Adaptive Wavelet Schwarz Methods for the Navier-Stokes Equation *

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Abstract

In this paper we are concerned with domain decomposition methods for the stationary incompressible Navier-Stokes equation. We construct an adaptive additive Schwarz method based on discretization by means of a divergence-free wavelet frame. We prove that the method is convergent and asymptotically optimal with respect to the degrees of freedom involved.

1 Introduction

Over the last years, adaptive wavelet methods for both linear and nonlinear partial differential equations have intensively been investigated, see, for instance, [2, 4]. One can often prove that these methods are not only convergent, but also asymptotically optimal. This means that the algorithm converges with the same rate as the best N-term wavelet approximation with respect to the degrees of freedom involved. The techniques used to show these results heavily rely on the properties of the underlying wavelet Riesz basis. This basis can be constructed such that its elements have vanishing moments, are piecewise smooth and characterize function spaces in the sense that weighted sequence norms of wavelet expansion coefficients are equivalent to smoothness norms such as Besov norms. Moreover, it is also possible to construct divergence-free wavelet bases, see [18, 24] which are very useful for the numerical treatment of incompressible flow problems, see [27, 28].

However, on more complicated domains, the design of such a wavelet basis becomes increasingly difficult and the condition numbers become worse. A way to facilitate the construction is to use redundant generating systems, namely wavelet frames, instead of bases, see [22]. To do so, let us assume that we can decompose the domain into overlapping subdomains that are affine images of the unit cube. Then, we can construct wavelet bases on each of the subdomains, which is significantly easier, and simply collect these bases. From this, we obtain a wavelet frame.

Because the construction of the wavelet frame applied in this paper is based on an overlapping domain decomposition, it is natural to combine wavelet methods with domain decomposition solvers such as Schwarz methods. These methods allow us to reduce the problem on the entire domain to a series of easier subproblems on the subdomains, that can moreover be parallelized very efficiently. For some early work on domain decomposition methods for nonlinear problems, we refer to [9, 25]. The combination

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with wavelet methods has proven to be effective for the numerical solution of linear elliptic problems, see [23], and has recently been generalized to a range of nonlinear problems, see [16]. Based on an idea from [20], in this paper we extend this approach to the stationary, incompressible Navier-Stokes equation. We show that the method is convergent and asymptotically optimal, albeit under the assumption of having a sufficiently small Reynolds number or sufficiently small datum. Since the latter assumption is rather restrictive, we consider our contribution as a first step in the development of more generally applicable Navier-Stokes solvers that are proven to converge with optimal rates.

In this work, by $C \lesssim D$ we will mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

2 The Navier-Stokes equations in frame coordinates

2.1 Navier-Stokes equations

We are concerned with the incompressible, steady-state, viscous Navier-Stokes equation on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \leq 4$, with Dirichlet boundary conditions

$$\begin{aligned}(u \cdot \nabla)u &= -\nabla p + \frac{1}{\mathrm{Re}}\Delta u + f && \text{in } \Omega,\\ \operatorname{div} u &= 0 && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega,\end{aligned}$$

where u denotes the velocity field of a fluid, p is the pressure term, f is the given inertial force and Re is the Reynolds number that describes the viscosity of the fluid. In addition, we normalize the pressure term p by $\int_\Omega p = 0$.

There are basically two general approaches for the numerical treatment of the variational form of this equation. One common approach is to solve for the velocity u and the pressure p simultaneously. Doing so leads to an indefinite saddle point problem, see, for instance, [11] for an overview in the context of finite element methods or [10] for a wavelet-based method.

An alternative approach is to reformulate the equation using a divergence-free ansatz space

$$V := \{v \in (H_0^1(\Omega))^d, \operatorname{div} v = 0\}.$$

The Leray weak formulation of the original problem then reduces to finding a $u \in V$ such that

$$\int_\Omega v(u \cdot \nabla)u = -\frac{1}{\mathrm{Re}}a(u,v) + \int_\Omega f \cdot v \quad (v \in V), \tag{1}$$

with $a(u,v) := \sum_{k=1}^d \int_\Omega \nabla u_k \nabla v_k$ or, equivalently,

$$a(u,v) + \mathrm{Re}\int_\Omega v(u \cdot \nabla)u = \mathrm{Re}\int_\Omega f \cdot v \quad (v \in V).$$

Note that in this formulation, the pressure term drops out, and we solve for the velocity field u only. For details of the derivation of the weak formulation, see, for instance, [26]. The formulation there coincides with (1) for our case $d \in \{2,3,4\}$. In particular, for any right-hand side f in the dual of $H_0^1(\Omega))^d$, the existence of a weak solution u is shown there. To guarantee uniqueness, it has to be assumed that the Reynolds number is sufficiently small or that the data f fulfills a smallness condition. We will equip V with the energy norm

$$|\!|\!|v|\!|\!| = a(v,v)^{\frac{1}{2}}.$$

In order to write the equation (1) as an infinite system of scalar equations, we apply a generating system for the space V. Hence, in the next subsections, we outline the construction of a divergence-free wavelet frame for this space.

