AN ADAPTIVE WAVELET METHOD FOR SEMI-LINEAR FIRST ORDER SYSTEM LEAST SQUARES

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Abstract. We design an adaptive wavelet scheme for solving first order system least squares formulations of second order elliptic PDEs that converge with the best possible rate in linear complexity. A wavelet Riesz basis is constructed for the space $\vec{H}_{0,\Gamma_N}(\mathrm{div};\Omega)$ on general polygons. The theoretical findings are illustrated by numerical experiments.

1. Introduction

Optimally converging adaptive wavelet schemes were developed by Cohen, Dahmen, and DeVore in a sequence of papers [CDD01, CDD02, CDD03a]. They can be applied for solving general operator equations $F(u) = 0$, where for some separable Hilbert space H, $F : H \to H'$ with an elliptic Fréchet derivative $DF(u)$. After equipping H with a Riesz basis $\Psi = \{\psi_\lambda : \lambda \in \nabla\}$ of wavelet type, the adaptive scheme produces a sequence of approximations from the span of the basis that converges to the solution with the best possible rate. In [XZ03, GHS07, Ste14], it was shown that this holds true without the need of a recurrent application of a coarsening of the iterands.

Least squares problems, including nonlinear ones fit into the above framework. Indeed, let $G(u) = 0$, where, for some separable Hilbert spaces H and K, $G : H \to K'$. This u can be found as a minimizer of $\frac{1}{2}\|G(v)\|^2_{K'}$. Necessarily it is a solution of the Euler-Lagrange equations $F(u)(h) := \langle DG(u)h, G(u)\rangle_{K'} = 0$ $(h \in H)$. Under the condition that $DG(u) : H \to K'$ is a boundedly invertible mapping with its image, $DF(u)$ is elliptic as required.

A key ingredient of the adaptive wavelet scheme is the approximation of the residual of the current approximation in wavelet coordinates. For simplicity restricting to affine F, i.e., $F(u) = f - Au$, the equation in wavelet coordinates reads as $\mathbf{Au} = \mathbf{f}$, where $\mathbf{A}$ is the infinite stiffness matrix, $\mathbf{f}$ is the infinite load vector, and $\mathbf{u}$ is the infinite coordinate vector of u w.r.t. the wavelet basis. Given a finitely supported $\mathbf{w} \approx \mathbf{u}$, the task is to approximate $\mathbf{r}(\mathbf{w}) := \mathbf{f} - \mathbf{Aw}$ within a fixed, sufficiently small *relative* tolerance $\delta > 0$. Traditionally, this is done by approximating $\mathbf{f}$ and $\mathbf{Aw}$ *separately*, both within *absolute* tolerance $\frac{1}{2}\delta\|\mathbf{r}\|$. The second task is performed by approximating the infinitely supported columns of $\mathbf{A}$ within tolerances that are inversely proportional with the modulus of the corresponding coefficient of $\mathbf{w}$. This approximate matrix-vector product is a *nonlinear* map, and its computation is quantitatively demanding.

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In the further discussion, we restrict ourselves to approximations from spans of sets $\{\psi_\lambda \colon \lambda \in \Lambda\}$ where, with a natural choice of the index set ∇, Λ is a finite *tree*. For functions in these spaces, one can switch between a wavelet and a locally finite single-scale representation in linear complexity. Furthermore, such functions are piecewise smooth w.r.t. a partition $\mathcal{T}_\Lambda$ of the underlying domain into $\mathcal{O}(\#\Lambda)$ cells. For non-affine F, in order to evaluate nonlinear terms at an appropriate complexity, it seems mandatory to make this restriction to tree approximation. The resulting approximation classes are known to be only slightly smaller than with unconstrained approximation ([CDDD01]).

In [Ste14], an efficient approximate residual evaluation was developed in the setting of least squares problems, when the space K is of the form $L_2(\Omega)^N$. In this case, we have $\mathbf{r}(\mathbf{w}) = [\langle DG(w)\psi_\mu, G(w)\rangle_{L_2(\Omega)^N}]_{\mu\in\nabla}$, where $w := \sum_{\lambda\in\Lambda} \mathbf{w}_\lambda \psi_\lambda$. The function $G(w)$ can be expected to inherit from w the property of being piecewise smooth w.r.t. $\mathcal{T}_\Lambda$. For given constant $k \in \mathbb{N}$, approximate residuals were considered that are defined by ignoring all ψ_μ whose levels minus k exceed the maximum level of this partition restricted to $\operatorname{supp}\psi_\mu$. It was proved that this approximate residual has a *relative* error less than $\delta = \delta(k) > 0$, with $\delta(k) \downarrow 0$ when $k \to \infty$. Moreover, this approximate residual can be computed efficiently by switching between multi- and single-scale representations. Note that in case of an affine G, it depends linearly on $\mathbf{w}$.

The analysis from [Ste14] applies under the additional assumption that the space H is of the form $H^{m_1}(\Omega) \times \cdots \times H^{m_M}(\Omega)$, or for these coordinate spaces being replaced by closed subspaces that incorporate essential boundary conditions. Semi-linear second order elliptic PDEs of the form

$$\begin{cases} -\operatorname{div} A\nabla p + N(p) = f & \text{on } \Omega \subset \mathbb{R}^n, \\ p = g & \text{on } \Gamma_D, \\ \vec{n}\cdot A\nabla p = h & \text{on } \Gamma_N, \end{cases}$$

can be formulated as first order system least squares for spaces H and K of the aforementioned types, under the additional assumption that the PDE is $H^2(\Omega)$-regular ([CMM97]).

To avoid the last, restrictive condition, in the *current work* we consider the common div-grad first order system least squares formulation of finding

$$\underset{(\vec{u},p)\in \vec{H}_{0,\Gamma_N}(\operatorname{div};\Omega)\times H^1_{0,\Gamma_D}(\Omega)}{\operatorname{argmin}} \|\vec{u} - A\nabla p\|^2_{L_2(\Omega)^n} + \|N(p) - \operatorname{div}\vec{u} - f\|^2_{L_2(\Omega)}, \tag{1.1}$$

where thus $H = \vec{H}_{0,\Gamma_N}(\operatorname{div};\Omega) \times H^1_{0,\Gamma_D}(\Omega)$, and $K = L_2(\Omega)^{n+1}$. This space H does not satisfy the assumptions made in [Ste14]. For $\Omega \subset \mathbb{R}^2$ being a polygon, we equip $\vec{H}_{0,\Gamma_N}(\operatorname{div};\Omega)$ with a wavelet Riesz basis. So far such bases were constructed on essentially product domains only, in [Urb01]. For the space $H^1_{0,\Gamma_D}(\Omega)$, several general applicable wavelet constructions are known.

We generalize the analysis of the efficient approximate residual evaluation from [Ste14] to the current setting. By doing so, we construct an adaptive wavelet method for solving (1.1) that converges with the best possible rate in linear computational complexity. Numerical examples are provided on an L-shaped domain that illustrate the optimal rates.

An often mentioned advantage of a least-squares formulation, when $K = L_2(\Omega)^n$, is that the residual provides an efficient and reliable a posteriori error estimator. In a finite element setting, an obvious construction of an adaptive method is to split the squared residual into the contributions from the individual elements, and to refine those elements that carry the largest local residuals, e.g., via a bulk chasing approach. Although this approach may give reasonable results in examples, where its implementation is actually not too much different from the method we propose, there is even no proof that it yields a convergent method. It is interesting to note that recently, in [CP15], an optimally convergent adaptive finite element was developed for (1.1), without a nonlinear term $N(p)$ though. It is based on a newly developed a posteriori error estimator, different from the obvious residual.

This paper is organised as follows. In Sect. 2, 3, and 6, we summarize findings from [Ste14] about adaptive wavelet schemes for solving well-posed linear or non-linear operator equations. In Sect. 4, we discuss well-posedness of the least-squares problem (1.1). A wavelet Riesz basis for $\vec{H}_{0,\Gamma_N}(\mathrm{div};\Omega)$ is constructed in Sect. 5. In Sect. 7, we adapt the analysis of an efficient residual evaluation from [Ste14] to the problem (1.1). Numerical experiments are presented in Sect. 8. Finally, in Sect. 9 we summarise our findings and give a short outlook on future applications of adaptive wavelet schemes.

In this paper, by $C \lesssim D$ we will mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

2. An adaptive wavelet-Galerkin method

For a real, separable Hilbert space H with dual H', and a possibly *non-affine* mapping $F\colon H \supset \mathrm{dom}(F) \to H'$, we search a solution u of

$$F(u) = 0.$$

We assume that such a solution u exists, and that

- (c_1) F is continuously Fréchet differentiable in a neighborhood of u,
- (c_2) $(v,w) \mapsto DF(u)(v)(w)$ is an inner product on H, with the associated norm being equivalent to the norm on H, i.e., $DF(u)$ is *elliptic*.

We assume to have available a *Riesz basis* $\Psi = \{\psi_\lambda \colon \lambda \in \nabla\}$ for H. Being a Riesz basis means that the *analysis* operator

$$\mathcal{F}\colon H' \to \ell_2(\nabla)\colon g \mapsto [g(\psi_\lambda)]_{\lambda\in\nabla}, \tag{2.1}$$

is boundedly invertible, and so is its adjoint, known as the *synthesis operator*,

$$\mathcal{F}'\colon \ell_2(\nabla) \to H\colon \mathbf{v} \mapsto \mathbf{v}^\top \Psi := \sum_{\lambda\in\nabla} v_\lambda \psi_\lambda.$$

Here, as usual, we have identified $\ell_2(\nabla)'$ with $\ell_2(\nabla)$. The norm on $\ell_2(\nabla)$, and similarly when ∇ reads as any other countable index set, will be simply denoted as $\|\cdot\|$. For any $\Lambda \subset \nabla$, we set $\ell_2(\Lambda) := \{\mathbf{v} \in \ell_2(\nabla)\colon \operatorname{supp}\mathbf{v} \subset \Lambda\}$. Only later it will be relevant that Ψ is a basis of *wavelet* type.

Writing $u = \mathcal{F}'\mathbf{u}$, and with

$$\mathbf{F} := \mathcal{F}F\mathcal{F}'\colon \ell_2(\nabla) \to \ell_2(\nabla),$$

an *equivalent* formulation of our operator equation is given by a coupled system of $\#\nabla$ many, i.e., usually infinitely many scalar equations

$$\mathbf{F}(\mathbf{u}) = 0.$$

Note that $\|u - \mathcal{F}'\mathbf{v}\|_H \eqsim \|\mathbf{u} - \mathbf{v}\|$, uniformly in $\mathbf{v} \in \ell_2(\nabla)$.

We are going to approximate $\mathbf{u}$, and so u, by a sequence of Galerkin approximations from the spans of increasingly larger sets of wavelets, which sets are created by an adaptive process. Thanks to (c_1) and (c_2), the following result dealing with Galerkin approximations can be proven.

Proposition 2.1 ([Ste14, Prop. 2.4, Lem. 2.7])**.** *There exists a neighborhood* $\mathbf{U}$ *of* $\mathbf{u}$ *in* $\ell_2(\nabla)$ *such that for any* $\Lambda \subset \nabla$ *with* $\inf_{\mathbf{v}_\Lambda \in \ell_2(\Lambda)} \|\mathbf{u} - \mathbf{v}_\Lambda\|$ *being sufficiently small, the equation* $\mathbf{F}(\mathbf{u}_\Lambda)|_\Lambda = 0$ *has a unique solution in* $\ell_2(\Lambda) \cap \mathbf{U}$*; and for any* $\mathbf{v}_\Lambda \in \ell_2(\Lambda) \cap \mathbf{U}$*, it holds that* $\|\mathbf{u}_\Lambda - \mathbf{v}_\Lambda\| \eqsim \sup_{0 \neq \mathbf{w}_\Lambda \in \ell_2(\Lambda)} \frac{|\mathbf{F}(\mathbf{v}_\Lambda)(\mathbf{w}_\Lambda)|}{\|\mathbf{w}_\Lambda\|}$.

An equivalent formulation of the equation $\mathbf{F}(\mathbf{u}_\Lambda)|_\Lambda = 0$ is $F(\mathcal{F}'\mathbf{u}_\Lambda)(v_\Lambda) = 0$ for all $v_\Lambda \in \operatorname{span}\{\psi_\lambda \colon \lambda \in \Lambda\}$, i.e., $\mathcal{F}'\mathbf{u}_\Lambda$ is the Galerkin approximation to u from $\operatorname{span}\{\psi_\lambda \colon \lambda \in \Lambda\}$, or equivalently, $\mathbf{u}_\Lambda$ is the Galerkin approximation to $\mathbf{u}$ from $\ell_2(\Lambda)$.

In order to be able to construct efficient algorithms, in particular when F is non-affine, it it will be needed to consider only sets Λ from a certain subset of all finite subsets of ∇. In our applications, this collection of so-called *admissible* Λ will consist of all finite *trees* which will be defined later. For the moment, it suffices when the collection of admissible sets is such that the union of any two admissible sets is again admissible.

To provide a benchmark to evaluate our adaptive algorithm, for $s > 0$, we define the nonlinear *approximation class*

(2.2)
$$\mathcal{A}^s := \Big\{\mathbf{u} \in \ell_2(\nabla) \colon \|\mathbf{u}\|_{\mathcal{A}^s} := \sup_{\varepsilon > 0} \varepsilon \times \min\big\{(\#\Lambda)^s \colon \Lambda \text{ is admissible}, \inf_{\mathbf{v} \in \ell_2(\Lambda)} \|\mathbf{u} - \mathbf{v}\| \leq \varepsilon\big\} < \infty\Big\}.$$

A vector $\mathbf{u} \in \mathcal{A}^s$ if and only if there exists a sequence of admissible $(\Lambda_i)_i$, with $\lim_{i \to \infty} \#\Lambda_i = \infty$, such that $\sup_i \inf_{\mathbf{v} \in \ell_2(\Lambda_i)} (\#\Lambda_i)^s \|\mathbf{u} - \mathbf{v}_i\| < \infty$. That is, $\mathbf{u}$ *can* be approximated at rate s by vectors supported on admissible sets, or, equivalently, u can be approximated at rate s from spaces of type $\operatorname{span}\{\psi_\lambda \colon \lambda \in \Lambda, \Lambda \text{ is admissible}\}$.

As shown in [CDDD01], the admissibility condition of Λ being a tree makes the approximation class $\mathcal{A}^s$ only "slightly" smaller than with unconstrained nonlinear approximation.

The adaptive wavelet Galerkin method (**awgm**) defined below produces a sequence of increasingly more accurate Galerkin approximations $\mathbf{u}_\Lambda$ to $\mathbf{u}$. The, generally, infinite residual $\mathbf{F}(\mathbf{u}_\Lambda)$ is used as an a posteriori error estimator to guide an appropriate enlargement of the current set Λ using a bulk chasing strategy, so that the sequence of approximations converge with the best possible rate to $\mathbf{u}$. To arrive at an implementable method, that is even of optimal computational complexity, both the Galerkin solution and its residual are allowed to be computed inexactly within sufficiently small (essentially) relative tolerances.

Algorithm 2.2 (awgm).

% Let $0 < \mu_0 \leq \mu_1 < 1$, $\delta, \gamma > 0$ *be constants,* $\Lambda_0 \subset \nabla$ *be admissible,*
% and $\mathbf{w}_{\Lambda_0} \in \ell_2(\Lambda_0)$*. Let* $\mathbf{U}$ *as in Proposition 2.1.*

`for` $i = 0, 1, \ldots$ `do`

(1) $\zeta := \frac{2\delta}{1+\delta} \|\mathbf{r}_{i-1}\|$. *% (Read* $\|\mathbf{r}_{-1}\|$ *as some scalar* $\eqsim \|\mathbf{u}\|$*.)*

`do` $\zeta := \zeta/2$; *Compute* $\mathbf{r}_i \in \ell_2(\nabla)$ *such that* $\|\mathbf{r}_i - \mathbf{F}(\mathbf{w}_{\Lambda_i})\| \leq \zeta$.
`until` $\zeta \leq \frac{\delta}{1+\delta}\|\mathbf{r}_i\|$.

(2) *Determine an admissible* $\Lambda_{i+1} \supset \Lambda_i$ *with* $\|\mathbf{r}_i|_{\Lambda_{i+1}}\| \geq \mu_0\|\mathbf{r}_i\|$ *such that* $\#(\Lambda_{i+1} \setminus \Lambda_i) \lesssim \#(\tilde{\Lambda} \setminus \Lambda_i)$ *for any admissible* $\tilde{\Lambda} \supset \Lambda_i$ *with* $\|\mathbf{r}_i|_{\tilde{\Lambda}}\| \geq \mu_1\|\mathbf{r}_i\|$.

(3) *Compute* $\mathbf{w}_{\Lambda_{i+1}} \in \ell_2(\Lambda_{i+1}) \cap \mathbf{U}$ *with* $\|\mathbf{F}(\mathbf{w}_{\Lambda_{i+1}})|_{\Lambda_{i+1}}\| \leq \gamma\|\mathbf{r}_i\|$.

