int_0^\pi \frac{J_\beta((x^2 + y^2r^2 + 2xyr\cos\phi)^{1/2})}{(x^2 + y^2r^2 + 2xyr\cos\phi)^{(1/2)\beta}} \, dm_{\alpha,\beta}(r,\phi), \qquad \alpha > \beta > -\tfrac{1}{2}. \tag{3.8}$$

This formula seems to be new.

It is surprising that the two product formulas (3.7) and (3.8), which seem to be much deeper results than the integral representation (3.1), can be derived from (3.1) so easily. Another surprising fact is that formula (3.1) implies (3.7) but is also a degenerate case of (3.7). In fact, one obtains (3.1) after dividing both sides of (3.7) by $R_n^{(\alpha,\beta)}(y)$ and then taking the limit for $y \to \infty$.

Formula (3.8) follows from (3.7) by applying the confluence relations

$$\lim_{n\to\infty} \frac{P_n^{(\alpha,\beta)}(1 - y^2/(2n^2))}{P_n^{(\alpha,\beta)}(1)} = 2^\alpha \Gamma(\alpha + 1) y^{-\alpha} J_\alpha(y) \tag{3.9}$$

and

$$\lim_{n\to\infty} \frac{P_n^{(\alpha,\beta)}(x^2/(2n^2) - 1)}{P_n^{(\alpha,\beta)}(-1)} = 2^\beta \Gamma(\beta + 1) x^{-\beta} J_\beta(x) \tag{3.10}$$

(cf. Erdélyi [8, vol. II, § 10, 8(41)]).

If $\beta \uparrow \alpha$ then the measure $dm_{\alpha,\beta}(r, \phi)$ defined in (3.2) degenerates to the measure

$$\frac{\Gamma(\alpha + 1)}{\sqrt{\pi}\,\Gamma(\alpha + \frac{1}{2})} \delta(1 - r)(\sin \phi)^{2\beta}\, dr\, d\phi.$$

Here $\delta(t)$ represents Dirac's delta function.

The degenerate forms of (3.1) and (3.7) for $\alpha = \beta$ are Gegenbauer's classical formulas for ultraspherical polynomials (cf. [8, vol. I, § 3, 15(22), (20)]). Formula (3.8) degenerates to the product formula

$$\begin{aligned} x^{-\beta} J_\beta(x) y^{-\beta} J_\beta(y) &= \frac{1}{2^\beta \sqrt{\pi}\,\Gamma(\beta + \frac{1}{2})} \\ &\cdot \int_0^\pi \frac{J_\beta((x^2 + y^2 + 2xy \cos \phi)^{1/2})}{(x^2 + y^2 + 2xy \cos \phi)^{(1/2)\beta}} (\sin \phi)^{2\beta}\, d\phi, \qquad \beta > -\tfrac{1}{2}. \end{aligned} \tag{3.11}$$

This is an integrated form of Gegenbauer's addition formula for Bessel functions (cf. Watson [17, § 11.4(2)]). It should be pointed out that new proofs are obtained for these two classical product formulas of Gegenbauer if one applies Bateman's identities (2.15) and (2.19) to (3.1) in the case $\alpha = \beta$.

Askey [1] derived the Laplace type integral representation (3.1) from its degenerate case $\alpha = \beta$ by using a fractional integral for Jacobi polynomials. In a similar way we can derive the product formula (3.8) from its special case (3.11) by applying Sonine's first integral

$$y^{-\alpha} J_\alpha(y) = \frac{1}{2^{\alpha-\beta-1}\Gamma(\alpha - \beta)} \int_0^1 (yr)^{-\beta} J_\beta(yr) r^{2\beta+1} (1 - r^2)^{\alpha-\beta-1}\, dr, \tag{3.12}$$

$\alpha > \beta > -1$ (see Watson [17, § 12.11(1)]). This method of reducing the case (α, β) to the case (β, β) fails for the product formula (3.7).