2.2 Frames

Recall that a countable collection $\Psi = \{\psi_\lambda : \lambda \in \Lambda\}$ in a Hilbert space V is called a *frame* for V when there exist two positive constants A_Ψ, B_Ψ such that

$$A_\Psi \|f\|_{V'} \leq \|(f(\psi_\lambda))_{\lambda\in\Lambda}\|_{\ell_2(\Lambda)} \leq B_\Psi \|f\|_{V'} \qquad (f \in V'). \tag{2}$$

As a consequence of (2), the frame operator

$$F : V' \to \ell_2(\Lambda) : f \mapsto f(\Psi) := (f(\psi_\lambda))_{\lambda\in\Lambda},$$

and so its adjoint

$$F' : \ell_2(\Lambda) \to V : \mathbf{c} \mapsto \mathbf{c}^\top \Psi := \sum_{\lambda\in\Lambda} c_\lambda \psi_\lambda,$$

are bounded with norm less than or equal to B_Ψ. The composition $F'F : V' \to V$ is boundedly invertible with $\|(F'F)^{-1}\|_{V\to V'} \leq A_\Psi^{-2}$. The collection $\tilde{\Psi} := (F'F)^{-1}\Psi$ is a frame for V' with frame operators

$$\tilde{F} := F(F'F)^{-1}, \qquad \tilde{F}' = (F'F)^{-1}F'$$

and frame constants B_Ψ^{-1}, A_Ψ^{-1}. Since $F'\tilde{F} = \mathrm{Id} = \tilde{F}'F$, $\tilde{\Psi}$ is called *a* dual frame for Ψ, known as the canonical dual frame. We have $\ell_2(\Lambda) = \operatorname{ran} F \oplus^\perp \ker F'$, and $F(F'F)^{-1}F'$ is the orthogonal projector onto $\operatorname{ran} F$. For these facts and further reading on frames, we refer to [1].

The key to the construction of a frame for a space of functions on a domain by means of an overlapping domain decomposition is the following result. We state the lemma, that seems folklore, together with a short and self-contained proof.

Lemma 2.1 *The property of a countable $\Psi \subset V$ being a frame for V with constants A_Ψ, B_Ψ is equivalent to $\overline{\operatorname{span} \Psi} = V$ and*

$$B_\Psi^{-1}\|u\|_V \leq \inf_{\{\mathbf{u}\in\ell_2(\Lambda),\, \mathbf{u}^\top\Psi = u\}} \|\mathbf{u}\|_{\ell_2(\Lambda)} \leq A_\Psi^{-1}\|u\|_V \qquad (u \in V). \tag{3}$$

Proof. If Ψ is a frame, then $(\tilde{\psi}_\lambda(u))_{\lambda\in\Lambda} = \arg\min\{\|\mathbf{u}\|_{\ell_2(\Lambda)} : \mathbf{u} \in \ell_2(\Lambda),\, \mathbf{u}^\top\Psi = u\}$, and (3) follows from $\tilde{\Psi}$ being a frame with frame constants B_Ψ^{-1}, A_Ψ^{-1}.

Conversely, let (3) be valid. Then from its first inequality, we deduce that $\mathbf{c} \mapsto \mathbf{c}^\top\Psi \in B(\ell_2(\Lambda), V)$, with norm less or equal to B_Ψ. From its second inequality, we infer that for $v \in V$,

$\sup_{0\neq \mathbf{d}\in\ell_2(\Lambda)} \frac{\langle \mathbf{d}^\top\Psi, v\rangle_V}{\|\mathbf{d}\|_{\ell_2(\Lambda)}\|v\|_V} \geq \frac{\|v\|_V}{\inf_{\{\mathbf{d}\in\ell_2(\Lambda),\,\mathbf{d}^\top\Psi=v\}}\|\mathbf{d}\|_{\ell_2(\Lambda)}} \geq A_\Psi$. Consequently, for given $u\in V$, there exists a unique solution $(\hat{\mathbf{u}}, w)\in\ell_2(\Lambda)\times V$ of the linear problem