`endfor`

In step (1), by means of a loop in which an absolute tolerance is decreased, the true residual $\mathbf{F}(\mathbf{w}_{\Lambda_i})$ is approximated within a relative tolerance δ. In step (2), bulk chasing is performed on the approximate residual. The idea is to find a smallest admissible $\Lambda_{i+1} \supset \Lambda_i$ with $\|\mathbf{r}_i|_{\Lambda_{i+1}}\| \geq \mu_0\|\mathbf{r}_i\|$. In order to be able to find an implementation that is of linear complexity, the condition of having a truly smallest Λ_{i+1} has been relaxed. Finally, in step (3), a sufficiently accurate approximation of the Galerkin solution on the new set Λ_{i+1} is determined.

Convergence of the adaptive wavelet Galerkin method, with the *best possible rate*, is stated in the following theorem.

Theorem 2.3 ([Ste14, Thm. 3.9])**.** *Assume conditions* (c_1) *and* (c_2)*. Let* μ_1, γ, δ, $\inf_{\mathbf{v}_{\Lambda_0} \in \ell_2(\Lambda_0)} \|\mathbf{u} - \mathbf{v}_{\Lambda_0}\|$*, and* $\|\mathbf{F}(\mathbf{w}_{\Lambda_0})|_{\Lambda_0}\|$ *all be sufficiently small. Then, for some* $\alpha = \alpha[\mu_0] < 1$*, the sequence* $(\mathbf{w}_{\Lambda_i})_i$ *produced by* **awgm** *satisfies*

$$\|\mathbf{u} - \mathbf{w}_{\Lambda_i}\| \lesssim \alpha^i\|\mathbf{u} - \mathbf{w}_{\Lambda_0}\|.$$

If, for whatever $s > 0$*,* $\mathbf{u} \in \mathcal{A}^s$*, then* $\#(\Lambda_{i+1} \setminus \Lambda_0) \lesssim \|\mathbf{u} - \mathbf{w}_{\Lambda_i}\|^{-1/s}$.

The *computation* of the approximate Galerkin solution $\mathbf{w}_{\Lambda_{i+1}}$ can be implemented by performing the simple fixed point iteration

$$\mathbf{w}_{\Lambda_{i+1}}^{(j+1)} = \mathbf{w}_{\Lambda_{i+1}}^{(j)} - \omega\mathbf{F}(\mathbf{w}_{\Lambda_{i+1}}^{(j)})|_{\Lambda_{i+1}}.$$

Taking $\omega > 0$ to be a sufficiently small constant and starting with $\mathbf{w}_{\Lambda_{i+1}}^{(0)} = \mathbf{w}_{\Lambda_i}$, a fixed number of iterations suffices to meet the condition $\|\mathbf{F}(\mathbf{w}_{\Lambda_{i+1}}^{(j+1)})|_{\Lambda_{i+1}}\| \leq \gamma\|\mathbf{r}_i\|$. This holds even true when each of the $\mathbf{F}()|_{\Lambda_{i+1}}$ evaluations is performed within an absolute tolerance that is a sufficiently small fixed multiple of $\|\mathbf{r}_i\|$.

Optimal *computational* complexity of the **awgm** –meaning that the work to obtain an approximation within a given tolerance $\varepsilon > 0$ can be bounded on some constant multiple of the bound on its support length from Thm. 2.3,– is guaranteed under the following two assumptions concerning the cost of the "bulk chasing" process, and that of the approximate residual evaluation, respectively.

Assumption 2.4. The determination of Λ_{i+1} in Algorithm 2.2 is performed in $\mathcal{O}(\#\operatorname{supp}\mathbf{r}_i + \#\Lambda_i)$ operations.

In case of unconstrained approximation, i.e., any finite $\Lambda \subset \nabla$ is admissible, the assumption is valid by collecting the largest entries in modulus of $\mathbf{r}_i$, where, to avoid a suboptimal complexity, an exact sorting should be replaced by an approximate sorting based on binning. With tree approximation, the assumption is valid by the application of the so-called *Thresholding Second Algorithm* from [BD04]. We refer to [Ste14, §3.4] for a discussion.

Assumption 2.5. For a sufficiently small, fixed $\eta > 0$, and for any admissible $\Lambda \subset \nabla$, $\mathbf{w}_\Lambda \in \ell_2(\Lambda) \cap \mathbf{U}$, and $\varepsilon \geq \eta \|\mathbf{F}(\mathbf{w}_\Lambda)\|$, there exists an $\mathbf{r} \in \ell_2(\nabla)$ with

$$\|\mathbf{r} - \mathbf{F}(\mathbf{w}_\Lambda)\| \leq \varepsilon,$$

that one can compute in $\mathcal{O}(\varepsilon^{-1/s} \max(\|\mathbf{w}_\Lambda\|_{\mathcal{A}^s}^{1/s}, 1) + \#\Lambda)$ operations.

Under both assumptions, the **awgm** has optimal computational complexity:

Theorem 2.6. *In the setting of Theorem 2.3, and under Assumptions 2.4 and 2.5, not only $\#\mathbf{w}_{\Lambda_i}$, but also the number of arithmetic operations required by* **awgm** *for the computation of $\mathbf{w}_{\Lambda_i}$ is $\mathcal{O}(\|\mathbf{u} - \mathbf{w}_{\Lambda_i}\|^{-1/s})$.*

Generally, the value of s for which $\mathbf{u} \in \mathcal{A}^s$ is *not* known. Therefore, in order to speak about **awgm** as a method of optimal computational complexity, it is needed to ensure Assumption 2.5 for any $s \in (0, s^*]$, where s^* is not less than than the largest possible s –that we will denote as $s_{\max}$–, for which, in view of the equation $F(u) = 0$ and the order of the wavelet basis Ψ, membership $\mathbf{u} \in \mathcal{A}^s$ can generally be expected.

In the literature, see [DSX00, CDD03b, XZ05, BU08, Vor09], for a fairly general class of nonlinear mappings F, Assumption 2.5 has been verified for any $s \in (0, s^*]$, where s^* has *some* positive value.

In Sect. 6, we will see that for a class of *least squares problems* it is possible to guarantee Assumption 2.5 with the bound on the cost even reading as $\mathcal{O}(\varepsilon^{-1/\tilde{s}} + \#\Lambda)$, with $\tilde{s}$ being the rate of approximation of the right-hand side by piecewise polynomials of arbitrary, fixed degree. Since the right-hand side is known, the rate $\tilde{s}$ is accessible, and usually it exceeds the value of s for which $\mathbf{u} \in \mathcal{A}^s$.

3. Least squares problems

In this section, we consider a much wider class of well-posed, generally nonlinear operator equations, where the derivative of the operator at the solution is not necessarily elliptic. First of all, this extends the scope of the **awgm**. Secondly, even for operators that do have an elliptic derivative at the solution, but that stem of a PDE of second (or higher) order, in view of designing an appropriate residual evaluation, i.e., one that satisfies Assumption 2.5, it turns out to be beneficial to reformulate it as a system of first order. Then ellipticity is lost, which, however, will be restored by forming nonlinear normal equations. The benefit will be that usually the first order operators can be applied on the span of the wavelets in *mild sense* (i.e., when applied to a wavelet, the operator lands in an L_2 space). This renders residuals that have some smoothness, and so have close to sparse representations w.r.t. to the dual wavelet basis.

For real Hilbert spaces H and K, and a possible *non-affine* mapping $G\colon H \supset \operatorname{dom}(G) \to K'$, let us search a solution u of

$$G(u) = 0. \tag{3.1}$$

We assume that such a solution u exists, and that

(c_1') G is two times continuously Fréchet differentiable in a neighborhood of u,

(c_2') $DG(u) \in B(H, K')$ is a homeomorphism onto its range, meaning that $\|DG(u)(v)\|_{K'} \eqsim \|v\|_H$ $(v \in H)$.

Note that even for $H = K$, the assumption (c_2') involving G is much weaker than the ellipticity assumption (c_2) from Sect. 2 involving F.

With $Q\colon \mathrm{dom}(G) \to \mathbb{R}$ defined as the *least-squares functional*

$$Q(v) := \tfrac{1}{2}\|G(v)\|_{K'}^2, \tag{3.2}$$

a necessary condition for u to be a solution of (3.1) is that it solves the least squares problem

$$u = \underset{v\in\mathrm{dom}(G)}{\operatorname{argmin}}\; Q(v). \tag{3.3}$$

A direct calculation shows that, in a neighborhood of u, Q is Fréchet differentiable with $DQ(v)(h) = \langle DG(v)h, G(v)\rangle_{K'}$. We conclude that with

$$F := DQ\colon H \supset \mathrm{dom}(Q) \to H' : v \mapsto (h \mapsto \langle DG(v)h, G(v)\rangle_{K'}),$$

a necessary condition for u to solve (3.1) is

$$F(u) = 0. \tag{3.4}$$

By (c_1'), a direct calculation shows that, in a neighborhood of u, $F\colon H \supset \mathrm{dom}(F) \to H'$ is continuously Fréchet differentiable with

$$DF(u)(h_1)(h_2) = \langle DG(u)h_1, DG(u)h_2\rangle_{K'} \quad (h_1, h_2 \in H).$$

From (c_2'), we infer that this F satisfies the conditions (c_1) and (c_2) from Sect. 2, in particular meaning that in a neighborhood of the solution u of (3.1), this u is the only root of F. So instead of solving (3.1), we can solve (3.4). Moreover, assuming that we have a Riesz basis Ψ for H available, assuming it is separable, we may apply the **awgm** to (3.4) to approximate u from the span of Ψ with the best possible (constrained) approximation rate, and, under Assumptions 2.4 and 2.5, in optimal computational complexity.

As we have seen, Assumption 2.4 can always be satisfied, and so below we are concerned with the verification of Assumption 2.5 concerning an efficient residual evaluation.

With $R \in B(K', K)$ being the Riesz map, defined by $f(k) = \langle k, Rf\rangle_K$ ($f \in K'$, $k \in K$), for $v \in \mathrm{dom}(G) \subset H$, and $h \in H$, we have $(DG(v)'RG(v)(h) = (DG(v)h)(RG(v)) = \langle RG(v), RDG(v)h\rangle_K = \langle G(v), DG(v)h\rangle_{K'}$, or

$$F(\cdot) = DG(\cdot)'RG(\cdot), \quad \text{and so} \quad \mathbf{F}(\cdot) = \mathcal{F}DG(\mathcal{F}'\cdot)'RG(\mathcal{F}'\cdot).$$

In applications, often K is a Cartesian product space, i.e., $K = K_1 \times \cdots \times K_N$, and so $G = (G_1, \ldots, G_N)$. Then with, for $1 \leq i \leq N$, $R_i \in B(K_i', K_i)$ being the Riesz map, we have

$$F(\cdot) = \sum_{i=1}^{N} DG_i(\cdot)'R_iG_i(\cdot), \quad \text{and so} \quad \mathbf{F}(\cdot) = \sum_{i=1}^{N} \mathcal{F}DG_i(\mathcal{F}'\cdot)'R_iG_i(\mathcal{F}'\cdot).$$

In the next proposition, a criterion is given for the verification of Assumption 2.5.

Proposition 3.1. *For $1 \leq i \leq N$, let R_i be factorized as $R_i = \hat{R}_i\check{R}_i$, where for some Hilbert space $\tilde{K}_i$, $\check{R}_i \in B(K_i', \tilde{K}_i)$ and $\hat{R}_i \in B(\tilde{K}_i, K_i)$ are boundedly invertible. Let the neighborhood $\mathbf{U}$ of $\mathbf{u} = (\mathcal{F}')^{-1}u \in \ell_2(\nabla)$ be small enough.*

Suppose that for sufficiently small, fixed $\eta > 0$, and some $s > 0$, for any $\varepsilon > 0$, admissible $\Lambda \subset \nabla$, and $\mathbf{w}_\Lambda \in \ell_2(\Lambda) \cap \mathbf{U}$, for $1 \leq i \leq N$ one can compute approximations $\check{R}_iG_i(\mathcal{F}'\mathbf{w}_\Lambda) \in \tilde{K}_i$, and $\mathcal{F}DG_i(\mathcal{F}'\mathbf{w}_\Lambda)'\hat{R}_i \in B(\tilde{K}_i, \ell_2(\nabla))$ such that

- $\|\check{R}_i G_i(\mathcal{F}'\mathbf{w}_\Lambda) - \widetilde{\check{R}_i G_i(\mathcal{F}'\mathbf{w}_\Lambda)}\|_{\tilde{K}_i} \leq \varepsilon$,
- $\|(\mathcal{F}DG_i(\mathcal{F}'\mathbf{w}_\Lambda)'\hat{R}_i - \widetilde{\mathcal{F}DG_i(\mathcal{F}'\mathbf{w}_\Lambda)'\hat{R}_i}\,)\widetilde{\check{R}_i G_i(\mathcal{F}'\mathbf{w}_\Lambda)}\| \leq \eta \|\widetilde{\check{R}_i G_i(\mathcal{F}'\mathbf{w}_\Lambda)}\|_{\tilde{K}_i}$,

taking $\mathcal{O}(\varepsilon^{-1/s}\max(\|\mathbf{w}_\Lambda\|_{\mathcal{A}^s}^{1/s}, 1) + \#\Lambda)$ *operations. Then*

$$\|\mathbf{F}(\mathbf{w}_\Lambda) - \sum_{i=1}^{N} \widetilde{\mathcal{F}DG_i(\mathcal{F}'\mathbf{w}_\Lambda)'\hat{R}_i}\;\widetilde{\check{R}_i G_i(\mathcal{F}'\mathbf{w}_\Lambda)}\| \lesssim \eta \|\mathbf{F}(\mathbf{w}_\Lambda)\| + \varepsilon,$$

and for this s*, Assumption 2.5 is satisfied.*

Proof. The proof is an easy consequence of $\|F(\mathcal{F}'\mathbf{v})\|_{H'} \eqsim \|G(\mathcal{F}'\mathbf{v})\|_{K'}$ ($\mathbf{v} \in \mathbf{U}$), that, as has been shown in [Ste14, Lemma 4.3], follows from the existence of a solution u of $G(u) = 0$, (c_2'), and the continuity of DG at u. □

Note that in Proposition 3.1, the applications of the generally nonlinear operators $\check{R}_i G_i$ are approximated within *absolute* tolerance ε, and the linear operators $\mathcal{F}DG_i(\mathcal{F}'\mathbf{w}_\Lambda)'\hat{R}_i$ are replaced by $\widetilde{\mathcal{F}DG_i(\mathcal{F}'\mathbf{w}_\Lambda)'\hat{R}_i}$ such that, when applied to the approximations $\widetilde{\check{R}_i G_i(\mathcal{F}'\mathbf{w}_\Lambda)}$, the *relative* error is less than or equal to η.

The definition of F depends on the choice of the norm on K', cf. (3.2), i.e., on the choice of the norm on $K = K_1 \times \cdots \times K_N$.

A *first possibility* that enables to evaluate the Riesz maps $R_i \in B(K_i', K)$ is to select Riesz bases $\Psi_{K_i} = \{\psi_\lambda^{K_i} : \lambda \in \nabla_{K_i}\}$ for the K_i, assuming they are separable, with analysis operators denoted as $\mathcal{F}_{K_i}$. Equipping the K_i with the *equivalent* norms $\|(\mathcal{F}_{K_i}')^{-1} \cdot\|$, and so the K_i' with $\|\mathcal{F}_{K_i} \cdot\|$, one verifies that $R_i = \mathcal{F}_{K_i}'\mathcal{F}_{K_i}$. Proposition 3.1 can be applied with $\hat{R}_i = \mathcal{F}_{K_i}'$, $\check{R}_i = \mathcal{F}_{K_i}$, and $\tilde{K}_i = \ell_2(\nabla_{K_i})$.

To avoid confusion, denoting (temporarily) the Riesz basis Ψ for H as $\Psi_H = \{\psi_\lambda^H : \lambda \in \nabla_H\}$, and its analysis operator as $\mathcal{F}_H$, we have $u = \mathcal{F}_H'\mathbf{u}$. Putting $\mathbf{G}_i := \mathcal{F}_{K_i} G_i \mathcal{F}_H'$, we find that $\mathbf{F} = \mathcal{F}_H F \mathcal{F}_H' = \sum_{i=1}^N \mathcal{F}_H DG_i(\mathcal{F}_H'\cdot)'\mathcal{F}_{K_i}'\mathcal{F}_{K_i} G_i(\mathcal{F}_H'\cdot) = \sum_{i=1}^N D\mathbf{G}_i(\cdot)^\top \mathbf{G}_i(\cdot)$, meaning that the **awgm** has to be applied to

$$\mathbf{F}(\mathbf{u}) = \sum_{i=1}^{N} D\mathbf{G}_i(\mathbf{u})^\top \mathbf{G}_i(\mathbf{u}) = 0,$$

for affine G known as the normal equations.