If $\beta \downarrow -\frac{1}{2}$ then the measure $dm_{\alpha,\beta}(r, \phi)$ degenerates to the measure

$$\frac{\Gamma(\alpha + 1)}{\sqrt{\pi}\,\Gamma(\alpha + \frac{1}{2})} (1 - r^2)^{\alpha - 1/2} (\delta(\phi) + \delta(\pi - \phi))\, dr\, d\phi.$$

The degenerate forms of (3.1) and (3.7) which are thus obtained are related to the degenerate forms for $\alpha = \beta$ by the quadratic transformation

$$\frac{P_n^{(\alpha,-1/2)}(2x^2-1)}{P_n^{(\alpha,-1/2)}(1)} = \frac{P_{2n}^{(\alpha,\alpha)}(x)}{P_{2n}^{(\alpha,\alpha)}(1)} \qquad \text{(see [8, vol. II, § 10.9(21)]).} \tag{3.13}$$

Formula (3.8) degenerates for $\beta = -\frac{1}{2}$ to

$$\cos x \cdot y^{-\alpha} J_\alpha(y) = \frac{1}{2^\alpha \sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})} \int_{-1}^{+1} \cos(x+yr)(1-r^2)^{\alpha-1/2}\,dr, \qquad \alpha > -\tfrac{1}{2}. \tag{3.14}$$

For $x = 0$, this is Poisson's integral ([17, § 3.3(1)])

$$y^{-\alpha} J_\alpha(y) = \frac{1}{2^\alpha \sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})} \int_{-1}^{+1} \cos(yr)(1-r^2)^{\alpha-1/2}\,dr \tag{3.15}$$

and, conversely, formula (3.14) immediately follows from (3.15). Thus, the double integral (3.8) connects (3.11) with Poisson's integral in a continuous way.

The remarks at the end of § 2 suggest that other integral formulas can be derived by the methods of § 3. One case, for Jacobi functions, is worked out in [9].

The left-hand sides of formulas (3.1), (3.7) and (3.8) can each be considered as the first term of an orthogonal expansion with respect to the measure $dm_{\alpha,\beta}(r, \phi)$. An orthogonal system of functions with respect to this measure is

$$f_{k,l}(r,\phi) = P_l^{(\alpha-\beta-1,\beta+k-l)}(2r^2-1)\,r^{k-l} P_{k-l}^{(\beta-1/2,\beta-1/2)}(\cos\phi), \qquad k \geqq l \geqq 0. \tag{3.16}$$

The expansion corresponding to formula (3.7) is called the addition formula for Jacobi polynomials (see Koornwinder [13]). The expansions corresponding to (3.1) and (3.8) can be obtained as degenerate cases of this addition formula. Recently, Gasper and the author independently gave analytic proofs of these expansions.

Gasper first derived the expansion corresponding to (3.1) in an elementary way and next applied (2.19) and (2.20) in order to obtain the addition formula. Similarly, one might prove the expansion corresponding to (3.8).

The author obtained the higher terms of the addition formula by doing integration by parts in (3.7). The same method might be applied to (3.1) and (3.8).

These two methods of proof will be published in the near future.

4. The integral representation of Braaksma and Meulenbeld. By interpreting Jacobi polynomials as spherical harmonics Braaksma and Meulenbeld [5] obtained an integral representation for Jacobi polynomials which is different from (3.1). Their formula is

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) = {} & \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\pi\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})} \frac{(\alpha+1)_n(\beta+1)_n}{(\frac{1}{2})_n n! 2^n} \\ & \cdot \int_0^\pi \int_0^\pi (i\sqrt{1-x}\cos\phi + \sqrt{1+x}\cos\psi)^{2n} \\ & \cdot (\sin\phi)^{2\alpha}(\sin\psi)^{2\beta}\,d\phi\,d\psi, \qquad \alpha > -\tfrac{1}{2}, \quad \beta > -\tfrac{1}{2}. \end{aligned} \tag{4.1}$$

As pointed out in [5], the analytic proof of (4.1) is easy.