$$\begin{cases} \langle \hat{\mathbf{u}}, \mathbf{d}\rangle_{\ell_2(\Lambda)} + \langle \mathbf{d}^\top\Psi, w\rangle_V &= 0 \qquad (\mathbf{d}\in\ell_2(\Lambda)), \\ \langle \hat{\mathbf{u}}^\top\Psi, v\rangle_V &= \langle u, v\rangle_V \quad (v\in V), \end{cases} \tag{4}$$

and $\hat{\mathbf{u}} = \arg\min\{\|\mathbf{u}\|_{\ell_2(\Lambda)} : \mathbf{u}\in\ell_2(\Lambda),\, \mathbf{u}^\top\Psi = u\}$. Defining $\tilde{\psi}_\lambda : u\mapsto \hat{\mathbf{u}}_\lambda \in V'$, (3) means that $\tilde{\Psi}$ is a frame for V' with frame constants B_Ψ^{-1}, A_Ψ^{-1}.

Next, we consider (4) for $u=\psi_\mu$, so that $\hat{\mathbf{u}}_\lambda = \tilde{\psi}_\lambda(\psi_\mu)$. With $R:\ell_2(\Lambda)\to\ell_2(\Lambda)'$, $B: V\to\ell_2(\Lambda)'$ being defined by $(R\mathbf{c})(\mathbf{d}) = \langle \mathbf{c},\mathbf{d}\rangle_{\ell_2(\Lambda)}$, $(Bw)(\mathbf{d}) = \langle \mathbf{d}^\top\Psi, w\rangle_V$, (4) with $u=\psi_\mu$ reads as

$$\begin{bmatrix} R & B \\ B' & 0\end{bmatrix}\begin{bmatrix}\hat{\mathbf{u}} \\ w\end{bmatrix} = \begin{bmatrix} 0 \\ B'\mathbf{e}_\mu\end{bmatrix}$$

and so $\hat{\mathbf{u}} = R^{-1}B(B'R^{-1}B)^{-1}B'\mathbf{e}_\mu$. We conclude that

$$\tilde{\psi}_\lambda(\psi_\mu) = \hat{\mathbf{u}}_\lambda = (R\hat{\mathbf{u}})(\mathbf{e}_\lambda) = (B(B'R^{-1}B)^{-1}B'\mathbf{e}_\mu)(\mathbf{e}_\lambda) = (B(B'R^{-1}B)^{-1}B'\mathbf{e}_\lambda)(\mathbf{e}_\mu) = \tilde{\psi}_\mu(\psi_\lambda). \tag{5}$$

The second equation in (4) shows that $\sum_{\lambda\in\Lambda}\tilde{\psi}_\lambda(u)\psi_\lambda = u$ $(u\in V)$, and so $\sum_{\lambda\in\Lambda}\tilde{\psi}_\lambda(u)\tilde{\psi}_\mu(\psi_\lambda) = \tilde{\psi}_\mu(u)$ $(u\in H,\ \mu\in\Lambda)$, or $\sum_{\lambda\in\Lambda}\tilde{\psi}_\mu(\psi_\lambda)\tilde{\psi}_\lambda = \tilde{\psi}_\mu$. Replacing $\tilde{\psi}_\mu(\psi_\lambda)$ by $\tilde{\psi}_\lambda(\psi_\mu)$ in the last equality because of (5), it reads as $\tilde{F}'\tilde{F}\psi_\mu = \tilde{\psi}_\mu$, with $\tilde{F}$ being the frame operator of $\tilde{\Psi}$. We conclude that Ψ is the canonical dual frame of $\tilde{\Psi}$, and thus in particular a frame, and which therefore has frame constants A_Ψ, B_Ψ. □

2.3 Domain decomposition

A wavelet frame will be obtained by decomposing the domain Ω into affine overlapping images of the unit cube, $\Omega = \bigcup_{i=0}^{m-1}\Omega_i$. Let us assume that such a decomposition exists and that we have wavelet bases $\Psi^{(i)} = \{\psi_\lambda^{(i)} : \lambda\in\Lambda_i\}$ for the spaces

$$V_i := \{v\in (H_0^1(\Omega_i))^d,\, \operatorname{div} v = 0\}.$$

Having these bases at hand, we simply set $\Psi := \bigcup_{i=0}^{m-1} E_i\Psi^{(i)}$, where E_i is the zero extension from V_i to V. The index set belonging to Ψ is denoted by $\Lambda := \cup_{i=0}^{m-1}\{i\}\times\Lambda_i$, so we can write $\Psi = \{\psi_\lambda : \lambda\in\Lambda\}$. Assuming that the subdomains Ω_i are overlapping in the sense that

$$H_0^1(\Omega) = H_0^1(\Omega_0) + \ldots + H_0^1(\Omega_{m-1}), \tag{6}$$

the following lemma ensures that the collection Ψ is indeed frame for the space V. This is important because condition (6) is easier to ensure in practice by construction of a partition of unity, compare [22], than to show an analogous property for the space V. The reason behind this is that, contrary to classical Sobolev spaces, the space V of divergence-free Sobolev functions is not closed under pointwise multiplication with smooth functions on $\bar{\Omega}$.