In order to do so, Proposition 3.1 learns us that is sufficient that for some sufficiently small, fixed $\eta > 0$, one is able to approximate –with the appropriate computational complexity–, for any $\varepsilon > 0$, admissible $\Lambda \subset \nabla_H$, and $\mathbf{w}_\Lambda \in \ell_2(\Lambda) \cap \mathbf{U}$, for all $1 \leq i \leq N$, $\mathbf{G}_i(\mathbf{w}_\Lambda) \in \ell_2(\nabla_{K_i})$ within tolerance $\varepsilon > 0$, and $D\mathbf{G}_i(\mathbf{w}_\Lambda)^\top \in B(\ell_2(\nabla_{K_i}), \ell_2(\nabla_H))$ within tolerance $\eta > 0$.

The second task is usually easy because of the near-sparsity of the linear operators in wavelet coordinates. The first task, i.e., the approximate evaluation of the generally nonlinear operator in wavelet coordinates with the appropriate computational complexity, is more demanding. We envisage, however, that for G stemming from a semi-linear PDO, under reasonable conditions this task can be performed whilst realizing Assumption 2.5 for any $s \in (0, s^*]$ with $s^* \geq s_{\max}$, whenever the G_i can be evaluated on any $\psi_\lambda^{K_i}$ in *mild sense*. This will be studied in future work.

In the current paper, we consider a special, although relevant situation where the latter condition is satisfied automatically. Indeed, as a *second possibility* that enables to evaluate the Riesz maps $R_i \in B(K_i', K)$, we consider the case that each K_i is of the form $L_2(\Omega)^{n_i}$. Equipping these spaces with the standard norm, we identify K_i' with K_i, and there is no need to equip the K_i with Riesz bases. We take $R_i = \hat{R}_i = \check{R}_i = I$, and $\tilde{K}_i = K_i = L_2(\Omega)^{n_i}$ in Proposition 3.1. In this setting, we have

$$F(\mathcal{F}'\cdot)(\circ) = \sum_{i=1}^{N} DG_i(\mathcal{F}'\cdot)' R_i G_i(\mathcal{F}'\cdot)(\circ) = \sum_{i=1}^{N} \langle G_i(\mathcal{F}'\cdot), DG_i(\mathcal{F}'\cdot)(\circ)\rangle_{L_2(\Omega)^{n_i}}$$

and so

$$\mathbf{F}(\cdot) = \mathcal{F}F\mathcal{F}'(\cdot) = \Big[\sum_{i=1}^{N} \big\langle G_i(\mathcal{F}'\cdot), DG_i(\mathcal{F}'\cdot)(\psi)\big\rangle_{L_2(\Omega)^{n_i}}\Big]_{\psi\in\Psi}.$$

As we will see, apart from making approximations to the right-hand side, needed in case of an inhomogeneous equation, and possibly the application of quadrature, it will be possible to evaluate the nonlinear terms $G_i(\mathcal{F}'\mathbf{w}_\Lambda)$ *exactly.* In this case, the construction of the approximate applications of the linear operators $DG_i(\mathcal{F}'\mathbf{w}_\Lambda)$ will require more attention.

In view of the doubling of the *order* of the operator equation when generating F from G, as applications we think in particular of G to correspond to a system of partial differential equations of *first order.* In this setting, the case $K = L_2(\Omega)^N$ enters naturally. In the next section, we give an example that will be central in this work.

4. Mixed formulation of semi-linear, elliptic PDEs of second order

As an application, on a domain $\Omega \subset \mathbb{R}^n$, we consider the semi-linear boundary value problem

$$\left\{\begin{aligned} -\operatorname{div} A\nabla p + N(p) &= f \quad \text{on } \Omega,\\ p &= g \quad \text{on } \Gamma_D,\\ \vec{n}\cdot A\nabla p &= h \quad \text{on } \Gamma_N,\end{aligned}\right. \tag{4.1}$$

where $\vec{n}$ is the outward pointing unit vector normal to the boundary, $N(p)$ depends generally nonlinearly on p and/or first order derivatives of p, $\Gamma_D \cup \Gamma_N = \partial\Omega$, $\Gamma_D \cap \Gamma_N = \emptyset$, $A = A^\top \in L_\infty(\Omega)^{n\times n}$ and

$$\xi^\top A(\cdot)\xi \eqsim \|\xi\|^2 \quad (\xi \in \mathbb{R}^n, a.e.). \tag{4.2}$$

We take $g = 0$ and $h = 0$. For a discussion how to deal with inhomogeneous boundary conditions, we refer to [Ste13, Corollary 3.1], and to [Ste14, §4.4] for issues related to the application of an adaptive wavelet scheme in that case.

Taking $f \in L_2(\Omega)$, and with

$$H^1_{0,\Gamma_D}(\Omega) := \left\{\begin{array}{ll} \{q \in H^1(\Omega)\colon q = 0 \text{ on } \Gamma_D\} & \text{when } |\Gamma_D| > 0,\\ H^1(\Omega)/\mathbb{R} & \text{when } |\Gamma_D| = 0,\end{array}\right. \tag{4.3}$$

where $|\cdot|$ denotes the Lebesgue measure of a set, we assume that

(d_1) there exists a solution $p \in H^1_{0,\Gamma_D}(\Omega)$ of

$$\int_\Omega A\nabla p\cdot\nabla q + N(p)q = \int_\Omega fq \quad (q \in H^1_{0,\Gamma_D}(\Omega)).$$

(d_2) $N\colon H^1_{0,\Gamma_D}(\Omega) \supset \operatorname{dom}(N) \to L_2(\Omega)$ is two times continuously differentiable in a neighbourhood of p,

(d_3) $H^1_{0,\Gamma_D}(\Omega) \to (H^1_{0,\Gamma_D}(\Omega))' \colon r \mapsto (q \mapsto \int_\Omega A\nabla r \cdot \nabla q + qDN(p)r)$ is boundedly invertible.

We consider the formulation of our boundary value problem as the first order system

$$\begin{cases} \vec{u} - A\nabla p &= 0 \quad \text{on } \Omega, \\ N(p) - \operatorname{div} \vec{u} &= f \quad \text{on } \Omega, \\ p &= 0 \quad \text{on } \Gamma_D, \\ \vec{n} \cdot \vec{n} &= 0 \quad \text{on } \Gamma_N, \end{cases}$$

and, with

$$\vec{H}_{0,\Gamma_N}(\operatorname{div};\Omega) := \{\vec{v} \in \vec{H}(\operatorname{div};\Omega) \colon \vec{n} \cdot \vec{v} = 0 \text{ on } \Gamma_N\},$$

define

$$\begin{aligned} G \colon \vec{H}_{0,\Gamma_N}(\operatorname{div};\Omega) \times H^1_{0,\Gamma_D}(\Omega) &\to L_2(\Omega)^n \times L_2(\Omega) \\ \colon (\vec{u}, p) &\mapsto (\vec{u} - A\nabla p, N(p) - \operatorname{div} \vec{u} - f). \end{aligned} \tag{4.4}$$

With p from (d_1), setting $\vec{u} = A\nabla p$, one infers that $\operatorname{div} \vec{u} = f - N(p) \in L_2(\Omega)$, because N maps $\operatorname{dom}(N) \subset H^1_{0,\Gamma_D}(\Omega)$ into $L_2(\Omega)$ by (d_2), as well as $\vec{n} \cdot \vec{v} = 0$ on Γ_N. We conclude that $G(\vec{u}, p) = 0$ has a solution.

Furthermore, (d_2) shows that G is two times continuously Fréchet differentiable in a neighborhood of $(\vec{u}, p)$, i.e., (c_1') is valid.

Similarly to the proof of [Ste13, Thm 3.1], from (4.2), $DN(p) \in B(H^1_{0,\Gamma_D}(\Omega), L_2(\Omega))$ by (d_2), and (d_3), it follows that

$$DG(\vec{u}, p) \colon \vec{H}_{0,\Gamma_N}(\operatorname{div};\Omega) \times H^1_{0,\Gamma_D}(\Omega) \to L_2(\Omega)^n \times L_2(\Omega) \text{ is boundedly invertible,}$$

which implies (c_2').

We conclude that $(\vec{u}, p)$ can be found as a solution of the *least squares problem* of solving

$$\underset{(\vec{u},p) \in \operatorname{dom}(G)}{\operatorname{argmin}} \frac{1}{2} \|G(\vec{u}, p)\|^2_{L_2(\Omega)^{n+1}},$$

or, equivalently, as the solution of

$$F(\vec{u}, p) := DG(\vec{u}, p)' G(\vec{u}, p) = 0, \tag{4.5}$$

being well-posed in the sense that it satisfies (c_1)-(c_2). Note that $\operatorname{dom}(F) = \operatorname{dom}(G) = \vec{H}_{0,\Gamma_N}(\operatorname{div};\Omega) \times \operatorname{dom}(N) \subset \vec{H}_{0,\Gamma_N}(\operatorname{div};\Omega) \times H^1_{0,\Gamma_D}(\Omega)$.

4.1. **Cubic nonlinearity.** A well-known example of an N for which (d_1)-(d_3) are satisfied is given by $N(p) = p^3$ for $n \leq 3$. In this case, the energy functional $\frac{1}{2}\int_\Omega A\nabla p \cdot \nabla p + \frac{1}{4}p^4 - fp$ is coercive and strictly convex, so that the solution p is even unique.

4.2. **Elliptic sine-Gordon equation.** As another example, for $N \colon \mathbb{R} \to \mathbb{R}$ being continuous and bounded, it is known, see e.g. [BS11], that (d_1) is satisfied, where now the solution is not necessarily unique. Obviously (d_2) is satisfied when N is two times continuously differentiable. A sufficient condition for (d_3) is that $\|DN(p)\|_{L_\infty(\Omega)}$ is less than the smallest eigenvalue of the operator $-\operatorname{div} A\nabla$.

In our experiments, that we will report on in Sect. 8, we will consider the elliptic sine-Gordon equation with Dirichlet boundary conditions, i.e. $A = I$, $N(p) = \sin p$, and $\Gamma_D = \partial\Omega$, on the L-shaped domain $(0,2)^2 \setminus (0,1] \times [1,2)$. For this domain, this smallest eigenvalue is known to be ≈ 9.5 (see e.g. [LO13]), so that (d_3) is satisfied.

5. Wavelet bases for $\vec{H}_{0,\Gamma_N}(\mathrm{div};\Omega)$ and $H^1_{0,\Gamma_D}(\Omega)$

In order to run the adaptive wavelet Galerkin method (**awgm**) to (4.5), we need to equip $\vec{H}_{0,\Gamma_N}(\mathrm{div};\Omega)$ and $H^1_{0,\Gamma_D}(\Omega)$ with (wavelet) Riesz bases.

Let Ω be a bounded, simply connected, Lipschitz polygon in $\mathbb{R}^n$, where

$$n = 2.$$

In [HSW96], a multi-level preconditioner has been constructed for the lowest order Raviart-Thomas finite element spaces. Heavily relying on ideas from that paper, here we construct a Riesz basis for $\vec{H}_{0,\Gamma_N}(\mathrm{div};\Omega)$. To the best of our knowledge, so far Riesz bases for $\vec{H}_{0,\Gamma_N}(\mathrm{div};\Omega)$ have been constructed basically on product domains only, cf. [Urb01].

Let $\mathcal{T}_0$ be a fixed, conforming initial triangulation of Ω, and let Γ_N be a, possibly empty, connected union of edges of $T \in \mathcal{T}_0$. For $\ell \in \mathbb{N} = \{1, 2, \ldots\}$, let $\mathcal{T}_\ell$ be created from $\mathcal{T}_{\ell-1}$ by breaking any $T \in \mathcal{T}_{\ell-1}$ into four triangles by connecting the midpoints of the edges of T ("red-refinement").

For $\ell \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, consider the finite element spaces

$$\begin{aligned} W_\ell &:= \{v \in L_2(\Omega) \colon v|_T \in P_0(T) \ (T \in \mathcal{T}_\ell)\}, \\ S_\ell &:= \{v \in H^1_{0,\Gamma_N}(\Omega) \colon v|_T \in P_1(T) \ (T \in \mathcal{T}_\ell)\}, \\ \vec{V}_\ell &:= \{\vec{v} \in \vec{H}_{0,\Gamma_N}(\mathrm{div};\Omega) \colon \vec{v}|_T \colon \vec{x} \mapsto c_T + d_T\vec{x} \text{ for some } c_T, d_T \in \mathbb{R} \ (T \in \mathcal{T}_\ell)\}, \end{aligned}$$

the last space being the lowest order Raviart-Thomas space w.r.t. $\mathcal{T}_\ell$. Recall the definition of $H^1_{0,\Gamma_N}(\Omega)$ given in (4.3), in particular for $|\Gamma_N| = 0$.

For $\ell \in \mathbb{N}_0$, let $\mathcal{E}_\ell$ denote the set of edges of $\mathcal{T}_\ell$ that are not on Γ_N. For use later, for $\ell \in \mathbb{N}$, we also introduce $\mathcal{E}_\ell^{\mathrm{new}}$ as being the set of $e \in \mathcal{E}_\ell$ that are not contained in an edge of $\mathcal{E}_{\ell-1}$.

Fixing some normal vector $\vec{n}_e$ on any $e \in \mathcal{E}_\ell$, the following result is well-known, see e.g. [HSW96, Lemma 2.1].

Lemma 5.1. *$\{\vec{\varphi}_{\ell,e} \colon e \in \mathcal{E}_\ell\} \subset \vec{V}_\ell$, defined by $\int_{e'} \vec{\varphi}_{\ell,e}\vec{n}_{e'} = \delta_{ee'}$ ($e' \in \mathcal{E}_\ell$), is an $L_2(\Omega)^n$-uniform Riesz basis for $\vec{V}_\ell$.*

Also the following result is basically well-known. Since we could not easily find a proof that covers the case $\Gamma_N = \partial\Omega$, for completeness we include one.

Lemma 5.2. *It holds that* $\mathrm{div}\,\vec{V}_\ell = \begin{cases} W_\ell \cap L_2(\Omega)/\mathbb{R} & \Gamma_N = \partial\Omega, \\ W_\ell & \textit{otherwise.} \end{cases}$

Proof. Obviously $\mathrm{div}\,\vec{V}_\ell \subset W_\ell$. Moreover, if $\Gamma_N = \partial\Omega$, then for $\vec{v} \in \vec{V}_\ell$, $\int_\Omega \mathrm{div}\,\vec{v} = 0$.

Given $f \in W_\ell$, or, in case $\Gamma_N = \partial\Omega$, $f \in W_\ell \cap L_2(\Omega)/\mathbb{R}$, let $\vec{v} = \nabla u$ with $u \in H^1_{0,\partial\Omega\setminus\Gamma_N}(\Omega)$ be the solution of $\int_\Omega \nabla u \cdot \nabla w = \int_\Omega fw$ ($w \in H^1_{0,\partial\Omega\setminus\Gamma_N}(\Omega)$). Then $\mathrm{div}\,\vec{v} = f$, and $\vec{v} \cdot \vec{n} = 0$ on Γ_N. Defining $\vec{v}_\ell \in \vec{V}_\ell$ by $\int_e \vec{v}_\ell = \int_e \vec{v}$ ($e \in \mathcal{E}_\ell$), for any $T \in \mathcal{T}_\ell$, we have $\int_T \mathrm{div}\,\vec{v}_\ell = \int_T \mathrm{div}\,\vec{v} = \int_T f$, and so, since $\mathrm{div}\,\vec{v}_\ell, f \in W_\ell$, it holds that $\mathrm{div}\,\vec{v}_\ell = f$. □

For $\ell \in \mathbb{N}$, we define

$$\vec{H}_\ell := \{\vec{v} \in \vec{V}_\ell \colon \int_{\partial T} \vec{v} \cdot \vec{n} = 0 \ (T \in \mathcal{T}_{\ell-1})\}.$$

Note that $\vec{H}_\ell = \mathrm{span}\{\vec{\varphi}_{\ell,e} \colon e \in \mathcal{E}_\ell^{\mathrm{new}}\}$.

Considering $\ell = 0$, let $\vec{H}_0$ be some subspace of $\vec{V}_0$ such that $\vec{V}_0 = \vec{H}_0 \oplus (\vec{V}_0 \cap \vec{H}(\operatorname{div} 0;\Omega))$, e.g., $\vec{H}_0$ being the orthoplement of $\vec{V}_0 \cap \vec{H}(\operatorname{div} 0;\Omega)$, and let $\vec{\Phi}_0$ be some basis for $\vec{H}_0$. As usual, $\vec{H}(\operatorname{div} 0;\Omega) := \{\vec{v} \in L_2(\Omega)^n \colon \operatorname{div} \vec{v} = 0\}$.