By using (2.19) a product formula can be derived from (4.1). The explicit form of the coefficients $b_{k,n}$ in (2.21) follows from

$$\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(-1)} = \frac{P_n^{(\beta,\alpha)}(-x)}{P_n^{(\beta,\alpha)}(1)} = {}_2F_1\left(-n, n+\alpha+\beta+1; \beta+1; \frac{1+x}{2}\right) = \sum_{k=0}^{n} \frac{(-n)_k(n+\alpha+\beta+1)_k}{(\beta+1)_k k!}\left(\frac{1+x}{2}\right)^k. \tag{4.2}$$

Hence,

$$\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(-1)}\frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)} = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\pi\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})} \cdot \int_0^\pi\int_0^\pi \sum_{k=0}^{n} \frac{(-n)_k(n+\alpha+\beta+1)_k}{(\frac{1}{2})_k k!\ 2^{2k}} \cdot (\sqrt{1-x}\sqrt{1-y}\cos\phi + \sqrt{1+x}\sqrt{1+y}\cos\psi)^{2k} \cdot (\sin\phi)^{2\alpha}(\sin\psi)^{2\beta}\,d\phi\,d\psi.$$

Let $C_{2n}^{\alpha+\beta+1}(t)$ denote a Gegenbauer polynomial. By using

$$\sum_{k=0}^{n} \frac{(-n)_k(n+\alpha+\beta+1)_k}{(\frac{1}{2})_k k!} t^{2k} = {}_2F_1(-n, n+\alpha+\beta+1; \tfrac{1}{2}; t^2) = \frac{P_n^{(-1/2,\alpha+\beta+1/2)}(1-2t^2)}{P_n^{(-1/2,\alpha+\beta+1/2)}(1)} = \frac{C_{2n}^{\alpha+\beta+1}(t)}{C_{2n}^{\alpha+\beta+1}(0)}$$

we conclude that

$$\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(-1)}\frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)} = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\pi\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})C_{2n}^{\alpha+\beta+1}(0)} \cdot \int_0^\pi\int_0^\pi C_{2n}^{\alpha+\beta+1}(\tfrac{1}{2}\sqrt{1-x}\sqrt{1-y}\cos\phi + \tfrac{1}{2}\sqrt{1+x}\sqrt{1+y}\cos\psi) \cdot (\sin\phi)^{2\alpha}(\sin\psi)^{2\beta}\,d\phi\,d\psi, \quad \alpha > -\tfrac{1}{2},\ \beta > -\tfrac{1}{2}. \tag{4.3}$$

Formula (4.3) was first obtained by Dijksma and Koornwinder [7]. They used similar group theoretic methods to those of Braaksma and Meulenbeld [5].

We can also derive from (2.15) and (4.1) that

$$x^{-\alpha}J_\alpha(x)y^{-\beta}J_\beta(y) = \frac{1}{2^{\alpha+\beta}\pi\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})} \cdot \int_0^\pi\int_0^\pi \cos(x\cos\phi + y\cos\psi)(\sin\phi)^{2\alpha}(\sin\psi)^{2\beta}\,d\phi\,d\psi, \quad \alpha > -\tfrac{1}{2},\ \beta > -\tfrac{1}{2}. \tag{4.4}$$

Writing

$$\cos(x\cos\phi + y\cos\psi) = \cos(x\cos\phi)\cos(y\cos\psi) - \sin(x\cos\phi)\sin(y\cos\psi),$$

we can reduce (4.4) to the product of two Poisson integrals (3.15).

5. Gasper's product formula. The right-hand sides of the formulas (3.1), (3.7) and (3.8) all have the form

$$\int_0^1 \int_0^\pi f(a^2r^2 + 2abr\cos\phi + b^2)\, dm_{\alpha,\beta}(r,\phi),$$

where the function f is continuous on $(0, \infty)$, the letters a and b represent positive real numbers and the measure $dm_{\alpha,\beta}(r, \phi)$ is defined by (3.2). By a transformation of the integration variables this integral can be rewritten in the so-called kernel form. We will prove that

$$\int_0^1 \int_0^\pi f(a^2r^2 + 2abr\cos\phi + b^2)\, dm_{\alpha,\beta}(r,\phi) = \int_0^\infty f(t^2)K_{\alpha,\beta}(a,b,t)t^{2\beta+1}\, dt, \tag{5.1}$$

where for $\alpha > \beta > -\frac{1}{2}$ the kernel $K_{\alpha,\beta}$ is defined by

$$K_{\alpha,\beta}(a,b,c) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})} \cdot a^{-2\alpha}\int_0^\pi (a^2 - b^2 - c^2 + 2bc\cos\psi)_+^{\alpha-\beta-1}(\sin\psi)^{2\beta}\, d\psi. \tag{5.2}$$

In formula (5.2) the notation

$$(x)_+^\lambda = \begin{cases} x^\lambda & \text{if } x > 0, \\ 0 & \text{if } x \leqq 0, \end{cases}$$

is used.