Lemma 2.2 *Assume the subdomains Ω_i are overlapping in the sense of* (6)*. Let $\Psi^{(i)}$ be frames or Riesz bases for V_i. Then, $\Psi := \bigcup_{i=0}^{m-1} E_i\Psi^{(i)}$ is a frame for V.*

Proof. We clearly have $H^1_0(\Omega)^d = H^1_0(\Omega_0)^d + \ldots + H^1_0(\Omega_{m-1})^d$. Hence, from [20, Lemma 2], it follows that even $V = V_0 + \ldots + V_{m-1}$. Now, from the partition lemma (see, for instance, [17]) we may conclude that there exists a stable splitting of V, which means that, uniformly in $v \in V$, we have

$$\begin{aligned}
\|v\|^2_{H^1(\Omega)^d} &\eqsim \inf_{\{(v_i)_i \in \prod_{i=0}^{m-1} V_i : v = \sum_{i=0}^{m-1} E_i v_i\}} \sum_{i=0}^{m-1} \|v_i\|^2_{H^1(\Omega_i)^d} \\
&\eqsim \inf_{\{(v_i)_i \in \prod_{i=0}^{m-1} V_i : v = \sum_{i=0}^{m-1} E_i v_i\}} \sum_{i=0}^{m-1} \inf_{\{\mathbf{v}_i \in \ell_2(\Lambda_i) : \mathbf{v}_i^\top \Psi^{(i)} = v_i\}} \|\mathbf{v}_i\|^2_{\ell_2(\Lambda_i)} \\
&\eqsim \inf_{\{\mathbf{v} \in \ell_2(\Lambda) : \mathbf{v}^\top \Psi = v\}} \|\mathbf{v}\|^2_{\ell_2(\Lambda)},
\end{aligned}$$

where we used Lemma 2.1 for the second $\eqsim$. The proof is completed by another application of this lemma. □

Remark 2.3 *The condition in Lemma 2.2 is fulfilled if there exists a smooth partition of unity with respect to the domain decomposition $\Omega = \bigcup_{i=0}^{m-1} \Omega_i$. However, it can also be shown for situations where such a partition does not exist. One important case of this kind is the prototype of a non-convex polygonal domain in two space dimensions, the L-shaped domain $\Omega = (-1,1)^2 \setminus [0,1)^2$ with subdomains $\Omega_0 = (-1,1) \times (-1,0)$ and $\Omega_1 = (-1,0) \times (-1,1)$, see [17, 30].*

2.4 Divergence-free wavelet bases on the subdomains

We consider subdomains that are hypercubes. More general subdomains can then be treated by applying the Piola transform. Furthermore, we restrict ourselves to the two-dimensional case, i.e., to the unit square $\mathcal{I}^2$, where $\mathcal{I} := (0,1)$. Divergence-free wavelet bases for $\{v \in H^1_0(\mathcal{I}^d) : \operatorname{div} v = 0\}$ were constructed in [24] for any dimension $d \geq 2$. These bases consist of *aniso*tropic wavelets, i.e., vectors of tensor products of univariate wavelets on arbitrary, unrelated scales. These anisotropic wavelet bases have the advantage that they give rise to approximation rates that are independent of the space dimension. On the other hand, the efficient approximate evaluation of nonlinear terms in anisotropic wavelet coordinates is yet not well understood. For that reason we present here a construction of a *iso*tropic divergence-free wavelet basis that applies to $d = 2$.

The construction starts with collections of univariate primal and dual wavelets, and, for $\ell \in \mathbb{N}_0$, collections of univariate primal and dual scaling functions

$$\Psi = \{\psi_\lambda : \lambda \in J\},\ \tilde{\Psi} = \{\tilde{\psi}_\lambda : \lambda \in J\},\ \Phi_\ell = \{\phi_{\ell,k} : 1 \leq k \leq N_\ell\},\ \tilde{\Phi}_\ell = \{\tilde{\phi}_{\ell,k} : 1 \leq k \leq N_\ell\},$$

such that

(a). $(\Psi, \tilde{\Psi})$ are $L_2(\mathcal{I})$-biorthogonal,

(b). $\{2^{-|\lambda|}\tilde{\psi}_\lambda : \lambda \in J\}$ is a Riesz basis for $H^1(\mathcal{I})$, where $|\lambda| \in \mathbb{N}_0$ is referred to as the level of λ,