Proposition 5.3. *For $\ell \in \mathbb{N}$, it holds that*

(i) $\|\cdot\|_{L_2(\Omega)^n} \eqsim 2^{-\ell} \|\operatorname{div} \cdot\|_{L_2(\Omega)}$ *on* $\vec{H}_\ell$,

(ii) $\dfrac{\|\sum_{e \in \mathcal{E}_\ell^{\text{new}}} c_e \operatorname{div} \vec{\varphi}_{\ell,e}\|^2_{L_2(\Omega)}}{\sum_{e \in \mathcal{E}_\ell^{\text{new}}} |c_e|^2 \|\operatorname{div} \vec{\varphi}_{\ell,e}\|^2_{L_2(\Omega)}} \in [\frac{1}{2}, 2]$ $(0 \neq (c_e)_{e \in \mathcal{E}_\ell^{\text{new}}})$,

(iii) $\operatorname{div} \vec{H}_\ell = W_\ell \cap W_{\ell-1}^{\perp_{L_2(\Omega)}}$ $(\subset L_2(\Omega)/\mathbb{R})$.

Proof. For a $T \in \mathcal{T}_{\ell-1}$, consider a numbering of the triangles in $\mathcal{T}_\ell$, and edges in $\mathcal{E}_\ell^{\text{new}}$ that are inside T as indicated in Figure 5. From, for $1 \leq i \leq 3$, $\int_{T_i} \operatorname{div} \vec{\varphi}_{\ell,e_i} = \pm 1$,

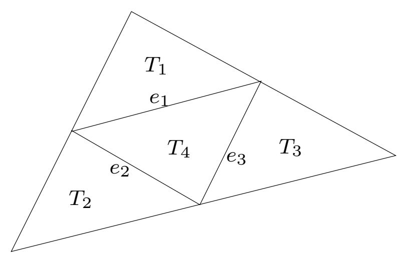

FIGURE 1. Numbering of the subtriangles and interior edges in a triangle.

$\int_{T_i \cup T_4} \operatorname{div} \vec{\varphi}_{\ell,e_i} = 0$, and $\operatorname{div} \vec{\varphi}_{\ell,e_i}|_{T_j} \in P_0(T_j)$ $(1 \leq j \leq 4)$, we infer that for $c_i \in \mathbb{R}$,

$$\|\operatorname{div} \sum_{i=1}^{3} c_i \vec{\varphi}_{\ell,e_i}\|^2_{L_2(T)} = \frac{4}{\operatorname{vol}(T)}(c_1^2 + c_2^2 + c_3^2 + (\pm c_1 \pm c_2 \pm c_3)^2),$$

and so, in particular, $\|\operatorname{div} \vec{\varphi}_{\ell,e_i}\|^2_{L_2(T)} = \frac{8}{\operatorname{vol}(T)}$. From $0 \leq (\pm c_1 \pm c_2 \pm c_3)^2 \leq 3(c_1^2 + c_2^2 + c_3^2)$, (ii) follows.

Together (ii), $\|\operatorname{div} \vec{\varphi}_{\ell,e_i}\|^2_{L_2(T)} = \frac{8}{\operatorname{vol}(T)}$, $\operatorname{vol}(T) \eqsim 2^{-(\ell-1)}$, and Lemma 5.1 show (i).

The proof of (iii) is obvious. $\square$

Corollary 5.4. (a). *The collection $\cup_{\vec{\varphi} \in \vec{\Phi}_0} \operatorname{div} \vec{\varphi} + \cup_{\ell \geq 1} \{2^{-\ell} \operatorname{div} \vec{\varphi}_{\ell,e} \colon e \in \mathcal{E}_\ell^{\text{new}}\}$ is a Riesz basis for $L_2(\Omega)$, or for $L_2(\Omega)/\mathbb{R}$ in case $\Gamma_N = \partial\Omega$.*

(b). *For square summable $(e_{\vec{\varphi}})_{\vec{\varphi} \in \vec{\Phi}_0}$ and $((d_{\ell,e})_{e \in \mathcal{E}_\ell^{\text{new}}})_{\ell \in \mathbb{N}}$, it holds that $\vec{v} := \sum_{\vec{\varphi} \in \vec{\Phi}_0} e_{\vec{\varphi}} \vec{\varphi} + \sum_{\ell \in \mathbb{N}} \sum_{e \in \mathcal{E}_\ell^{\text{new}}} d_{\ell,e} \vec{\varphi}_{\ell,e} \in \vec{H}_{0,\Gamma_N}(\operatorname{div};\Omega)$, and $\|\vec{v}\|_{\vec{H}(\operatorname{div};\Omega)} \lesssim \|\operatorname{div} \vec{v}\|_{L_2(\Omega)}$.*

Proof. (a) follows from $\{2^{-\ell} \operatorname{div} \vec{\varphi}_{\ell,e} \colon e \in \mathcal{E}_\ell^{\text{new}}\}$ being an $L_2(\Omega)$-uniform Riesz basis for $W_\ell \cap W_{\ell-1}^{\perp_{L_2(\Omega)}}$ by Proposition 5.3, and $\cup_{\vec{\varphi} \in \vec{\Phi}_0} \operatorname{div} \vec{\varphi}$ being a basis for $\operatorname{div} \vec{V}_0 = \begin{cases} W_0 \cap L_2(\Omega)/\mathbb{R} & \Gamma_N = \partial\Omega, \\ W_0 & \text{otherwise,} \end{cases}$ by Lemma 5.2.

(b). Since by Proposition 5.3(i), for $\vec{h}_\ell \in \vec{H}_\ell$,

$$\sum_{\ell,k\in\mathbb{N}_0} \langle \vec{h}_\ell, \vec{h}_k\rangle_{L_2(\Omega)^n} \leq \sum_{\ell,k\in\mathbb{N}_0} \|\vec{h}_\ell\|_{L_2(\Omega)^n}\|\vec{h}_k\|_{L_2(\Omega)^n}$$
$$\lesssim \sum_{\ell,k\in\mathbb{N}_0} 2^{-(\ell+k)}\|\operatorname{div}\vec{h}_\ell\|_{L_2(\Omega)}\|\operatorname{div}\vec{h}_k\|_{L_2(\Omega)} \lesssim \sum_{\ell\in\mathbb{N}_0}\|\operatorname{div}\vec{h}_\ell\|^2_{L_2(\Omega)},$$

an application of (a) completes the proof of this part. □

Remark 5.5. The collection from Corollary 5.4(a) can even easily be turned into an orthonormal basis without jeopardising the local supports.

Theorem 5.6. *For Σ_N being some Riesz basis for $H^1_{0,\Gamma_N}(\Omega)$,*

$$\cup_{\vec{\varphi}\in\vec{\Phi}_0}\vec{\varphi} + \cup_{\ell\geq 1}\{2^{-\ell}\vec{\varphi}_{\ell,e}\colon e\in\mathcal{E}^{\text{new}}_\ell\} + \cup_{\sigma\in\Sigma_N}\,\mathbf{curl}\,\sigma$$

is a Riesz basis for $\vec{H}_{0,\Gamma_N}(\operatorname{div};\Omega)$.

Proof. We show that the synthesis operator $\mathcal{F}'\colon \ell_2 \to \vec{H}_{0,\Gamma_N}(\operatorname{div};\Omega)$ associated with the collection is bounded, injective, and surjective, and so boundedly invertible.

For any square summable $(c_\sigma)_{\sigma\in\Sigma_N}$, we have $\sum_{\sigma\in\Sigma_N} c_\sigma\sigma \in H^1_{0,\Gamma_N}(\Omega)$, and so $\mathbf{curl}\sum_{\sigma\in\Sigma_N} c_\sigma\sigma = \sum_{\sigma\in\Sigma_N} c_\sigma\,\mathbf{curl}\,\sigma \in L_2(\Omega)$, $\operatorname{div}\sum_{\sigma\in\Sigma_N} c_\sigma\,\mathbf{curl}\,\sigma = 0$, and thus

$$\|\sum_{\sigma\in\Sigma_N} c_\sigma\,\mathbf{curl}\,\sigma\|^2_{\vec{H}(\operatorname{div};\Omega)} = \|\sum_{\sigma\in\Sigma_N} c_\sigma\,\mathbf{curl}\,\sigma\|^2_{L_2(\Omega)^n} = |\sum_{\sigma\in\Sigma_N} c_\sigma\sigma|^2_{H^1(\Omega)} \eqsim \sum_{\sigma\in\Sigma_N}|c_\sigma|^2.$$

An application of Corollary 5.4 completes the proof of boundedness of $\mathcal{F}'$.

For square summable $(e_{\vec{\varphi}})_{\vec{\varphi}\in\vec{\Phi}_0}$, $((d_{\ell,e})_{e\in\mathcal{E}^{\text{new}}_\ell})_{\ell\in\mathbb{N}}$, and $(c_\sigma)_{\sigma\in\Sigma_N}$, let

$$\sum_{\vec{\varphi}\in\vec{\Phi}_0} e_{\vec{\varphi}}\vec{\varphi} + \sum_{\ell\geq 1}\sum_{e\in\mathcal{E}^{\text{new}}_\ell} d_{\ell,e}2^{-\ell}\vec{\varphi}_{\ell,e} + \sum_{\sigma\in\Sigma_N} c_\sigma\,\mathbf{curl}\,\sigma = 0.$$

Then $\sum_{\vec{\varphi}\in\vec{\Phi}_0} e_{\vec{\varphi}}\operatorname{div}\vec{\varphi} + \sum_{e\in\mathcal{E}^{\text{new}}_\ell} d_{\ell,e}2^{-\ell}\operatorname{div}\vec{\varphi}_{\ell,e} = 0$, and so $(e_{\vec{\varphi}})_{\vec{\varphi}\in\vec{\Phi}_0} = 0 = ((d_{\ell,e})_{e\in\mathcal{E}^{\text{new}}_\ell})_{\ell\in\mathbb{N}}$ by Corollary 5.4(a). Consequently, $0 = \|\sum_{\sigma\in\Sigma_N} c_\sigma\,\mathbf{curl}\,\sigma\|_{L_2(\Omega)^n} = |\sum_{\sigma\in\Sigma_N} c_\sigma\sigma|_{H^1(\Omega)}$, which implies $(c_\sigma)_{\sigma\in\Sigma_N} = 0$. We conclude that $\mathcal{F}'$ is injective.

Given $\vec{v} \in \vec{H}_{0,\Gamma_N}(\operatorname{div};\Omega)$, Corollary 5.4(a) shows that there exist square summable $(e_{\vec{\varphi}})_{\vec{\varphi}\in\vec{\Phi}_0}$, and $((d_{\ell,e})_{e\in\mathcal{E}^{\text{new}}_\ell})_{\ell\in\mathbb{N}}$ with

$$\operatorname{div}\vec{v} = \sum_{\vec{\varphi}\in\vec{\Phi}_0} e_{\vec{\varphi}}\operatorname{div}\vec{\varphi} + \sum_{\ell\geq 1}\sum_{e\in\mathcal{E}^{\text{new}}_\ell} d_{\ell,e}2^{-\ell}\operatorname{div}\vec{\varphi}_{\ell,e}.$$

By Corollary 5.4(b), $\vec{v} - \big(\sum_{\vec{\varphi}\in\vec{\Phi}_0} e_{\vec{\varphi}}\vec{\varphi} + \sum_{\ell\geq 1}\sum_{e\in\mathcal{E}^{\text{new}}_\ell} d_{\ell,e}2^{-\ell}\vec{\varphi}_{\ell,e}\big) \in \vec{H}_{0,\Gamma_N}(\operatorname{div};\Omega)\cap \vec{H}(\operatorname{div}0;\Omega)$. Since Ω is a simply connected bounded Lipschitz domain, from [GR79, Thm. 3.1] we know that there exists a $\omega \in H^1(\Omega)$, unique up to a constant, with $\mathbf{curl}\,\omega$ being equal to this difference. Necessarily $\frac{\partial\omega}{\partial\tau}|_{\Gamma_N} = 0$. Since Γ_N is connected, we can select this constant so that $\omega \in H^1_{0,\Gamma_N}(\Omega)$. We conclude that there exists a square summable $(c_\sigma)_{\sigma\in\Sigma_N}$ such that $\omega = \sum_{\sigma\in\Sigma_N} c_\sigma\sigma$, and thus $\mathbf{curl}\,\omega = \sum_{\sigma\in\Sigma_N} c_\sigma\,\mathbf{curl}\,\sigma$, which shows that $\mathcal{F}'$ is surjective, and thus completes the proof. □

So far Σ_N can be any Riesz basis for $H^1_{0,\Gamma_N}(\Omega)$, and so it might not have any relation to the sequences of partitions $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$, or, in particular, to the Raviart-Thomas spaces $(\vec{V}_\ell)_{\ell\in\mathbb{N}_0}$. In view of an efficient implementation, however, it is beneficial when such relations do exist.

Proposition 5.7. *Let* $\Sigma_N = \cup_{\ell\in\mathbb{N}_0}\Sigma_N^{(\ell)}$ *be such that* $\operatorname{span}\cup_{k=0}^{\ell}\Sigma_N^{(k)} = S_\ell$. *Then*

$$\cup_{\vec{\varphi}\in\vec{\Phi}_0}\vec{\varphi} + \cup_{k=1}^{\ell}\{2^{-k}\vec{\varphi}_{k,e} \colon e\in\mathcal{E}_k^{\rm new}\} + \cup_{k=0}^{\ell}\cup_{\sigma\in\Sigma_N^{(k)}}\mathbf{curl}\,\sigma$$

is a basis for $\vec{V}_\ell$, *i.e., all basis functions with "levels" up to* ℓ *span the space* $\vec{V}_\ell$.

Proof. Since $\mathbf{curl}\,S_\ell \subset W_\ell\times W_\ell \cap \vec{H}_{0,\Gamma_N}(\operatorname{div};\Omega)$, obviously all these functions are in $\vec{V}_\ell$.

Now let $\vec{v}_\ell\in\vec{V}_\ell$ be given. Then from $\operatorname{div}\vec{V}_\ell\subset W_\ell$, Corollary 5.4(a), and Proposition 5.3(iii), it follows that there exist $(e_{\vec{\varphi}})_{\vec{\varphi}\in\vec{\Phi}_0}$, and $((d_{\ell,e})_{e\in\mathcal{E}_k^{\rm new}})_{1\le k\le\ell}$, such that

$$\operatorname{div}\vec{v}_\ell = \sum_{\vec{\varphi}\in\vec{\Phi}_0} e_{\vec{\varphi}}\operatorname{div}\vec{\varphi} + \sum_{k=1}^{\ell}\sum_{e\in\mathcal{E}_k^{\rm new}} d_{k,e}2^{-k}\operatorname{div}\vec{\varphi}_{k,e},$$

and so $\vec{v}_\ell - \big(\sum_{\vec{\varphi}\in\vec{\Phi}_0} e_{\vec{\varphi}}\vec{\varphi} + \sum_{k=1}^{\ell}\sum_{e\in\mathcal{E}_k^{\rm new}} d_{k,e}2^{-k}\vec{\varphi}_{k,e}\big) = \mathbf{curl}\,\omega$ for some $\omega\in H^1_{0,\Gamma_N}(\Omega)$. Since $\vec{v}_\ell - \big(\sum_{\vec{\varphi}\in\vec{\Phi}_0} e_{\vec{\varphi}}\vec{\varphi} + \sum_{k=1}^{\ell}\sum_{e\in\mathcal{E}_k^{\rm new}} d_{k,e}2^{-k}\vec{\varphi}_{k,e}\big)\in\vec{V}_\ell\cap\vec{H}(\operatorname{div}0;\Omega)$, it is in $W_\ell\times W_\ell$. This means that ω is piecewise linear w.r.t. $\mathcal{T}_\ell$, and so $\omega\in S_\ell$. □

What is left is the specification of a Riesz basis Σ_N for $H^1_{0,\Gamma_N}(\Omega)$ of the type as in Proposition 5.7. A possibility is a prewavelet basis, i.e., a basis such that $\operatorname{span}\Sigma_N^{(k)} = S_k\cap S_{k-1}^{\perp_{L_2(\Omega)}}$ ($S_{k-1} := \{0\}$). Compactly supported continuous piecewise linear prewavelets for general initial triangulations $\mathcal{T}_0$ were constructed in [Ste98a, FQ99].

A more efficient construction is possible by relaxing the level-wise $L_2(\Omega)$-orthogonality by an orthogonality w.r.t. a discrete inner product. In [Ste98b], the following proposition was proven about the resulting so-called *three-point hierarchical basis*. We refer to [LO00] for results in the shift-invariant case concerning the full range of stability of this basis in the scale of Sobolev spaces.

Theorem 5.8. *Let* $\mathcal{N}_\ell$ *denote the set of nodes of* $\mathcal{T}_\ell$ *that are not on* Γ_N. *For* $v\in\mathcal{N}_\ell$, *let* $\phi_{\ell,v}\in S_\ell$ *be the scaled nodal basis function defined by* $\phi_{\ell,v}(v') = 2^{\ell/2}\delta_{v,v'}$ ($v'\in\mathcal{N}_\ell$). *For* $\ell\ge 1$ *and* $v\in\mathcal{N}_\ell\setminus\mathcal{N}_{\ell-1}$, *define*

$$\sigma_{\ell,v} = \phi_{\ell,v} - \sum_{\{v'\in\mathcal{N}_{\ell-1}\colon |\operatorname{supp}\phi_{\ell,v}\cap\operatorname{supp}\phi_{\ell,v'}|>0\}} \frac{\int_\Omega\phi_{\ell,v}}{2\int_\Omega\phi_{\ell,v'}}\phi_{\ell,v'}.$$

Then, with $\Sigma_N^{(0)}$ *being some basis for* $\mathcal{S}_0$,

$$\Sigma_N := \cup_{\sigma\in\Sigma_N^{(0)}}\sigma + \cup_{\ell\ge1}\{2^{-\ell}\sigma_{\ell,v}\colon v\in\mathcal{N}_\ell\setminus\mathcal{N}_{\ell-1}\}$$

is a Riesz basis for $H^1_{0,\Gamma_N}(\Omega)$ (*and* $\operatorname{span}\Sigma_N^{(0)} + \cup_{\ell\ge1}\{2^{-\ell}\sigma_{\ell,v}\colon v\in\mathcal{N}_\ell\setminus\mathcal{N}_{\ell-1}\} = S_\ell$).