Formula (5.1) can be proved by successively performing the following transformations of variables to the left-hand side of (5.1). First, we put

$$x = r\cos\phi, \quad y = r\sin\phi,$$

next,

$$x' = ax + b, \quad y' = ay,$$

and finally,

$$x' = t\cos\psi, \quad y' = t\sin\psi.$$

Thus we obtain the equalities

$$\int_0^1 \int_0^\pi f(|ar\,e^{i\phi} + b|^2)(1 - r^2)^{\alpha-\beta-1} r^{2\beta+1} (\sin\phi)^{2\beta}\, dr\, d\phi$$

$$= \int_{-\infty}^{\infty} \int_0^{\infty} f((ax+b)^2 + (ay)^2)(1 - x^2 - y^2)_+^{\alpha-\beta-1} y^{2\beta}\, dx\, dy$$

$$= a^{-2\alpha} \int_{-\infty}^{+\infty} \int_0^{\infty} f((x')^2 + (y')^2)(a^2 - b^2 - (x')^2 - (y')^2 + 2bx')_+^{\alpha-\beta-1} \cdot (y')^{2\beta}\, dx'\, dy'$$

$$= a^{-2\alpha} \int_0^{+\infty} \int_0^{\pi} f(t^2)(a^2 - b^2 - t^2 + 2bt\cos\psi)_+^{\alpha-\beta-1} t^{2\beta+1}\, dt\, d\psi.$$

Formula (5.1) follows by substitution of (3.2) and (5.2).

The kernel $K_{\alpha,\beta}$, defined by (5.2), is clearly nonnegative. Putting $f(x) \equiv 1$ in (5.1) we find

$$\int_0^\infty K_{\alpha,\beta}(a, b, t) t^{2\beta+1}\, dt = 1. \tag{5.3}$$

The analytic form of the kernel K was studied by Macdonald (see Watson [17, p. 412]) and by Gasper [11]. It turns out that three different cases have to be distinguished. Let

$$B \equiv \frac{b^2 + c^2 - a^2}{2bc}. \tag{5.4}$$

Then (5.2) takes the form

$$K_{\alpha,\beta}(a, b, c) = \frac{2^{\alpha-\beta}\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})} a^{-2\alpha}(bc)^{\alpha-\beta-1} \cdot \int_{-1}^{+1} (s - B)_+^{\alpha-\beta-1}(1 - s^2)^{\beta-1/2}\, ds. \tag{5.5}$$

Case I. $a < |b - c|$. Here $1 < B$, and $K_{\alpha,\beta}(a, b, c) = 0$.

Case II. $|b - c| < a < b + c$. Here $-1 < B < 1$, and

$$K_{\alpha,\beta}(a, b, c) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})} a^{-2\alpha}(bc)^{\alpha-\beta-1}(1 - B^2)^{\alpha-1/2} \cdot {}_2F_1\left(\alpha+\beta, \alpha-\beta; \alpha+\tfrac{1}{2}; \frac{1-B}{2}\right). \tag{5.6}$$

Case III. $b + c < a$. Here $B < -1$, and

$$K_{\alpha,\beta}(a, b, c) = \frac{2^{\alpha-\beta}\Gamma(\alpha+1)}{\Gamma(\alpha-\beta)\Gamma(\beta+1)} a^{-2\alpha}(bc)^{\alpha-\beta-1} \frac{(1 - B)^{\alpha-1/2}}{(-1 - B)^{\beta+1/2}} \cdot {}_2F_1\left(\alpha+\beta, \beta+\tfrac{1}{2}; 2\beta+1; \frac{2}{1+B}\right). \tag{5.7}$$

For these results, cf. [17, p. 412] and [11].

