EXPLICIT FORMULAS FOR SPECIAL FUNCTIONS RELATED TO SYMMETRIC SPACES

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It is our purpose to derive new explicit formulas for special functions by group theoretic interpretation. These formulas should be significant in the context of harmonic analysis for the classical expansions. Here we sketch a proof for the addition formula for Jacobi polynomials (see [8], [9]).

The radial part of the Laplace-Beltrami operator on a compact symmetric space of rank one G/K is

$$\omega_{p,q} = d^2/d\theta^2 + (p \cot\theta + 2q \cot 2\theta)\, d/d\theta, \tag{1}$$

where $0<\theta<\pi/2$ and p, q are certain nonnegative integers. The eigenfunctions of $\omega_{p,q}$, regular in $\theta=0$ and $\pi/2$, are Jacobi polynomials $P_n^{(\alpha,\beta)}(\cos 2\theta)$ $(\alpha=\frac{1}{2}(p+q-1),\ \beta=\frac{1}{2}(q-1))$. It was pointed out by Gangolli [4] that the convolution for radial functions on G/K implies a positive convolution structure for Jacobi series for certain values of α and β.

The radial convolution structure on G/K is closely connected with the product formula

$$\varphi(x)\,\varphi(y) = \int_K \varphi(xky)\, dk \tag{2}$$

for spherical functions φ. Let

$$\varphi(xky) = \sum_{\delta\in\hat{K}} \varphi_\delta(x, k, y) \tag{3}$$

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be the corresponding expansion of $\varphi(xky)$ as a function of $k \in K$. We call this expansion the addition formula for φ. In the case of $SO(q)/SO(q-1)$, formulas (2) and (3), when written in analytic form, are well-known results for Gegenbauer polynomials [**2**, §3.15, (19) and (20)].

We obtain the addition formula for Jacobi polynomials by generalizing the method used in the case $SO(q)/SO(q-1)$. Let G/K be of rank one, let V be a subspace of $L^2(G/K)$, irreducible under G, and let $\varphi \in V$ be a spherical function. When $e \in G/K$ is the K-invariant point then the function $\varphi(\xi)$ only depends on the distance $d(\xi, e)$, and we write $\varphi(\xi) = p(d(\xi, e))$ for $\xi \in G/K$. Let the functions $f_k(\xi)$ $(k=1, \ldots, N)$ form an arbitrary orthonormal base of V. Then

$$\sum_{k=1}^{N} f_k(\xi)\, \overline{f_k(\eta)} = \text{const}\, p(d(\xi, \eta)), \tag{4}$$

$p(d(\xi, \eta))$ is the kernel function for V. Next, we decompose V with respect to K, and we have $V = \sum_{\delta \in \hat{K}} V_\delta$. Let the functions $f_{k_\delta}(\xi)$ $(k_\delta = 1, \ldots, N_\delta)$ form an orthonormal base of V_δ and write

$$\psi_\delta(\xi, \eta) = \sum_{k_\delta = 1}^{N_\delta} f_{k_\delta}(\xi)\, \overline{f_{k_\delta}(\eta)}. \tag{5}$$

Then the sum

$$\sum_{\delta \in \hat{K}} \psi_\delta(\xi, \eta) = \text{const}\, p(d(\xi, \eta)) \tag{6}$$

is the required addition formula. The problem is to choose suitable coordinates and to express the functions $\psi_\delta(\xi, \eta)$ in terms of special functions.

We have solved this problem for the complex projective space $SU(q)/U(q-1)$ with spherical functions $P_n^{(q-2,0)}$. Observe that the homogeneous space $U(q)/U(q-1)$ is the unit sphere in the complex vector space $\mathbf{C}^q$. The functions on this sphere which are invariant under scalar multiplication by $e^{i\psi}$ are precisely the functions on $SU(q)/U(q-1)$. Ikeda and Kayama ([**6**], [**7**]) developed a theory for functions on $U(q)/U(q-1)$ analogous to the classical theory of spherical harmonics. By using their results we obtained the explicit addition formula

$$\begin{aligned} &P_n^{(q-2,0)}(2|\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 r e^{i\varphi}|^2 - 1) \\ &\quad = \sum_{k=0}^{n} \sum_{l=0}^{k} c_{n,k,l}^{(q-2,0)} f_{n,k,l}^{(q-2,0)}(\cos 2\theta_1)\, f_{n,k,l}^{(q-2,0)}(\cos 2\theta_2)\, P_{k,l}^{(q-2,0)}(re^{i\varphi}), \end{aligned} \tag{7}$$

where $c_{n,k,l}^{(q-2,0)}$ are constants,

$$f_{n,k,l}^{(q-2,0)}(\cos 2\theta)=(\sin\theta)^{k+l}(\cos\theta)^{k-l}\,P_{n-k}^{(q-2+k+l,k-l)}(\cos 2\theta)$$

and

$$P_{k,l}^{(q-2,0)}(re^{i\varphi})=P_l^{(q-3,k-l)}(2r^2-1)\,r^{k-l}\cos(k-l)\,\varphi.$$

The functions $P_{k,l}^{(q-2,0)}(x+iy)$ are polynomials in x and y, orthogonal in the unit disk with respect to the measure $(1-x^2-y^2)^{q-3}\,dx\,dy$. Formula (7) is an orthogonal expansion in terms of these polynomials.

Let $G=KAK$ be the Cartan decomposition associated with the symmetric space G/K. Suppose that the expansion (3) can be written as

$$\varphi(a_1ka_2)=\sum_{\delta\in\hat{K}} f_\delta(a_1)\,f_\delta(a_2)\,p_\delta(k) \qquad (a_1,a_2\in A,\ k\in K). \tag{8}$$

This is the case for $SU(q)/U(q-1)$. Let M be the centralizer of A in K. Then the function $p_\delta(k)$ is a spherical function on the homogeneous space K/M. In our example we have $K/M=U(q-1)/U(q-2)$, which is not a symmetric space. The $U(q-2)$-orbits in $U(q-1)/U(q-2)$ depend on two parameters. This explains the double expansion in (7).

The addition formula for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with general α and β can be obtained by repeated differentiation of both sides of (7) with respect to φ and by doing analytic continuation with respect to α and β. When $\alpha\geqq\beta\geqq-\frac{1}{2}$, the addition formula implies a product formula and the product formula immediately gives the positivity of the convolution structure for Jacobi series. This result was earlier obtained by Gasper [**5**]. He obtained the kernel K in

$$\frac{P_n^{(\alpha,\beta)}(x)\,P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)}=\int_{-1}^{+1} K(x,y,z)\,P_n^{(\alpha,\beta)}(z)\,(1-z)^\alpha\,(1+z)^\beta\,dz \tag{9}$$

explicitly as a nonnegative function. Formula (9) also follows from our product formula (see [**11**]).

There is an interpretation of Jacobi polynomials $P_n^{(p/2-1,q/2-1)}$ as spherical harmonics of degree $2n$ in $q+p$ dimensions which are invariant under the rotation group $SO(q)\times SO(p)$ (see Zernike and Brinkman [**12**], Braaksma and Meulenbeld [**1**]). In this interpretation another group theoretic proof can be given for our formulas (Flensted-Jensen [**3**], Koornwinder [**10**]).

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