(c). $\{2^{-|\lambda|}\psi_\lambda : \lambda \in J\}$ is a Riesz basis for $H^1_0(\mathcal{I})$,

(d). Ψ is local, meaning that both $\operatorname{diam}\operatorname{supp}\psi_\lambda \lesssim 2^{-|\lambda|}$, and each interval of length $2^{-\ell}$ intersects the supports of an at most uniformly bounded number of ψ_λ for $|\lambda| = \ell$; and $\tilde{\Psi}$ is local,

(e). there is a $\underline{\lambda} \in J$ with $|\underline{\lambda}| = 0$, such that $\tilde{\psi}_{\underline{\lambda}}$ is a multiple of the constant function $\mathbb{1}$, (f). $\operatorname{span}\{\psi_\lambda : \lambda \in J, |\lambda| \le \ell\} = \operatorname{span} \Phi_\ell$, $\operatorname{span}\{\tilde{\psi}_\lambda : \lambda \in J, |\lambda| \le \ell\} = \operatorname{span} \tilde{\Phi}_\ell$,

(g). Φ_ℓ and $\tilde{\Phi}_\ell$ are biorthogonal, uniform (in ℓ) $L_2(\mathcal{I})$-Riesz bases for their spans,

(h). Φ_ℓ is uniformly (in ℓ) local, meaning that both $\operatorname{diam} \operatorname{supp} \phi_{\ell,k} \lesssim 2^{-\ell}$, and each interval of length $2^{-\ell}$ intersects the supports of an at most uniformly bounded number of $\phi_{\ell,k}$ for $|\lambda| = \ell$; and $\tilde{\Phi}$ is uniformly local.

(i). for each ℓ, $\int_{\mathcal{I}} \phi_{\ell,k}$ is independent of $1 \le k \le N_\ell$, and $\inf \operatorname{supp} \phi_{k,\ell} \le \inf \operatorname{supp} \phi_{k+1,\ell}$ for $1 \le k \le N_\ell - 1$.

With the exception of (e) and (i), all conditions are standard, and biorthogonal wavelets and scaling functions that satisfy them have been constructed in e.g. [6, 7, 21]. To satisfy (e), it is sufficient that $\mathbb{1} \in \operatorname{span}\{\tilde{\psi}_\lambda : \lambda \in J, |\lambda| = 0\}$, which in view of (b) is a natural condition that is satisfied by the constructions in these references. Indeed, when this holds true then by means of a simple basis transformation, that only involves primal and dual wavelets on the lowest level, (e) is satisfied.

The second condition in (i) just fixes an ordering of the scaling functions by their supports. The first condition in (i) can always be satisfied by a rescaling of the scaling functions. However, in order that this rescaling does not jeopardize $\|\phi_{\ell,k}\|_{L_2(\mathcal{I})} \eqsim 1$, initially, for each ℓ, the $\int_{\mathcal{I}} \phi_{\ell,k}$'s should have comparable values. In view of $\int_{\mathcal{I}} \phi_{\ell,k} \lesssim 2^{-\ell/2}$, sufficient is $\int_{\mathcal{I}} \phi_{\ell,k} \gtrsim 2^{-\ell/2}$, which is satisfied by the B-spline scaling functions in the aforementioned references.

Note that the conditions (a)–(i) do not require that the wavelets and scaling functions are constructed by adapting a stationary multi-resolution analysis on the line to a bounded interval. In this sense, our construction generalizes earlier ones as e.g. [13]. Earlier work towards a more abstract setting can be found in [29].

Remark 2.4 *Because of $[H^1(\mathcal{I})', H^1_0(\mathcal{I})]_{\frac{1}{2},2} \simeq L_2(\mathcal{I})$, with the space at the left hand side being the real interpolation space obtained by the K-method, the conditions* (a), (b), *and* (c) *imply that Ψ, and so $\tilde{\Psi}$, are Riesz bases for $L_2(\mathcal{I})$.*

Next, from $(\Psi, \tilde{\Psi})$ we construct a new pair of biorthogonal wavelets $(\overset{+}{\Psi}, \bar{\tilde{\Psi}})$, and corresponding primal and dual scaling functions, by means of integration or differentiation at primal or dual side, respectively. This generalizes the construction in [19, Proposition 7] for the shift-invariant case on $\mathbb{R}$.