Note that for $v\in\mathcal{N}_\ell\setminus\mathcal{N}_{\ell-1}$, $\#\{v'\in\mathcal{N}_{\ell-1}\colon |\operatorname{supp}\phi_{\ell,v}\cap\operatorname{supp}\phi_{\ell,v'}| > 0\}$ is either 2, see Figure 5, in which case $\int_\Omega\sigma_{\ell,v} = 0$, or less than 2 in case v is the midpoint of an edge of an $T\in\mathcal{T}_{\ell-1}$ that has one or two endpoints on Γ_N.

Remark 5.9. The construction of a Riesz basis for $\vec{H}_{0,\Gamma_N}(\operatorname{div};\Omega)$ outlined in this section does not directly extend to $n=3$, the main obstacle being that the curl operator in three dimension has an infinite dimensional kernel.

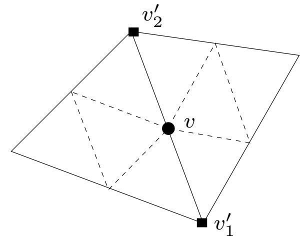

FIGURE 2. A three-point hierarchical basis function associated to $v \in \mathcal{N}_\ell \setminus \mathcal{N}_{\ell-1}$, being a linear combination of the nodal basis functions $\phi_{\ell,v}$, $\phi_{\ell,v_1'}$, and $\phi_{\ell,v_2'}$.

Remark 5.10. As expressed by its title, the second topic of this section is the construction of a Riesz basis for $H^1_{0,\Gamma_D}(\Omega)$. Obviously, the construction of Theorem 5.8 applies with Γ_N replaced by Γ_D, yielding the three-point hierarchical basis Σ_D for $H^1_{0,\Gamma_D}(\Omega)$.

6. Residual evaluation with least squares problems when $K = L_2(\Omega)^N$

In this section, we verify Assumption 2.5 on the cost of the residual evaluation inside the adaptive wavelet Galerkin method, in case this method is applied to a least squares problem, where the residual lives in $L_2(\Omega)^N$ (i.e., the second possibility discussed at the end of Sect. 3).

We consider $G\colon H \supset \operatorname{dom}(G) \to K'$, where for some domain $\Omega \subset \mathbb{R}^n$ and $N \in \mathbb{N}$,

$$K = K' = L_2(\Omega)^N,$$

i.e., $G = (G_1, \ldots, G_N)$, and, for some $M \in \mathbb{N}$ and $m_1, \ldots, m_n \in \mathbb{N}_0$,

$$H = H^{m_1}(\Omega) \times \cdots \times H^{m_M}(\Omega).$$

To verify Assumption 2.5, we will check the conditions of Proposition 3.1, which can be done for each G_i *separately*.

Remark 6.1. The space H is *not* of the type as with our model problem discussed in Sections 3 and 4, where $H = \vec{H}_{0,\Gamma_N}(\operatorname{div};\Omega) \times H^1_{0,\Gamma_D}(\Omega)$. So, some massage will be needed to apply the results that will be derived in the current section to that problem. This will be done in Section 7. At this point, we note that the results from the current section remain valid when, for $1 \le j \le M$, $H^{m_j}(\Omega)$ is replaced by a closed subspace defined by the incorporation of homogeneous boundary conditions.

Generally, the equation $G_i(u) = 0$ is inhomogeneous, and we write

$$G_i(\cdot) = G_i^{\mathrm{h}}(\cdot) - f_i,$$

with the operator G_i^{h} thus being homogeneous, and $f_i \in L_2(\Omega)$.

With, for $1 \le j \le M$, $\Psi^{(j)} = \{\psi^{(j)}_\lambda \colon \lambda \in \nabla^{(j)}\}$ being a Riesz basis of (isotropic) wavelet type for $H^{m_j}(\Omega)$, we select the Riesz basis for H as

$$\vec{\Psi} := \bigcup_{j=1}^{M} \Psi^{(j)} \vec{e}_j. \tag{6.1}$$

We start with collecting a number of (standard) assumptions on $\vec{\Psi}$:

(w_1) There exists a collection $\{\omega_\nu : \nu \in \mathcal{O}\}$ (*independent of* j) of uniformly Lipschitz domains, e.g., simplices or hyperrectangles, such that, with $|\nu| \in \mathbb{N}_0$ being the *level* of ν, $\omega_\nu \cap \omega_\vartheta = \emptyset$ when $|\nu| = |\vartheta|$ and $\nu \neq \vartheta$; for any $\ell \in \mathbb{N}_0$, $\bar\Omega = \cup_{|\nu|=\ell} \bar\omega_\nu$; $\operatorname{diam} \omega_\nu \eqsim 2^{-|\nu|}$; and $\bar\omega_\nu$ is the union of some $\bar\omega_\vartheta$ with $|\vartheta| = |\nu| + 1$.

(w_2) $\operatorname{Supp} \psi_\lambda^{(j)}$ is contained in a connected union of a uniformly bounded number of $\bar\omega_\nu$'s with $|\nu| = |\lambda|$.

(w_3) Each ω_ν is intersected by the supports of a uniformly bounded number of $\psi_\lambda^{(j)}$'s with $|\lambda| = |\nu|$.

(w_4) $\bar\Omega = \cup_{\{\lambda \in \nabla^{(j)} : |\lambda| = \ell\}} \operatorname{supp} \psi_\lambda^{(j)}$.

We assume that $\operatorname{clos}_{H^{m_j}(\Omega)} \operatorname{span}\{\psi_\lambda^{(j)} : |\lambda| \leq \ell\}$ can be equipped with a "single-scale" basis $\Phi_\ell^{(j)} = \{\phi_\lambda^{(j)} : \lambda \in \Delta_\ell^{(j)}\}$. Setting, for $\lambda \in \Delta_\ell^{(j)}$, the level $|\lambda| := \ell$, we assume that

(w_5) $\operatorname{Supp} \phi_\lambda^{(j)}$ is contained in a connected union of a uniformly bounded number of $\bar\omega_\nu$ with $|\nu| = |\lambda|$.

(w_6) Each ω_ν is intersected by the supports of a uniformly bounded number of $\phi_\lambda^{(j)}$ with $|\lambda| = |\nu|$.

(w_7) $\{\phi_\lambda^{(j)}|_{\omega_\nu} : |\lambda| = |\nu|, \phi_\lambda^{(j)}|_{\omega_\nu} \neq 0\}$ is independent.

W.l.o.g. we assume that $\Delta_\ell^{(j)} \cap \Delta_{\ell'}^{(j)} = \emptyset$ when $\ell \neq \ell'$, and set

$$\Delta^{(j)} = \cup_{\ell \in \mathbb{N}_0} \Delta_\ell^{(j)},$$

being the index set of all "scaling functions" over all levels.

Although this has not yet been imposed, the idea behind the introduction of the subdomains ω_ν is that the functions from $\Phi_\ell^{(j)}$ are *piecewise smooth* w.r.t. the partition $\bar\Omega = \cup_{|\nu|=\ell} \bar\omega_\nu$.

To be able to find, in linear complexity, a representation of a function, given as linear combination of wavelets, in terms of a locally finite set of scaling functions –this in view of an efficient evaluation of nonlinear terms–,we will impose a tree condition on the underlying set of wavelet indices. A similar approach was followed earlier in [DSX00, CDD03b, XZ05, BU08, Vor09].

Definition 6.2. To each $\lambda \in \nabla^{(j)}$ with $|\lambda| > 0$, we associate one $\mu \in \nabla^{(j)}$ with $|\mu| = |\lambda| - 1$ and $|\operatorname{supp} \psi_\lambda^{(j)} \cap \operatorname{supp} \psi_\mu^{(j)}| > 0$. We will call μ the *parent* of λ and so λ a *child* of μ.

We call $\Lambda \subset \nabla^{(j)}$ a *tree* when it contains $\{\lambda \in \nabla^{(j)} : |\lambda| = 0\}$, and when the parent of any $\lambda \in \Lambda$ with $|\lambda| > 0$ is in Λ.

Definition 6.3. A collection $\mathcal{T} \subset \{\omega_\nu : \nu \in \mathcal{O}\}$ such that any two domains from $\mathcal{T}$ have empty intersection and $\bar\Omega = \cup_{\omega_\nu \in \mathcal{T}} \bar\omega_\nu$ will be called a *tiling* (of Ω). The smallest common refinement of tilings $\mathcal{T}_1$ and $\mathcal{T}_2$ will be denoted as $\mathcal{T}_1 \oplus \mathcal{T}_2$.

A proof of the following proposition, as well as an algorithm to apply the multi-to-single-scale transformation $\mathbf{T}_\Lambda^{(j)}$ that is mentioned, is given in [Ste14, §4.3].

Proposition 6.4. *Given a finite tree* $\Lambda \subset \nabla^{(j)}$*, there exists a* $\Delta_\Lambda^{(j)} \subset \Delta^{(j)}$ *such that*

(i) $\operatorname{span}\{\psi_\lambda^{(j)} : \lambda \in \Lambda\} \subset \operatorname{span}\{\phi_\lambda^{(j)} : \lambda \in \Delta_\Lambda^{(j)}\}$,

(ii) *the representation of this embedding w.r.t. the collections* $\{\psi_\lambda^{(j)} : \lambda \in \Lambda\}$ *and* $\{\phi_\lambda^{(j)} : \lambda \in \Delta_\Lambda^{(j)}\}$*, denoted as* $\mathbf{T}_\Lambda^{(j)}$*, can be applied to any vector in* $\ell_2(\Lambda)$ *in* $\mathcal{O}(\#\Lambda)$ *operations* (*and so,* in particular, $\#\Delta_\Lambda^{(j)} \lesssim \#\Lambda$),

(iii) *the difference in levels of any two functions from* $\{\phi_\lambda^{(j)} : \lambda \in \Delta_\Lambda^{(j)}\}$ *whose supports have non-empty intersection is uniformly bounded, say by a constant* L.

(iv) *there exists a tiling* $\mathcal{T}_\Lambda^{(j)}$ *with* $\#\mathcal{T}_\Lambda^{(j)} \lesssim \#\Lambda$*, such that if for* $\lambda \in \Delta_\Lambda^{(j)}$ *and* $\omega_\nu \in \mathcal{T}_\Lambda^{(j)}$, $\operatorname{supp} \phi_\lambda^{(j)} \cap \omega_\nu \neq \emptyset$*, then* $|\nu| - L \leq |\lambda| \leq |\nu|$.

When the $\Phi_\ell^{(j)}$ are indeed piecewise smooth w.r.t. $\bar{\Omega} = \cup_{|\nu|=\ell}\bar{\omega}_\nu$, the last inequality in (iv) means that $\phi_\lambda^{(j)}$ for $\lambda \in \Delta_\Lambda^{(j)}$, and thus $\psi_\lambda^{(j)}$ for $\lambda \in \Lambda^{(j)}$, are *piecewise smooth* w.r.t. $\mathcal{T}_\Lambda^{(j)}$; whereas the one but last inequality means that $\mathcal{T}_\Lambda^{(j)}$ is not unnecessarily fine for having this property.

W.l.o.g. we assume that for $1 \leq j \neq \jmath \leq M$, $\nabla^{(j)} \cap \nabla^{(\jmath)} = \emptyset$ and $\Delta^{(j)} \cap \Delta^{(\jmath)} = \emptyset$, and set

$$\nabla := \cup_{j=1}^M \nabla^{(j)}, \quad \Delta := \cup_{j=1}^M \Delta^{(j)},$$

and for $\lambda \in \nabla$, $\mu \in \Delta$,

$$\vec{\psi}_\lambda := \psi_\lambda^{(j)} \vec{e}_j \text{ when } \lambda \in \nabla^{(j)}, \quad \vec{\phi}_\mu := \phi_\mu^{(j)} \vec{e}_j \text{ when } \mu \in \Delta^{(j)}.$$

The basis for H from (6.1) now reads as

$$\vec{\Psi} = \{\vec{\psi}_\lambda : \lambda \in \nabla\}.$$

Definition 6.5. We define a finite $\Lambda \subset \nabla$ to be *admissible*, cf. (2.2), when $\Lambda = \cup_{j=1}^M \Lambda_j$ with $\Lambda_j \subset \nabla^{(j)}$ being finite trees.

Remark 6.6. Equipping ∇ with a tree structure by calling μ a parent of λ when, for some $1 \leq j \leq M$, $\mu, \lambda \in \nabla^{(j)}$ and μ is the parent of λ w.r.t. the tree structure on $\nabla^{(j)}$, $\Lambda \subset \nabla$ being admissible just means that Λ is a finite tree.

Given an admissible $\Lambda = \cup_{j=1}^M \Lambda_j \subset \nabla$, setting

$$\Delta_\Lambda := \cup_{j=1}^M \Delta_{\Lambda_j}^{(j)}$$

(cf. Proposition 6.4), we have

$$\vec{\mathcal{S}}_\Lambda := \operatorname{span}\{\vec{\psi}_\lambda : \lambda \in \Lambda\} \subset \operatorname{span}\{\vec{\phi}_\mu : \mu \in \Delta_\Lambda\}.$$

The representation of this embedding w.r.t. $\{\vec{\psi}_\lambda : \lambda \in \Lambda\}$ and $\{\vec{\phi}_\mu : \mu \in \Delta_\Lambda\}$ will be denoted as $\mathbf{T}_\Lambda$. With the $\mathcal{T}_{\Lambda_j}^{(j)}$ as defined in Proposition 6.4(iv), we define the tiling

$$\mathcal{T}_\Lambda := \oplus_{j=1}^M \mathcal{T}_{\Lambda_j}^{(j)}. \tag{6.2}$$

For $\Lambda \subset \nabla$, we define $I_\Lambda \in B(\ell_2(\Lambda), \ell_2(\nabla))$ as the extension with zeros, so that its adjoint I'_Λ restricts a vector in $\ell_2(\nabla)$ to its indices in Λ.

The idea behind the approximate evaluation of

$$\mathcal{F} DG_i(\mathcal{F}'\mathbf{w}_\Lambda)'(G_i^{\mathrm{h}}(\mathcal{F}'\mathbf{w}_\Lambda) - f_i) = \left[\left\langle G_i^{\mathrm{h}}(\mathcal{F}'\mathbf{w}_\Lambda) - f_i, DG_i(\mathcal{F}'\mathbf{w}_\Lambda)\vec{\psi}_\lambda\right\rangle_{L_2(\Omega)}\right]_{\lambda\in\nabla},$$

for given admissible $\Lambda \subset \nabla$ and $\mathbf{w}_\Lambda \in \ell_2(\Lambda)$, which will be outlined in Theorem 6.9 and Corollary 6.10, can be sketched as follows: With common wavelet constructions,

$\mathcal{F}'\mathbf{w}_\Lambda$ is piecewise smooth, e.g. polynomial, w.r.t. $\mathcal{T}_\Lambda$, and wavelets with supports not at $\partial\Omega$ have a vanishing moment (forthcoming condition (w_8)). Under mild conditions on G_i^{h}, $G_i^{\mathrm{h}}(\mathcal{F}'\mathbf{w}_\Lambda)$ inherits some piecewise smoothness w.r.t. $\mathcal{T}_\Lambda$, as well as some higher order global integrability (i.e., membership of $L_{\xi'}(\Omega)$ for some $\xi' > 2$) from its argument.

Now given $\eta > 0$ and $\varepsilon > 0$ as in Proposition 3.1, the first step is to approximate the forcing function $f_i \in L_2(\Omega)$ within tolerance ε by a function $\tilde{f}_i$ that is piecewise smooth, e.g. polynomial, w.r.t. a tiling $\mathcal{T}_\varepsilon$, say with $\#\mathcal{T}_\varepsilon \lesssim \varepsilon^{-1/\tilde{s}}$. Then what is left to approximate is $\mathcal{F}DG_i(\mathcal{F}'\mathbf{w}_\Lambda)'(G_i^{\mathrm{h}}(\mathcal{F}'\mathbf{w}_\Lambda) - \tilde{f}_i)$, i.e.,

$$[\langle G_i^{\mathrm{h}}(\mathcal{F}'\mathbf{w}_\Lambda) - \tilde{f}_i, DG_i(\mathcal{F}'\mathbf{w}_\Lambda)\vec{\psi}_\lambda\rangle_{L_2(\Omega)}]_{\lambda\in\nabla}$$

within tolerance $\eta\|G_i^{\mathrm{h}}(\mathcal{F}'\mathbf{w}_\Lambda) - \tilde{f}_i\|_{L_2(\Omega)}$.