Next, we will rewrite the formulas (3.1), (3.7) and (3.8) in kernel form using formula (5.1). It follows from (3.1) that

$$(5.8) \qquad R_n^{(\alpha,\beta)}(x) = \frac{1}{2}\int_0^\infty y^{n+\beta} K_{\alpha,\beta}\left(\sqrt{(x-1)/2}, \sqrt{(x+1)/2}, \sqrt{y}\right) dy, \qquad x > 1.$$

A Mehler type integral for Jacobi functions (also for complex n) which follows from (5.8) leads to an explicit expression for the Radon transform for Jacobi function expansions (to be published by the author). The analogous Mehler type integral for Jacobi polynomials was independently obtained by Gasper (yet unpublished). He applied the formulas [8, vol. I, § 2.4(3) and § 2.8(11)]. The kernel form of (3.7) was first obtained by Gasper [10]. It is

$$(5.9) \qquad \begin{aligned} R_n^{(\alpha,\beta)}(\cos 2\theta_1) R_n^{(\alpha,\beta)}(\cos 2\theta_2) = \int_0^{\pi/2} & R_n^{(\alpha,\beta)}(\cos 2\theta_3) K_{\alpha,\beta}(\sin\theta_1 \sin\theta_2, \\ & \cos\theta_1 \cos\theta_2, \cos\theta_3)(\cos\theta_3)^{2\beta+1} \sin\theta_3 \, d\theta_3, \\ & 0 < \theta_2 < \frac{\pi}{2}, \quad 0 < \theta_2 < \frac{\pi}{2}, \quad \alpha > \beta > -\tfrac{1}{2}. \end{aligned}$$

Here, the range of integration is restricted, because $a = \sin\theta_1 \sin\theta_2$, $b = \cos\theta_1 \cos\theta_2$ and $c > 1$ would imply the condition of Case I.

Formula (3.8) can be rewritten as

$$(5.10) \qquad \frac{J_\alpha(x)}{x^\alpha} \frac{J_\beta(y)}{y^\beta} = \frac{1}{2^\alpha \Gamma(\alpha+1)} \int_0^\infty \frac{J_\beta(z)}{z^\beta} K_{\alpha,\beta}(x, y, z) z^{2\beta+1} \, dz.$$

It follows by the homogeneity of $K_{\alpha,\beta}$ that

$$(5.11) \qquad J_\alpha(\lambda x) J_\beta(\lambda y) \lambda^{-\alpha} = \int_0^\infty \frac{x^\alpha y^\beta z^\beta K_{\alpha,\beta}(x, y, z)}{2^\alpha \Gamma(\alpha+1)} J_\beta(\lambda z) z \, dz.$$

By duality it follows from (5.9) that

$$(5.12) \qquad \begin{aligned} & \sum_{n=0}^\infty h_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos 2\theta_1) R_n^{(\alpha,\beta)}(\cos 2\theta_2) R_n^{(\alpha,\beta)}(\cos 2\theta_3) \\ & \quad = \frac{K_{\alpha,\beta}(\sin\theta_1 \sin\theta_2, \cos\theta_1 \cos\theta_2, \cos\theta_3)}{2^{\alpha+\beta+2}(\sin\theta_3)^{2\alpha}}, \end{aligned}$$

where

$$(h_n^{(\alpha,\beta)})^{-1} = \int_{-1}^{+1} (R_n^{(\alpha,\beta)}(x))^2 (1-x)^\alpha (1+x)^\beta \, dx$$

and $\cos\theta_3 \neq |\cos(\theta_1 \pm \theta_2)|$. It follows from (5.11) that

$$(5.13) \qquad \int_0^\infty J_\alpha(\lambda x) J_\beta(\lambda y) J_\beta(\lambda z) \lambda^{1-\alpha} \, d\lambda = \frac{x^\alpha y^\beta z^\beta K_{\alpha,\beta}(x, y, z)}{2^\alpha \Gamma(\alpha+1)}$$

for $z \neq |x \pm y|$.

In order to prove (5.12) and (5.13) by duality one has to use that the function $K_{\alpha,\beta}(a, b, t)$ is continuous differentiable on the intervals $(0, |a-b|)$, $(|a-b|, a+b)$

and $(a + b, \infty)$. In the Jacobi case, the equiconvergence theorem for Jacobi series (Szegö [16, Thm. 9.1.2]) and well-known convergence properties of Fourier-cosine series then can be applied. In the Bessel case, the tool is Hankel's inversion theorem [17, p. 456].