Proposition 2.5 *With $\mathring{J} := J \setminus \{\underline{\lambda}\}$, we define*

$$\overset{+}{\Psi} = \{\overset{+}{\psi}_\lambda : \lambda \in \mathring{J}\}, \quad \bar{\tilde{\Psi}} = \{\bar{\tilde{\psi}}_\lambda : \lambda \in \mathring{J}\},$$

by

$$\overset{+}{\psi}_\lambda : x \mapsto \int_0^x 2^{|\lambda|} \psi_\lambda(y) dy, \quad \bar{\tilde{\psi}}_\lambda = -2^{-|\lambda|} \tilde{\psi}'_\lambda.$$

Then

(i). $\overset{+}{\Psi}, \bar{\tilde{\Psi}}$ *are $L_2(\mathcal{I})$-biorthogonal collections,*

(ii). $\overset{+}{\Psi}$ *is a Riesz basis for $L_2(\mathcal{I})$, and*

(iii). $\{4^{-|\lambda|} \overset{+}{\psi}_\lambda : \lambda \in \mathring{J}\}$ *is a Riesz basis for $H^2_0(\mathcal{I})$.*

(iv). $\overset{+}{\Psi}$ *and* $\bar{\tilde{\Psi}}$ *are local.*

Proof. Obviously $\operatorname{supp} \bar{\tilde{\psi}}_\lambda \subset \operatorname{supp} \tilde{\psi}_\lambda$. Since for $\lambda \in \mathring{J}$, $\int_{\mathcal{I}} \psi_\lambda = 0$ by (a) and (e), we infer that $\operatorname{supp} \overset{+}{\psi}_\lambda \subset \operatorname{convhull}(\operatorname{supp} \psi_\lambda)$, showing (iv), as well as $\overset{+}{\psi}_\lambda \in H_0^1(\mathcal{I})$, the latter showing that

$$\langle \overset{+}{\psi}_\lambda, \bar{\tilde{\psi}}_\mu \rangle_{L_2(\mathcal{I})} = \langle \overset{+}{\psi}_\lambda, -2^{-|\mu|} \tilde{\psi}'_\mu \rangle_{L_2(\mathcal{I})} = 2^{|\lambda|-|\mu|} \langle \psi_\lambda, \tilde{\psi}_\mu \rangle_{L_2(\mathcal{I})},$$

and thus (i) by (a).

Since $H_0^1(\mathcal{I}) \cap \operatorname{span}\{\mathbb{1}\}^{\perp_{L_2(\mathcal{I})}} \to H_0^2(\mathcal{I}) : g \mapsto (x \mapsto \int_0^x g(y)dy)$ is bounded, with bounded inverse $f \mapsto f'$, $\{2^{-|\lambda|}\psi_\lambda : \lambda \in \mathring{J}\}$ being a Riesz basis for $H_0^1(\mathcal{I}) \cap \operatorname{span}\{\mathbb{1}\}^{\perp_{L_2(\mathcal{I})}}$, by (c), (e), (a), is equivalent to $\{4^{-|\lambda|}\overset{+}{\psi}_\lambda : \lambda \in \mathring{J}\}$ being a Riesz basis for $H_0^2(\mathcal{I})$, i.e., (iii).

Since $H^1(\mathcal{I}) \cap \operatorname{span}\{\psi_{\underline{\lambda}}\}^{\perp_{L_2(\mathcal{I})}} \to L_2(\mathcal{I}) : f \mapsto f'$ is bounded, with bounded inverse $g \mapsto \Big(x \mapsto \int_0^x g(y)dy - \frac{\int_0^1 \int_0^z g(y)dy \psi_{\underline{\lambda}}(z)dz}{\int_0^1 \psi_{\underline{\lambda}}(z)dz}\Big)$, $\{2^{-|\lambda|}\tilde{\psi}_\lambda : \lambda \in \mathring{J}\}$ being a Riesz basis for $H^1(\mathcal{I}) \cap \operatorname{span}\{\psi_{\underline{\lambda}}\}^{\perp_{L_2(\mathcal{I})}}$, by (b) and (a), is equivalent to $\bar{\tilde{\Psi}}$ being a Riesz basis for $L_2(\mathcal{I})$, i.e., (ii) by (i). □

Remark 2.6 *From* $[L_2(\mathcal{I}), H_0^2(\mathcal{I})]_{\frac{1}{2},2} = H_0^1(\mathcal{I})$, *(ii) and (iii) imply that* $\{2^{-|\lambda|}\overset{+}{\psi}_\lambda : \lambda \in \mathring{J}\}$ *is a Riesz basis for* $H_0^1(\mathcal{I})$*.*

The somewhat technical proof of the following proposition is postponed to the appendix.