Let us for the moment think of $H = L_2(\Omega)$, and $DG_i(\mathcal{F}'\mathbf{w}_\Lambda)$ being the identity. For the general case, we will use locality of $DG_i(\mathcal{F}'\mathbf{w}_\Lambda)$, and impose some mild additional continuity conditions (conditions (1) and (2) of Thm. 6.9). Let us consider all entries of $[\cdots]_\lambda$ for which, for some *fixed*, sufficienty large $k \in \mathbb{N}_0$, $|\lambda| > |\nu| + k$ for any $\nu \in \mathcal{T}_\Lambda \oplus \mathcal{T}_\varepsilon$ with $|\operatorname{supp}\psi_\lambda \cap \omega_\nu| > 0$.

For those entries for which additionally both $\operatorname{supp}\psi_\lambda \subset \bar{\omega}_\nu$ for some $\nu \in \mathcal{T}_\Lambda \oplus \mathcal{T}_\varepsilon$, and ψ_λ has a vanishing moment, the smoothness of $(G_i^{\mathrm{h}}(\mathcal{F}'\mathbf{w}_\Lambda) - \tilde{f}_i)|_{\omega_\nu}$ (condition (4) of Thm. 6.9), together with $\int_\Omega \psi_\lambda = 0$ show that their total contribution is less or equal to $\frac{\eta}{2}\|G_i^{\mathrm{h}}(\mathcal{F}'\mathbf{w}_\Lambda) - \tilde{f}_i\|_{L_2(\Omega)}$, assuming k be sufficiently large.

For those entries that do not satisfy these additional conditions, either $\operatorname{supp}\psi_\lambda$ is at the boundary, or $(\operatorname{supp}\psi_\lambda)^{\mathrm{int}}$ intersects $\partial\omega_\nu$ for a $\nu \in \mathcal{T}_\Lambda \oplus \mathcal{T}_\varepsilon$. Together, the fact that, with $\frac{1}{\xi} + \frac{1}{\xi'} = 1$, $\|\psi_\lambda\|_{L_\xi(\Omega)} \lesssim 2^{\lambda(\frac{n}{2} - \frac{n}{\xi})}$ by Hölder's inequality, the higher order integrability of $G_i^{\mathrm{h}}(\mathcal{F}'\mathbf{w}_\Lambda) - \tilde{f}_i$ (condition (5) of Thm. 6.9), and $\partial\omega_\nu$ being a lower dimensional manifold, show that also the total contribution from these entries is less or equal to $\frac{\eta}{2}\|G_i^{\mathrm{h}}(\mathcal{F}'\mathbf{w}_\Lambda) - \tilde{f}_i\|_{L_2(\Omega)}$, assuming k being sufficiently large.

Extending the set of all remaining indices to an admissible set, which has cardinality $\mathcal{O}(\#\Lambda + \varepsilon^{-1/\tilde{s}})$, and computing the corresponding entries $[\cdots]_\lambda$, first in terms of a locally finite set of single scale coordinates, and then by applying a locally single-scale to multi-scale transformation, the cost of computing them is $\mathcal{O}(\varepsilon^{-1/\tilde{s}} + \#\Lambda)$, meaning that Assumption 2.5 is satisfied when $\tilde{s} \geq s$.

This sketch of the approximate residual evaluation motivates the following definition.

Definition 6.7. Given a tiling $\mathcal{T}$ and a $k \in \mathbb{N}_0$, we set

$$\begin{cases} \Lambda_{\mathcal{T},k} \subset \nabla \text{ to be the smallest enlargement to an admissible set} \\ \text{of } \{\lambda \in \nabla \colon \exists \omega_\nu \in \mathcal{T} \text{ with } |\operatorname{supp}\psi_\lambda \cap \omega_\nu| > 0 \wedge |\lambda| \leq |\nu| + k\}. \end{cases} \tag{6.3}$$

For a proof of the next proposition, one may consult [Ste14, Prop. 4.15].

Proposition 6.8. *For a tiling $\mathcal{T}$ and $k \in \mathbb{N}_0$, $\#\Lambda_{\mathcal{T},k} \lesssim \#\mathcal{T}$ (dependent on k).*

To continue, we add one more wavelet assumption:

(w_8) $\int_\Omega \psi_\lambda^{(j)} dx = 0$, possibly with the exception of those λ with $\operatorname{dist}(\operatorname{supp}\psi_\lambda^{(j)}, \partial\Omega) \lesssim 2^{-|\lambda|}$, or with $|\lambda| = 0$.

In the next theorem, one could think of the tiling $\mathcal{T} \supset \mathcal{T}_\Lambda$, and the function g, as being $\mathcal{T}_\Lambda \oplus \mathcal{T}_\varepsilon$ and $\tilde{f}_i$, respectively. They will be specified in Corollary 6.10.

Theorem 6.9 ([Ste14, Thm. 4.16])**.** (a). *Let $G_i^{\rm h}$ be a* local *operator, meaning that $G_i^{\rm h}(\vec{w})(x)$ depends only on $\vec{w}$ and derivatives of $\vec{w}$ at x. For some $\theta \in (0,1]$ and $\xi \in [1,2)$, and with $\frac{1}{\xi}+\frac{1}{\xi'}=1$, we assume that, uniformly for $\vec{w}$ in some neighborhood U of the solution $\vec{u}$ of $G(\vec{u})=0$ in $H^{m_1}(\Omega)\times\cdots\times H^{m_M}(\Omega)$,*

(1) $D_jG_i(\vec{w}) \in B(W_\xi^{m_j}(\Omega), L_\xi(\Omega))$,

(2) $D_jG_i(\vec{w})|_{H_0^{m_j}(\Omega)}$ *has an extension to a mapping in* $B(H_0^{m_j-\theta}(\Omega), H^\theta(\Omega)')$, *where, relevant for the case $m_j=0$, $H_0^s(\Omega)$ for $s<0$ should be read as $H^{-s}(\Omega)'$.*

Furthermore, we assume that for any admissible $\Lambda\subset\nabla$, tilings $\mathcal{T}\supseteq\mathcal{T}_\Lambda$, $\omega_\nu\in\mathcal{T}$, $g\in V(\omega_\nu)$ –being some function space on ω_ν–, and $\vec{w}_\Lambda\in\vec{\mathcal{S}}_\Lambda$,

(3) $\|G_i^{\rm h}(\vec{w}_\Lambda)-g\|_{L_{\xi'}(\omega_\nu)} \lesssim 2^{|\nu|(\frac{n}{2}-\frac{n}{\xi'})}\|G_i^{\rm h}(\vec{w}_\Lambda)-g\|_{L_2(\omega_\nu)}$,

(4) $\|G_i^{\rm h}(\vec{w}_\Lambda)-g\|_{H^\theta(\omega_\nu)} \lesssim 2^{|\nu|\theta}\|G_i^{\rm h}(\vec{w}_\Lambda)-g\|_{L_2(\omega_\nu)}$.

Then for any $\eta>0$, there exists a $k\in\mathbb{N}_0$ such that for any admissible $\Lambda\subset\nabla$, $\vec{w}_\Lambda\in\vec{\mathcal{S}}_\Lambda\cap U$, any tiling $\mathcal{T}\supseteq\mathcal{T}_\Lambda$, and $g\in\prod_{\omega_\nu\in\mathcal{T}}V(\omega_\nu)$,

$$\|(\mathrm{Id}-I_{\Lambda_{\mathcal{T},k}}I'_{\Lambda_{\mathcal{T},k}})\mathcal{F}DG_i(\vec{w}_\Lambda)'(G_i^{\rm h}(\vec{w}_\Lambda)-g)\| \le \eta\|G_i^{\rm h}(\vec{w}_\Lambda)-g\|_{L_2(\Omega)}. \tag{6.4}$$

(b). *Let*

(5) $\sup_{\nu\in\mathcal{O}}\dim V(\omega_\nu)<\infty$, $V(\omega_\nu)\subset\prod_{\{|\vartheta|=|\nu|+1:\,\bar{\omega}_\vartheta\subset\bar{\omega}_\nu\}}V(\omega_\vartheta)$.

Assume that for any $\nu\in\mathcal{O}$, $\mu\in\Delta$ with $|\mu|=|\nu|$, $\vec{v}\in\mathrm{span}\{\vec{\phi}_\gamma\colon\gamma\in\Delta,|\gamma|=|\nu|\}$, and $g\in V(\omega_\nu)$,

(6) $\langle G_i^{\rm h}(\vec{v})-g, DG_i(\vec{v})\vec{\phi}_\mu\rangle_{L_2(\omega_\nu)}$ *can be computed in $\mathcal{O}(1)$ operations.*

Then, for $\vec{w}_\Lambda$ and g as in (a)*, by expressing $\vec{w}_\Lambda=\mathbf{w}_\Lambda^\top\Psi$, where $\mathbf{w}_\Lambda\in\ell_2(\Lambda)$, w.r.t. $\{\vec{\phi}_\gamma\colon\gamma\in\Delta_{\Lambda_{\mathcal{T},k}}\}$ by an application of $\mathbf{T}_{\Lambda_{\mathcal{T},k}}$ to $\mathbf{w}_\Lambda$, the latter extended with zeros on $\Lambda_{\mathcal{T},k}\backslash\Lambda$; and for any $\omega_\nu\in\mathcal{T}_{\Lambda_{\mathcal{T},k}}$, by expressing $g|_{\omega_\nu}$ w.r.t. some basis of $V(\omega_\nu)$, and subsequently by computing the approximate residual $I'_{\Lambda_{\mathcal{T},k}}\mathcal{F}DG_i(\vec{w}_\Lambda)'(G_i^{\rm h}(\vec{w}_\Lambda)-g)$ as*

$$\mathbf{T}_{\Lambda_{\mathcal{T},k}}^\top\Big[\big\langle G_i^{\rm h}(\vec{w}_\Lambda)-g, DG_i(\vec{w}_\Lambda)\vec{\phi}_\mu\big\rangle_{L_2(\Omega)}\Big]_{\mu\in\Delta_{\Lambda_{\mathcal{T},k}}}, \tag{6.5}$$

its evaluation takes $\mathcal{O}(\#\mathcal{T})$ operations.

Corollary 6.10. *Assume the conditions of Theorem 6.9, and that f_i is such that for some $\tilde{s}\ge s$, for any $\varepsilon>0$ a tiling $\mathcal{T}_\varepsilon$ and an $\widetilde{f}_i\in\prod_{\omega_\nu\in\mathcal{T}_\varepsilon}V(\omega_\nu)$ can be found with $\#\mathcal{T}_\varepsilon\lesssim\varepsilon^{-1/s}$ and $\|f_i-\widetilde{f}_i\|_{L_2(\Omega)}\le\varepsilon$. Then by applying Theorem 6.9 with $\mathcal{T}=\mathcal{T}_\Lambda\oplus\mathcal{T}_\varepsilon$ and $g=\widetilde{f}_i$, the conditions of Proposition 3.1 on the approximate evaluation of $\mathcal{F}DG_i(\vec{w}_\Lambda)'(G_i^{\rm h}(\vec{w}_\Lambda)-f_i)$ are fulfilled, and so Assumption 2.5 is valid, with the bound for the cost even reading as $\mathcal{O}(\#\Lambda+\varepsilon^{-1/\tilde{s}})$.*

Proof. The first condition of Proposition 3.1 reads as $\|f_i-\widetilde{f}_i\|_{L_2(\Omega)}\le\varepsilon$, and the second one is (6.4). The condition of the number of operations is a consequence of $\#\mathcal{T}\le\#\mathcal{T}_\Lambda+\#\mathcal{T}_\varepsilon\lesssim\#\Lambda+\varepsilon^{-1/\tilde{s}}$ by an application of Proposition 6.4(iv), in combination with the last statement of Theorem 6.9. □

We conclude that under the conditions of Corollary 6.10, Theorem 2.6 shows that the **awgm** produces a sequence of approximations to the solution u of the least squares problem (3.4) from the span of the Riesz basis for H, with K and

H as in this current section, that converge with the best possible (constrained) approximation rate in optimal computational complexity.

7. Residual evaluation for the least squares problems from Sect. 4

For G from (4.4), and with the Riesz bases as constructed in Sect. 5, in this section, using the techniques from the previous section, we verify the conditions of Proposition 3.1 on the approximate residual evaluation. Although we consider general space dimensions n, recall that our current construction of a basis for $\vec{H}_{0,\Gamma_N}(\mathrm{div};\Omega)$ only applies for $n=2$.

Writing $A=[\vec{a}_1\cdots\vec{a}_n]$, for $(\vec{v},q)\in\vec{H}_{0,\Gamma_N}(\mathrm{div};\Omega)\times H^1_{0,\Gamma_D}(\Omega)$, we have

$$G_i(\vec{v},q)=\begin{cases} v_i-\vec{a}_i\cdot\nabla q & \text{when } 1\leq i\leq n,\\ N(q)-\mathrm{div}\,\vec{v}-f & \text{when } i=n+1,\end{cases}$$

and so, for $(\vec{z},r)\in\vec{H}_{0,\Gamma_N}(\mathrm{div};\Omega)\times H^1_{0,\Gamma_D}(\Omega)$,

$$\begin{aligned}((DG_i(\vec{v},q)'G_i(\vec{v},q))(\vec{z},r)&=\langle G_i(\vec{v},q),DG_i(\vec{v},q)(\vec{z},r)\rangle_{L_2(\Omega)}\\ &=\begin{cases}\langle v_i-\vec{a}_i\cdot\nabla q, z_i-\vec{a}_i\cdot\nabla r\rangle_{L_2(\Omega)} & \text{when } 1\leq i\leq n,\\ \langle N(q)-\mathrm{div}\,\vec{v}-f, DN(q)r-\mathrm{div}\,\vec{z}\rangle_{L_2(\Omega)} & \text{when } i=n+1,\end{cases}\end{aligned}$$

We will assume that the $\vec{a}_i$ are piecewise polynomial w.r.t. the initial triangulation $\mathcal{T}_0$, and, although actually not necessary, that they are globally continuous.

Let $\vec{\Psi}^{(1)}=\{\vec{\psi}^{(1)}_\lambda\colon\lambda\in\nabla^{(1)}\}$ be the Riesz basis for $\vec{H}_{0,\Gamma_N}(\mathrm{div};\Omega)$ as introduced in Section 5. It is the union of $\mathbf{curl}\,\Sigma_N$, where Σ_N is the three-point hierarchical basis for $H^1_{0,\Gamma_N}(\Omega)$ constructed in Theorem 5.8, and $\cup_{\vec{\varphi}\in\vec{\Phi}_0}\vec{\varphi}+\cup_{\ell\geq 1}\{2^{-\ell}\vec{\varphi}_{\ell,e}\colon e\in\mathcal{E}^{\mathrm{new}}_\ell\}$ as defined in Lemma 5.1 (with $\vec{\Phi}_0$ some basis of some complement space of the divergence-free subspace of the lowest order Raviart-Thomas space w.r.t $\mathcal{T}_0$). Let $\nabla^{(1)}=\nabla^{(1,A)}\cup\nabla^{(1,B)}$ be the corresponding splitting of the index set.

Let $\Psi^{(2)}=\{\psi^{(2)}_\lambda\colon\lambda\in\nabla^{(2)}\}$ be the three-point hierarchical Riesz basis Σ_D for $H^1_{0,\Gamma_D}(\Omega)$ as introduced in Remark 5.10 at the end of Section 5.

Although elements of $\vec{\Psi}^{(1)}$ are vector-valued, Definition 6.2 of a subset $\Lambda\subset\nabla^{(1)}$ being a tree applies without modification, and so does Definition 6.5 of a subset $\Lambda\subset\nabla:=\nabla^{(1)}\cup\nabla^{(2)}$ being admissible. Also Proposition 6.4 applies, with the tiling from (6.2) now being a subset of the infinite union of triangulations created by red-refinement starting from some initial conforming triangulation $\mathcal{T}_0$ of Ω.