Combination of (5.12) and (5.13) gives

$$\begin{aligned}
&\sum_{n=0}^{\infty} h_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos 2\theta_1) R_n^{(\alpha,\beta)}(\cos 2\theta_2) R_n^{(\alpha,\beta)}(\cos 2\theta_3) \\
&\quad = 2^{-\beta-2}\Gamma(\alpha+1)(\sin\theta_1 \sin\theta_2 \sin^2\theta_3)^{-\alpha}(\cos\theta_1\cos\theta_2\cos\theta_3)^{-\beta} \\
&\quad\quad \cdot \int_0^{\infty} J_\alpha(\lambda \sin\theta_1 \sin\theta_2) J_\beta(\lambda\cos\theta_1\cos\theta_2) J_\beta(\lambda\cos\theta_3)\lambda^{1-\alpha}\,d\lambda, \\
&\qquad\qquad \alpha > \beta > -\tfrac{1}{2}, \quad \cos\theta_3 \neq |\cos(\theta_1 \pm \theta_2)|.
\end{aligned} \tag{5.14}$$

For (5.13) and (5.14) see Watson [17, pp. 411, 413]. Gasper [10] obtained (5.12) by combining these two formulas of Watson. Formula (5.13) was applied by Copson [6, p. 352] to the Riemann–Green function for the hyperbolic analogue of (2.1).

REFERENCES

[1] R. ASKEY, *Jacobi polynomials, I. New proofs of Koornwinder's Laplace type integral representation and Bateman's bilinear sum*, this Journal, 5 (1974), pp. 119–124.

[2] W. N. BAILEY, *Generalized Hypergeometric Series*, Cambridge Tracts in Math. and Math. Physics No. 32, Cambridge University Press, Cambridge, 1935.

[3] H. BATEMAN, *A generalisation of the Legendre polynomial*, Proc. London Math. Soc. (2), 3 (1905), pp. 111–123.

[4] ———, *Partial Differential Equations of Mathematical Physics*, Cambridge University Press, Cambridge, 1932.

[5] B. L. J. BRAAKSMA AND B. MEULENBELD, *Jacobi polynomials as spherical harmonics*, Indag. Math., 30 (1968), pp. 384–389.

[6] E. T. COPSON, *On a singular boundary value problem for an equation of hyperbolic type*, Arch. Rational Mech. Anal., 1 (1957), pp. 349–356.

[7] A. DIJKSMA AND T. H. KOORNWINDER, *Spherical harmonics and the product of two Jacobi polynomials*, Indag. Math., 33 (1971), pp. 191–196.

[8] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Higher Transcendental Functions*, vols. I, II, III, McGraw-Hill, New York, 1953 and 1955.

[9] M. FLENSTED-JENSEN AND T. H. KOORNWINDER, *The convolution structure for Jacobi function expansions*, Ark. Mat., to appear.

[10] G. GASPER, *Positivity and the convolution structure for Jacobi series*, Ann. of Math., 93 (1971), pp. 112–118.

[11] ———, *Banach algebras for Jacobi series and positivity of a kernel*, Ibid., 95 (1972), pp. 261–280.

[12] P. HENRICI, *Addition theorems for general Legendre and Gegenbauer functions*, J. Rational Mech. Anal., 4 (1955), pp. 983–1018.

[13] T. H. KOORNWINDER, *The addition formula for Jacobi polynomials, I. Summary of results*, Indag. Math., 34 (1972), pp. 188–191.

[14] ———, *The addition formula for Jacobi polynomials, II. The Laplace type integral representation and the product formula*, Math. Centrum Amsterdam Afd. Toegepaste Wisk., Rep. TW 133, 1972.

[15] ———, *The addition formula for Jacobi polynomials and spherical harmonics*, SIAM J. Appl. Math., 25 (1973), pp. 236–246.

[16] G. SZEGÖ, *Orthogonal Polynomials*, Colloquium Publications, vol. 23, Amer. Math. Soc., Providence, R.I., 1972.

[17] G. N. WATSON, *Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1944.