Proposition 2.7 *The collections* $\overset{+}{\Phi}_\ell = \{\overset{+}{\phi}_{\ell,k} : 1 \leq k \leq N_\ell - 1\}$, $\bar{\tilde{\Phi}}_\ell = \{\bar{\tilde{\phi}}_{\ell,k} : 1 \leq k \leq N_\ell - 1\}$, *defined by*

$$\overset{+}{\phi}_{\ell,k} : x \mapsto \int_0^x 2^{\ell+1}(\phi_{\ell,k+1}(y) - \phi_{\ell,k}(y))dy, \quad \bar{\tilde{\phi}}_{\ell,k} = -2^{-(\ell+1)} \sum_{p=k+1}^{N_\ell} \tilde{\phi}'_{\ell,p},$$

are biorthogonal, uniformly local, uniform $L_2(\mathcal{I})$*-Riesz bases for* $\operatorname{span}\{\overset{+}{\psi}_\lambda : \lambda \in \mathring{J}, |\lambda| \leq \ell\}$, $\operatorname{span}\{\bar{\tilde{\psi}}_\lambda : \lambda \in \mathring{J}, |\lambda| \leq \ell\}$*, respectively.*

Having two biorthogonal multi-resolution analyses related by integration or differentiation at hand, we are ready to construct a wavelet Riesz basis for $\{v \in H_0^1(\mathcal{I}^2)^2 : \operatorname{div} v = 0\}$, as well as a corresponding dual basis. Let us denote these bases here by Σ and $\tilde{\Sigma}$, respectively. With $\tilde{\Sigma}$ being a dual basis, we mean that $\tilde{\Sigma} \subset H^{-1}(\mathcal{I}^2)^2$, $\langle \tilde{\Sigma}, \Sigma \rangle_{H^{-1}(\mathcal{I}^2)^2 \times H_0^1(\mathcal{I}^2)^2} = \mathrm{Id}$, and

$$v \mapsto \langle \tilde{\Sigma}, v \rangle_{H^{-1}(\mathcal{I}^2)^2 \times H_0^1(\mathcal{I}^2)^2} \in B(H_0^1(\mathcal{I}^2)^2, \ell_2(\tilde{\Sigma})). \tag{7}$$

Consequently, $v \mapsto \langle \tilde{\Sigma}, v \rangle^\top_{H^{-1}(\mathcal{I}^2)^2 \times H_0^1(\mathcal{I}^2)^2} \Sigma$ is a bounded projection on $H_0^1(\mathcal{I}^2)^2$, with its image being equal to $\{v \in H_0^1(\mathcal{I}^2)^2 : \operatorname{div} v = 0\}$. Note that such a dual basis is not unique.

Although our bases are similar to those constructed in [18] in the shift-invariant case on $\mathbb{R}^d$, working on a bounded domain causes some difficulties by which this construction of the isotropic divergence-free wavelets is restricted to two space dimensions. We refer to [24] for a further discussion of this point.

In the following, we set $I_\ell = \{1 \leq k \leq N_\ell - 1\}$, and for $\lambda \in \mathring{J}$, we write $\lambda = (\ell, k)$ where $\ell = |\lambda|$ and k runs over an index set J_ℓ, so that $\mathring{J} = \cup_{\ell=0}^\infty \{\ell\} \times J_\ell$. The lengthy proof of the following proposition is given in the appendix.

Proposition 2.8 *The collection*

$$\bigcup_{\ell\in\mathbb{N}_0} 2^{-\ell}\Big(\Big\{[-\overset{+}{\psi}_{\ell+1,k}\otimes(\phi_{\ell,m+1}-\phi_{\ell,m}),\psi_{\ell+1,k}\otimes\overset{+}{\phi}_{\ell,m}]^\top : k\in J_{\ell+1},\, m\in I_\ell\Big\}$$
$$\cup\Big\{[-\overset{+}{\phi}_{\ell,k}\otimes\psi_{\ell+1,m},(\phi_{\ell,k+1}-\phi_{\ell,k})\otimes\overset{+}{\psi}_{\ell+1,m}]^\top : k\in I_\ell,\, m\in J_{\ell+1}\Big\} \tag{8}$$
$$\cup\Big\{[-\overset{+}{\psi}_{\ell,k}\otimes\psi_{\ell,m},\psi_{\ell,k}\otimes\overset{+}{\psi}_{\ell,m}]^\top : k\in J_\ell,\, m\in J_\ell\Big\}\Big)$$

is a Riesz basis for $\{v\in H_0^1(\mathcal{I}^2)^2 : \operatorname{div} v = 0\}$.