Let $\Lambda\subset\nabla$ be admissible, i.e., $\Lambda=\Lambda^{(1)}\cup\Lambda^{(2)}$ with $\Lambda^{(1)}\subset\nabla^{(1)}$ and $\Lambda^{(2)}\subset\nabla^{(2)}$ being trees. Let $\mathbf{v}_{\Lambda^{(1)}}\in\ell_2(\Lambda^{(1)})$ and $\mathbf{q}_{\Lambda^{(2)}}\in\ell_2(\Lambda^{(2)})$. Let $\mathcal{T}_\Lambda$ be the tiling as in (6.2), so that $\vec{v}_{\Lambda^{(1)}}:=\mathbf{v}^\top_{\Lambda^{(1)}}\vec{\Psi}^{(1)}$ and $q_{\Lambda^{(2)}}:=\mathbf{q}^\top_{\Lambda^{(2)}}\Psi^{(2)}$ are piecewise linear w.r.t $\mathcal{T}_\Lambda$. Given $\varepsilon>0$, let $\tilde{f}$ be a piecewise polynomial, of some fixed degree, w.r.t. a tiling $\mathcal{T}_\varepsilon$, with $\#\mathcal{T}_\varepsilon\lesssim\varepsilon^{-1/\tilde{s}}$, such that $\|f-\tilde{f}\|_{L_2(\Omega)}\leq\varepsilon$. With $\mathcal{T}:=\mathcal{T}_\Lambda\oplus\mathcal{T}_\varepsilon$, and $k\in\mathbb{N}$, let $\Lambda_{\mathcal{T},k}\subset\nabla$ be as in (6.3). Given an $\eta>0$, we have to show that for k being a sufficienty large constant,

$$\begin{aligned}\|(\mathrm{Id}-I_{\Lambda_{\mathcal{T},k}}I'_{\Lambda_{\mathcal{T},k}})\mathcal{F}DG_i(\vec{v}_{\Lambda^{(1)}},q_{\Lambda^{(2)}})'(G^{\mathrm{h}}_i(\vec{v}_{\Lambda^{(1)}},q_{\Lambda^{(2)}})-\tilde{f}_i)\|\\ \leq\eta\|G^{\mathrm{h}}_i(\vec{v}_{\Lambda^{(1)}},q_{\Lambda^{(2)}})-\tilde{f}_i\|_{L_2(\Omega)},\end{aligned}\tag{7.1}$$

where for $1\leq i\leq n$, $\tilde{f}_i:=0$ and $G^{\mathrm{h}}_i:=G_i$, and $\tilde{f}_{n+1}:=\tilde{f}$ and $G^{\mathrm{h}}_{n+1}:=G_{n+1}+f$.

Because of the occurrence of the vector-valued Sobolev space $\vec{H}_{0,\Gamma_N}(\mathrm{div};\Omega)$, and the fact that all basis functions of type $2^{-\ell}\vec{\varphi}_{\ell,e}$ have no vanishing moment, i.e., (w_8) is not valid, Theorem 6.9 cannot be applied directly. We will be saved by the facts that $\mathrm{div}\, 2^{-\ell}\vec{\varphi}_{\ell,e}$ has a vanishing moment, and $\|2^{-\ell}\vec{\varphi}_{\ell,e}\|_{L_2(\Omega)^n} \lesssim 2^{-\ell}$.

Corresponding to the splitting of ∇ into $\nabla^{(1,A)} \cup \nabla^{(1,B)} \cup \nabla^{(2)}$, we split $\Lambda_{\mathcal{T},k}$ into $\Lambda^{(1,A)}_{\mathcal{T},k}$, $\Lambda^{(1,B)}_{\mathcal{T},k}$, and $\Lambda^{(2)}_{\mathcal{T},k}$, and prove (7.1) for $1 \leq i \leq n+1$, with $\mathrm{Id} - I_{\Lambda_{\mathcal{T},k}} I'_{\Lambda_{\mathcal{T},k}}$ reading as the restriction of the wavelets indices from ∇ to those from subsequently $\nabla^{(1,A)} \setminus \Lambda^{(1,A)}_{\mathcal{T},k}$ (case$_{1,A}$), $\nabla^{1,B} \setminus \Lambda^{(1,B)}_{\mathcal{T},k}$ (case$_{1,B}$), and $\nabla^{(2)} \setminus \Lambda^{(2)}_{\mathcal{T},k}$ (case$_2$).

For $1 \leq i \leq n$, with some massage, case$_{1,A}$ fits into the setting analyzed in Theorem 6.9, when, in this theorem, we substitute "$H^{m_i}(\Omega)$"$= H^1_{0,\Gamma_N}(\Omega)$, equipped with Riesz basis Σ_N, "$G^{\mathrm{h}}_i(\vec{w}_\Lambda) - g$"$= (\vec{v}_{\Lambda^{(1)}})_i - \vec{a}_i \cdot \nabla q_{\Lambda^{(2)}}$, and "$DG_i(\vec{w})$"$\colon H^1_{0,\Gamma_N}(\Omega) \to L_2(\Omega)\colon \sigma \mapsto (\mathbf{curl}\,\sigma)_i$. Thanks to the assumption of the $\vec{a}_i$ being piecewise polynomial w.r.t $\mathcal{T}_0$, the conditions (3) and (4) are standard inverse inequalities on spaces of polynomials. Since (1) and (2) are also valid, case$_{1,A}$ follows for $1 \leq i \leq n$.

Since the levels of the wavelets in $\nabla^{(1,B)} \setminus \Lambda^{(1,B)}_{\mathcal{T},k}$ are in any case greater or equal to k, from $\|((\langle\cdot, 2^{-\ell}(\boldsymbol{\varphi}_{\ell,e})_i\rangle_{L_2(\Omega)})_{e\in\mathcal{E}^{\mathrm{new}}_\ell})_{\ell\geq k}\| \lesssim 2^{-k}\|\cdot\|_{L_2(\Omega)}$, being a consequence of Lemma 5.1, case$_{1,B}$ follows for $1 \leq i \leq n$.

For $1 \leq i \leq n$, also case$_2$ fits into the setting analyzed in Theorem 6.9, when we substitute "$H^{m_i}(\Omega)$"$= H^1_{0,\Gamma_D}(\Omega)$, equipped with Riesz basis Σ_D, "$G^{\mathrm{h}}_i(\vec{w}_\Lambda) - g$"$= (\vec{v}_{\Lambda^{(1)}})_i - \vec{a}_i \cdot \nabla q_{\Lambda^{(2)}}$, and "$DG_i(\vec{w})$"$\colon H^1_{0,\Gamma_D}(\Omega) \to L_2(\Omega)\colon r \mapsto \vec{a}_i \cdot \nabla r$. Conditions (1)-(4) are valid, with the first two following from $\vec{a}_i \in H^1(\Omega)$ (actually, inspection of the proof of Theorem 6.9 reveals that this global smoothness condition on $\vec{a}_i$ can be avoided). We conclude that case$_2$ is valid for $1 \leq i \leq n$.

Since the elements of $\mathbf{curl}\,\Sigma_N$ are divergence-free, case$_{1,A}$ is obviously valid for $i = n+1$.

Case$_{1,B}$ for $i = n+1$ fits into the setting of Theorem 6.9, when we substitute "$H^{m_{n+1}}(\Omega)$"$= L_2(\Omega)$, equipped with Riesz basis $\cup_{\vec{\varphi}\in\vec{\Phi}_0} \mathrm{div}\,\vec{\varphi} + \cup_{\ell\geq 1}\{2^{-\ell}\,\mathrm{div}\,\vec{\varphi}_{\ell,e}\colon e \in \mathcal{E}^{\mathrm{new}}_\ell\}$, "$G^{\mathrm{h}}_i(\vec{w}_\Lambda) - g$"$= N(q_{\Lambda^{(2)}}) - \mathrm{div}\,\vec{v}_{\Lambda^{(1)}} - \tilde{f}$ and "$DG_i(\vec{w})$"$=$ Id. Obviously (1) and (2) are valid.

In view of the selection of the basis for $\vec{H}_{0,\Gamma_N}(\mathrm{div};\Omega) \times H^1_{0,\Gamma_D}(\Omega)$, we cannot expect the solution $(\mathbf{u}, p)$ to be in $\mathcal{A}^s$ for an $s > 1$. Therefore, it is sufficient to consider $\tilde{f}$ being piecewise constant w.r.t. the tiling $\mathcal{T}$, meaning that $\tilde{s}$ cannot be expected to exceed 1, but, on the other hand, that $\tilde{s} = 1$ under the mild Besov smoothness condition $f \in B^1_\varrho(L_\tau(\Omega))$ for arbitrary $\varrho > 0$ and $\frac{1}{\tau} < \frac{1}{n} + \frac{1}{2}$ ([BDDP02]).

Restricting to local operators N, (3) and (4), and so case$_{1,B}$ for $i = n+1$, will follow from

$$\|N(z) - g\|_{L_{\xi'}(T)} \lesssim \mathrm{diam}(T)^{\frac{n}{2}-\frac{n}{\xi'}}\|N(z) - g\|_{L_2(T)},$$
$$\|N(z) - g\|_{H^\theta(T)} \lesssim \mathrm{diam}(T)^{-\theta}\|N(z) - g\|_{L_2(T)},$$

for some $\xi' > 2$ and $\theta > 0$, uniformly in $T \in \cup_\ell \mathcal{T}_\ell$, $z \in P_1(T)$, $g \in P_0(T)$. By a transformation of variables, and an application of Sobolev's inequality, these estimates are reduced to the condition that on some reference triangle T,

$$|N(z)|_{H^1(T)} \lesssim \|N(z) - g\|_{L_2(T)} \quad (z \in P_1(T),\ g \in P_0(T)). \tag{7.2}$$

Case$_2$ for $i = n+1$ fits into the setting of Theorem 6.9, when we substitute "$H^{m_{n+1}}(\Omega)$"$= H^1_{0,\Gamma_D}(\Omega)$, equipped with Riesz basis Σ_D, "$G^{\mathrm{h}}_i(\vec{w}_\Lambda) - g$"$= N(q_{\Lambda^{(2)}}) -$

$\operatorname{div} \vec{v}_{\Lambda^{(1)}} - \tilde{f}$ and "$DG_i(\vec{w})$": $H^1_{0,\Gamma_D}(\Omega) \to L_2(\Omega)\colon r \mapsto DN(q_{\Lambda^{(2)}})r$. Conditions (1) and (2) will follow from

$$r \mapsto DN(q)r \in B(L_2(\Omega), H^1(\Omega)'), \tag{7.3}$$

$$r \mapsto DN(q)r \in B(W^1_\xi(\Omega), L_\xi(\Omega)), \tag{7.4}$$

for $\frac{1}{\xi'} + \frac{1}{\xi} = 1$, i.e., $\xi < 2$, uniformly in q in some neighbourhood of the solution $p \in H^1_{0,\Gamma_D}(\Omega)$; and (3) and (4) will follow from (7.2).

For our examples of mappings N considered in §4.1 and 4.2, we will verify (7.2)-(7.4) in the next two subsections.

7.1. **Verification of** (7.2)**-**(7.4) **for** $N(q) = q^3$ **and** $n \leq 3$**.** Obviously, in this case (7.2) follows from the equivalence of norms on finite dimensional spaces.

For $\frac{1}{t} + \frac{1}{t'} = 1$, and any $\xi \geq 1$, Hölder's inequality gives

$$\Big(\int_\Omega |3q^2 r|^\xi\Big)^{\frac{1}{\xi}} \leq 3\Big(\int_\Omega |q^2|^{\xi t}\Big)^{\frac{1}{\xi t}}\Big(\int_\Omega |r|^{\xi t'}\Big)^{\frac{1}{\xi t'}}.$$

Sobolev's inequality shows that for $\frac{1}{\varrho} \geq \frac{1}{\xi} - \frac{1}{n}$, $W^1_\xi(\Omega) \hookrightarrow L_\varrho(\Omega)$. So taking $\frac{1}{\xi t'} = \frac{1}{\xi} - \frac{1}{n}$, $(\int_\Omega |r|^{\xi t'})^{1/\xi t'} \lesssim \|r\|_{W^1_\xi(\Omega)}$. With this choice, $\frac{1}{2\xi t} \geq \frac{1}{2} - \frac{1}{n}$, and so $(\int_\Omega |q^2|^{\xi t})^{1/\xi t} \lesssim \|q\|^2_{H^1(\Omega)}$, if and only if $n \leq 3$. So (7.4) is valid for $n \leq 3$.

As above, for $n \leq 3$, $(\int_\Omega |3q^2 v|^2)^{\frac{1}{2}} \lesssim \|q\|^2_{H^1(\Omega)}\|v\|_{H^1(\Omega)}$, and so $\|3q^2 h\|_{H^1(\Omega)'} = \sup_{0 \neq v \in H^1(\Omega)} \frac{\langle 3q^2 h, v\rangle_{L_2(\Omega)}}{\|v\|_{H^1(\Omega)}} \lesssim \|q\|^2_{H^1(\Omega)}\|h\|_{L_2(\Omega)}$, showing (7.3).

7.2. **Verification of** (7.2)**-**(7.4) **for** $N(q) = \sin q$**.** Clearly, the mapping $DN(q)\colon r \mapsto r\cos q$ satisfies (7.3) and (7.4).

To show (7.2), we take a reference triangle T with $(0,0) \in T$, and write $z(\vec{x}) = \vec{a} \cdot \vec{x} + b$.

For the L-shaped domain $(0,2) \setminus [1,2)^2$ and right-hand side $f = 1$ that we have used in our numerical experiments, we *observed* that the numerical approximations $(\tilde{\mathbf{u}}, \tilde{p})$ to the exact solution $(\mathbf{u}, p)$ satisfy $\|\tilde{p}\| \leq \frac{\pi}{2} - \delta$ for some $\delta > 0$..

Using this information, it is sufficient to verify (7.2) for $\|z\|_{L_\infty(T)} \leq \frac{\pi}{2} - \delta$, and so for $\|\vec{a}\| \lesssim 1$, and $b \ \leq \frac{\pi}{2} - \delta$.

For any $\varepsilon > 0$, $\inf_{\|\vec{a}\| \in [\varepsilon, 1], b, g \in \mathbb{R}} \|\sin z - g\|_{L_2(T)} \gtrsim 1$ by a continuity and compactness argument. From $\partial_i \sin z(\vec{x}) = a_i \cos z(\vec{x})$, we have $|\sin z|_{H^1(T)} \lesssim \|\vec{a}\| \lesssim 1$, and so it remains to consider $\|\vec{a}\| < \varepsilon$.

From $\sin z(\vec{x}) = \sin b + \vec{a} \cdot \vec{x} \cos b + \mathcal{O}(\|\vec{a}\|^2)$, we arrive at

$$\inf_{\|\vec{a}\| < \varepsilon, |b| \leq \frac{\pi}{2} - \delta, g \in \mathbb{R}} \frac{\|\sin z - g\|_{L_2(T)}}{\|\vec{a}\|} = \inf_{\|\vec{a}\| = 1, |b| \leq \frac{\pi}{2} - \delta, \bar{g} \in \mathbb{R}} \|\bar{g} - \vec{a} \cdot \vec{x} \cos b + \mathcal{O}(\varepsilon)\|_{L_2(T)} > 0$$

for ε being sufficiently small, which completes the proof of (7.2).

8. Numerics

On the L-shaped domain $\Omega = (0,2)^2 \setminus (0,1] \times [1,2)$, and given f, we consider the problem of finding p such that

$$\begin{cases} -\triangle p + N(p) = f & \text{on } \Omega, \\ p = 0 & \text{on } \partial\Omega, \end{cases}$$

i.e., (4.1) with $A = I$, $g = 0$, and $\Gamma_D = \partial\Omega$. Reformulated as a system of first order, it reads as finding $(\vec{u}, p)$ such that

$$\begin{cases} \vec{u} - \nabla p &= 0 \quad \text{on } \Omega, \\ N(p) - \operatorname{div} \vec{u} &= f \quad \text{on } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

We take $f = 1$, and consider both $N(p) = p^3$ or $N(p) = \sin p$.

The definition of G from (4.4) now reads as

$$G \colon \vec{H}(\operatorname{div}; \Omega) \times H_0^1(\Omega) \to L_2(\Omega)^2 \times L_2(\Omega) \colon (\vec{u}, p) \mapsto (\vec{u} - \nabla p, N(p) - \operatorname{div} \vec{u} - f),$$

so that $DG(\vec{u}, p) \colon (\vec{v}, q) \mapsto (\vec{v} - \nabla q, N'(p)q - \operatorname{div} \vec{v})$.

We equip $\vec{H}(\operatorname{div}; \Omega)$ and $H_0^1(\Omega)$ with the Riesz bases $\vec{\Psi}^{(1)} = \{\vec{\psi}_\lambda^{(1)} : \lambda \in \nabla^{(1)}\}$ and $\Psi^{(2)} = \{\psi_\lambda^{(2)} : \lambda \in \nabla^{(2)}\}$ constructed in Sect. 5, and recalled at the beginning of Sect. 7. The elements of $\vec{\Psi}^{(1)}$ are lowest order Raviart-Thomas functions, and those of $\Psi^{(2)}$ are continuous piecewise linears, each of them w.r.t. some uniform dyadic ("red") refinement of a conforming initial triangulation of Ω.

The **awgm**, i.e. Algorithm 2.2, is applied to the coupled system of the infinitely many scalar equations of finding $(\mathbf{u}, \mathbf{p}) \in \ell_2(\nabla^{(1)}) \times \ell_2(\nabla^{(2)})$ such that, with $(\vec{u}, p) := (\mathbf{u}^\top \vec{\Psi}^{(1)}, \mathbf{p}^\top \Psi^{(2)})$,

$$\Big[\big\langle G_1(\vec{u}, p), DG_1(\vec{u}, p)(\vec{\psi}^{(1)}, \psi^{(2)})\big\rangle_{L_2(\Omega)^2} + \big\langle G_2(\vec{u}, p), DG_2(\vec{u}, p)(\vec{\psi}^{(1)}, \psi^{(2)})\big\rangle_{L_2(\Omega)}\Big]_{(\lambda,\mu) \in \nabla^{(1)} \times \nabla^{(2)}} = 0,$$

i.e.,

$$\mathbf{r}(\mathbf{u}, \mathbf{p}) := \Big[\big\langle \vec{u} - \nabla p, \vec{\psi}_\lambda^{(1)} - \nabla \psi_\mu^{(2)}\big\rangle_{L_2(\Omega)^2} + \big\langle N(p) - \operatorname{div} \vec{u} - f, N'(p)\psi_\mu^{(2)} - \operatorname{div} \vec{\psi}_\lambda^{(1)}\big\rangle_{L_2(\Omega)}\Big]_{(\lambda,\mu) \in \nabla^{(1)} \times \nabla^{(2)}} = 0.$$

For running this algorithm, for any given finite trees $\Lambda_1 \subset \nabla^{(1)}$, $\Lambda_2 \subset \nabla^{(2)}$, and $\mathbf{w}$ and $\mathbf{q}$ with $\operatorname{supp} \mathbf{w} \subset \Lambda_1$, $\operatorname{supp} \mathbf{q} \subset \Lambda_2$, one should be able to evaluate $\mathbf{r}(\mathbf{w}, \mathbf{q})|_{\Lambda_1 \times \Lambda_2}$ (for the Galerkin solve, step (3) in Alg. 2.2), or $\mathbf{r}(\mathbf{w}, \mathbf{q})|_{\bar{\Lambda}_1 \times \bar{\Lambda}_2}$, where $\bar{\Lambda}_1$ and $\bar{\Lambda}_2$ are trees that are (slightly) inflated versions of Λ_1 and Λ_2 (for the approximate residual computation, step (1) in Alg. 2.2).

For our convenience, we took $f = 1$, so that there is no need to approximate f by a piecewise polynomial, and $G_1(\vec{w}, q)$ and $G_2(\vec{w}, q)$ can be evaluated exactly. Consequently, both $\mathbf{r}(\mathbf{w}, \mathbf{q})|_{\Lambda_1 \times \Lambda_2}$, and $\mathbf{r}(\mathbf{w}, \mathbf{q})|_{\bar{\Lambda}_1 \times \bar{\Lambda}_2}$ can be evaluated exactly (ignoring some quadrature issues for $N(p) = \sin p$). In this situation, the loop (1) in Algorithm 2.2 can be replaced by *one* evaluation of $\mathbf{r}(\mathbf{w}, \mathbf{q})|_{\bar{\Lambda}_1 \times \bar{\Lambda}_2}$, when $\bar{\Lambda}_1$ and $\bar{\Lambda}_2$ are taken sufficiently large so that the relative error in this approximation of the infinite residual $\mathbf{r}(\mathbf{w}, \mathbf{q})$ is less than a sufficiently small constant $\delta > 0$.

Let us briefly summarize the construction of suitable $\bar{\Lambda}_1$ and $\bar{\Lambda}_2$ that was outlined in Sect. 5. Thanks to Λ_1 being a tree, there exists a tiling $\mathcal{T}_{\Lambda_1}$, i.e., an essentially disjoint covering by closed triangles from all dyadic grids, with respect to which each $\vec{\psi}_\lambda^{(1)}$ is piecewise polynomial, and that satisfies $\#\mathcal{T}_{\Lambda_1} \lesssim \#\Lambda^{(1)}$. There is a similar tiling $\mathcal{T}_{\Lambda_2}$ associated to Λ_2 and $\Psi^{(2)}$. Let $\mathcal{T}$ be the smallest common refinement of both these tilings. For $k \in \mathbb{N}$, the set $\Lambda_{\mathcal{T},k}^{(1)}$ is defined as the smallest enlargement to a tree of the set of $\lambda \in \Lambda_1$ for which $\operatorname{supp} \vec{\psi}_\lambda^{(1)}$ intersects a tile from

$\mathcal{T}$ whose level plus k is not less than $|\lambda|$ (the level of $\vec{\psi}_\lambda$). The set $\Lambda^{(2)}_{\mathcal{T},k}$ is defined similarly. It holds that $\#\Lambda^{(1)}_{\mathcal{T},k} + \#\Lambda^{(2)}_{\mathcal{T},k} \lesssim \#\Lambda^{(1)} + \#\Lambda^{(2)}$, only dependent on k. As was shown in Sect. 6–7, for any $\delta > 0$, there exist a *constant* k such that with $(\bar{\Lambda}^{(1)}, \bar{\Lambda}^{(2)}) := (\Lambda^{(1)}_{\mathcal{T},k}, \Lambda^{(2)}_{\mathcal{T},k})$, the relative error in the resulting approximate residual evaluation is less than δ.

In our experiments, it turned that it suffices to take

$$k = 1,$$

in order to get an optimally converging **awgm**. So our implementation of step (1) of Alg. 2.2 consists of replacing the exact residual $\mathbf{r}(\mathbf{w}, \mathbf{q})$ by $\mathbf{r}(\mathbf{w}, \mathbf{q})|_{\bar{\Lambda}_1 \times \bar{\Lambda}_2}$, where $(\bar{\Lambda}^{(1)}, \bar{\Lambda}^{(2)}) := (\Lambda^{(1)}_{\mathcal{T},1}, \Lambda^{(2)}_{\mathcal{T},1})$. With this choice of k, the quotient $\frac{\#\mathbf{r}(\mathbf{w},\mathbf{q})}{\#\mathbf{w}+\#\mathbf{q}}$ never exceeded 7.

The *computation* of $\mathbf{r}(\mathbf{w}, \mathbf{q})|_{\Lambda_1 \times \Lambda_2}$ can be performed in linear computational complexity. Indeed, again thanks to Λ_1, Λ_2 being a trees, there exist collections of single-scale functions, i.e., standard basis functions for the Raviart Thomas spaces or nodal hat functions, that have cardinalities of order $\#\Lambda_1$ or $\#\Lambda_2$, and whose spans contain $\operatorname{span}\{\vec{\psi}^{(1)} \colon \lambda \in \Lambda_1\}$ or $\operatorname{span}\{\psi^{(2)} \colon \lambda \in \Lambda_2\}$. Now $\mathbf{r}(\mathbf{w}, \mathbf{q})|_{\Lambda_1 \times \Lambda_2}$ is found by first expressing $w := \mathbf{w}^\top \vec{\Psi}^{(1)}$ and $p := \mathbf{p}^\top \Psi^{(2)}$ in these single-scale coordinates by applying multi-to-single scale transformations; then applying the residual, viewed as an element in $\big(\vec{H}(\mathrm{div};\Omega) \times H^1_0(\Omega)\big)'$, to all pairs of single-scale functions (i.e. replacing in the expression for $\mathbf{r}(\mathbf{w}, \mathbf{q})$, the pairs $(\vec{\psi}^{(1)}_\lambda, \psi_\mu)$ by all pairs of single-scale functions); and finally, by applying the transpose of the multi-to-single scale transformations. In the second of these three steps, it is needed to evaluate $N(p)$ and $N'(p)$ in selected points. Thanks to the representation of p in single-scale coordinates, each of these evaluations takes $\mathcal{O}(1)$ operations.

Since $\bar{\Lambda}_1$, $\bar{\Lambda}_2$ are trees as well, a similar computation yields $\mathbf{r}(\mathbf{w}, \mathbf{q})|_{\bar{\Lambda}_1 \times \bar{\Lambda}_2}$ in linear computational complexity.

Step (2) of Alg. 2.2 consists of a bulk chasing step on the approximate residual $\mathbf{r}(\mathbf{w}, \mathbf{q})|_{\bar{\Lambda}_1 \times \bar{\Lambda}_2}$ which has to output finite trees in $\nabla^{(1)}$ and $\nabla^{(2)}$. Instead of applying the provable optimal strategy from [BD04], for our convenience, we simply selected the smallest $\Lambda_1 \cup \Lambda_2 \subset \tilde{\Lambda}_1 \cup \tilde{\Lambda}_2 \subset \bar{\Lambda}_1 \cup \bar{\Lambda}_2$ such that $\|\mathbf{r}(\mathbf{w}, \mathbf{q})|_{\tilde{\Lambda}_1 \times \tilde{\Lambda}_2}\| \geq \mu_0 \|\mathbf{r}(\mathbf{w}, \mathbf{q})|_{\bar{\Lambda}_1 \times \bar{\Lambda}_2}\|$, and then extended $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ to trees. As bulk chasing parameter, we took $\mu_0 = 0.4$.

In step (3), the Galerkin system is solved approximately. Vectors $\mathbf{w}$ and $\mathbf{q}$ with $\operatorname{supp} \mathbf{w} \subset \Lambda_1$, $\operatorname{supp} \mathbf{q} \subset \Lambda_2$ have to be determined such that $\|\mathbf{r}(\mathbf{w}, \mathbf{q})|_{\Lambda_1 \times \Lambda_2}\|$ is less or equal to a constant γ times the norm of the approximate residual of the (approximate) Galerkin solution corresponding to the previous sets Λ_1 and Λ_2. We took the value $\gamma = 0.5$. To determine $\mathbf{w}$ and $\mathbf{q}$, we run the damped Richardson iteration started with the previous (approximate) Galerkin solution, and damping parameter $\omega = 0.1$. In our experiments, the number of these Richardson iterations needed never exceeded 4.

In Figure 3, for both $N(p) = p^3$ or $N(p) = \sin p$, the norm of the approximate residual (divided by the norm of the initial residual), being a quantity that is equivalent to the relative error in $\vec{H}(\mathrm{div};\Omega) \times H^1_0(\Omega)$-norm, vs. the number of wavelets N is illustrated. For both cases, one observes an error decay that is proportional with $N^{-\frac{1}{2}}$. In view of the wavelets that are applied, the norm in which the error is measured, and the space dimension $n = 2$, this rate $\frac{1}{2}$ is the best

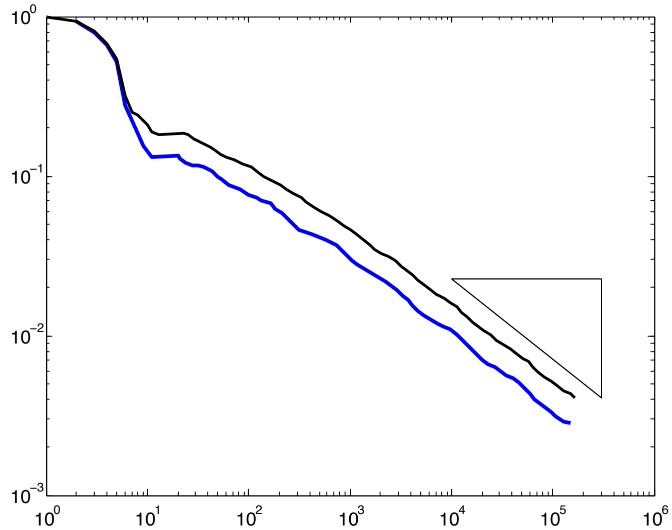

FIGURE 3. Norm of residual vs. number of wavelets in log-log scale, for $N(p) = p^3$ (black, upper curve) or $N(p) = \sin p$ (blue, lower curve). The hypotenuse of the triangle has a slope of $-\frac{1}{2}$.

that generally can be expected, even if the solution would be smooth (which it is not, due to the re-entrant corner).

In Figure 4 and 5 the centers of the supports of the wavelets that were adaptively

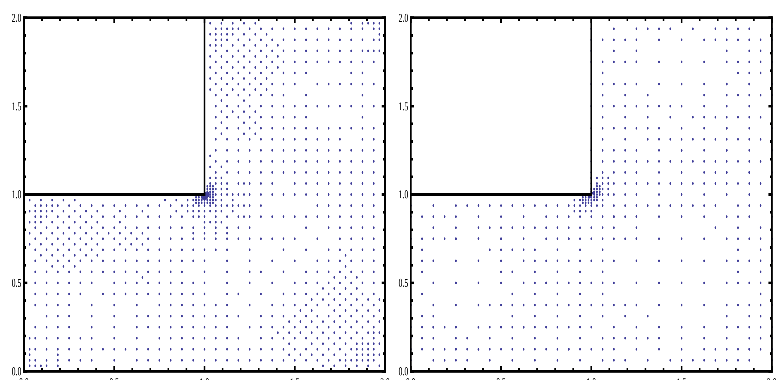

FIGURE 4. Centers of the supports of the wavelets in $H_0^1(\Omega)$ for the approximation of p (left, 930 wavelets), or the wavelets in $H(\text{div};\Omega)$ for the approximation of $\vec{u}$ (right, 631 wavelets) produced by **awgm** after 39 iterations for $N(p) = p^3$.

selected are illustrated for the cases of $N(p) = p^3$ and $N(p) = sinp$, respectively. One observes a strong refinement near the re-entrant corner.

Finally, in Figure 6, approximate solutions are illustrated.

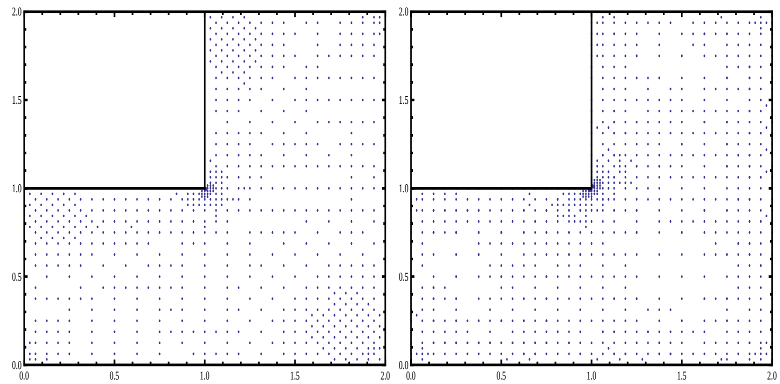

FIGURE 5. Centers of the supports of the wavelets in $H^1_0(\Omega)$ for the approximation of p (left, 673 wavelets), or the wavelets in $H(\text{div};\Omega)$ for the approximation of $\vec{u}$ (right, 954 wavelets) produced by **awgm** after 40 iterations for $N(p) = \sin p$.

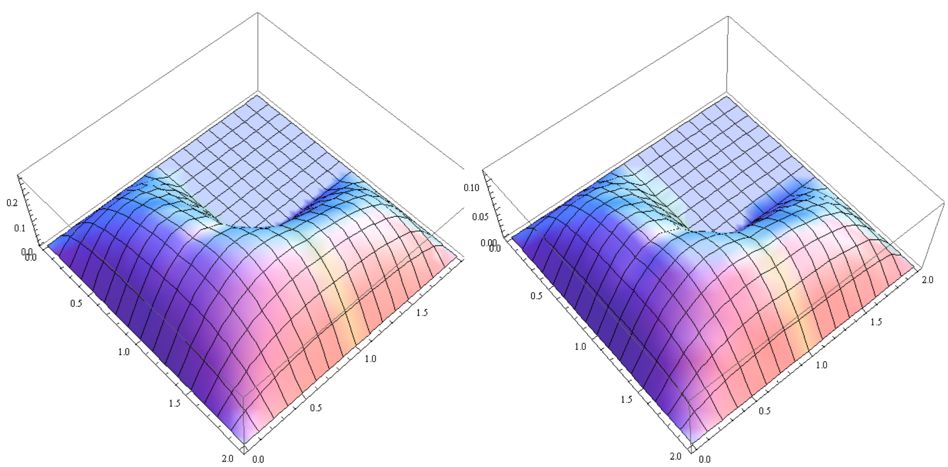

FIGURE 6. Approximate solutions of $-\triangle p + N(p) = 1$ on Ω, $p = 0$ on $\partial\Omega$, for $N(p) = p^3$ (left) or $N(p) = \sin p$ (right), as a linear combination of approximately 200 wavelets. Note the difference in vertical scale in both pictures.

9. Summary and outlook

We have constructed an adaptive wavelet method for solving a first order system least squares formulation of a second order semi-linear elliptic PDE, that converges with the best possible rate in linear computational complexity. A key ingredient is an also quantitatively efficient approximate residual evaluation. This approximate residual is the restriction of the infinite residual to those indices that correspond to wavelets that are located in a thin shell around the tree of the current set of active wavelets.

The analysis of this scheme uses the property that in our least-squares formulation the residual is measured in an L_2-space. We envisage, however, that this restriction can be relatex to the property that the operator -when applied to any wavelet- lands in L_2. This will extend the scope to a much wider class of first order system least squares formulations of PDEs.

We plan to apply this approach to simultaneous space-time variational formulations of (parabolic) time-evolution problems. In this setting, the arising function spaces are Bochner spaces, that can only be equipped with a basis that is a tensor product of bases in space and time. This rules out finite element type a posteriori error estimation based on integration-by-parts over mesh cells. On the other hand, the tensor product basis allows to solve the time evolution problem at a complexity of solving the corresponding stationary problem.

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