A dual basis is given by

$$\bigcup_{\ell\in\mathbb{N}_0} 2^{\ell}\Big(\Big\{[0,\tilde{\psi}_{\ell+1,k}\otimes\bar{\tilde{\phi}}_{\ell,m}]^\top : k\in J_{\ell+1},\, m\in I_\ell\Big\}$$
$$\cup\Big\{[-\bar{\tilde{\phi}}_{\ell,k}\otimes\tilde{\psi}_{\ell+1,m},0]^\top : k\in I_\ell,\, m\in J_{\ell+1}\Big\} \tag{9}$$
$$\cup\Big\{[-\bar{\tilde{\psi}}_{\ell,k}\otimes\tilde{\psi}_{\ell,m},\tilde{\psi}_{\ell,k}\otimes\bar{\tilde{\psi}}_{\ell,m}]^\top : k\in J_\ell,\, m\in J_\ell\Big\}\Big).$$

Both the primal and dual basis are local, meaning both that the diameter of the support of a wavelet on "level ℓ" is $\lesssim 2^{-\ell}$, and that each ball of diameter $2^{-\ell}$ intersects the supports of an at most uniformly bounded number of wavelets on level ℓ.

2.5 The Navier-Stokes equations as an infinite system of scalar nonlinear equations

With the wavelet frame $\Psi = \{\psi_\lambda : \lambda\in\Lambda\} = \bigcup_{i=0}^{m-1} E_i\Psi^{(i)}$ for V, with the $\Psi^{(i)}$ as given in (8), and corresponding frame operator $F : V' \to \ell_2(\Lambda)$, we can now reformulate equation (1) as an equivalent infinite system of scalar nonlinear equations. Because of $\overline{\operatorname{span}\Psi} = V$, the weak form is equivalent to finding a vector $\mathbf{u}\in\ell_2(\Lambda)$ such that

$$\mathbf{A}\mathbf{u} + \operatorname{Re}\mathbf{G}(\mathbf{u}) = \operatorname{Re}\mathbf{f},$$

where $\mathbf{A}$ is the infinite-dimensional stiffness matrix $\mathbf{A} = \{a(\psi_\lambda,\psi_\mu)\}_{\mu,\lambda\in\Lambda}$, $\mathbf{G}$ is the discrete nonlinearity $\mathbf{G}(\mathbf{u}) = (\int_\Omega \psi_\lambda\cdot(u\cdot\nabla)u)_{\lambda\in\Lambda}$, $u = \mathbf{u}^\top\Psi$ and $\mathbf{f}$ is the discrete right-hand side $\mathbf{f} = (\int_\Omega f\psi_\lambda)_{\lambda\in\Lambda}$.

From a discrete solution $\mathbf{u}$, the continuous solution can be retrieved by $u = F'\mathbf{u} = \mathbf{u}^\top\Psi$. It is important to note that, because the operator F' is not injective unless Ψ is a basis, uniqueness of the continuous solution u does not imply uniqueness of the discrete solution $\mathbf{u}$.

2.6 Approximation spaces

In the following, we recall the concept of asymptotic optimality of adaptive wavelet methods, see, for instance, [2, 4]. Assume that the original problem has a solution $u = \mathbf{u}^\top\Psi$, which has *some* discrete representation $\mathbf{u}\in\ell_2(\Lambda)$ in the given wavelet frame Ψ that can be approximated with rate s with respect to the degrees of freedom, i.e.,

$$\sup_{N\in\mathbb{N}} N^s \inf_{\#\operatorname{supp}\mathbf{v}\le N} \|\mathbf{u}-\mathbf{v}\|_{\ell_2(\Lambda)} < \infty.$$

Then, we expect our algorithm to achieve the same rate s.

Dealing with nonlinear problems it has turned out that we have to confine ourselves to approximations supported on a tree-type index sets. In the present context of wavelet frames constructed from wavelet bases in the fashion described above, we say that a set $T = \bigcup_{i=0}^{m-1}\{i\} \times T_i \subset \Lambda = \bigcup_{i=0}^{m-1}\{i\} \times \Lambda_i$ has an *aggregated tree* structure, if all the T_i are trees. Here we call $T_i \subset \Lambda_i$ a *tree*, when for any $\lambda \in T_i$ with $|\lambda| > 0$, $\operatorname{supp} \psi_\lambda^{(i)}$ is covered by the supports of $\psi_\mu^{(i)}$ for some $\mu \in T_i$ with $|\mu| = |\lambda| - 1$. In the literature, slightly different definitions of tree index sets can be found, but the differences are harmless.

Now we define

$$\Sigma_{N,\mathcal{AT}} := \{\mathbf{v} \in \ell_2(\Lambda),\, \#\operatorname{supp}\mathbf{v} \leq N,\, \operatorname{supp}\mathbf{v} \text{ has aggregated tree structure}\}$$

and set $\sigma_{N,\mathcal{AT}}(\mathbf{v}) := \inf_{\mathbf{w}\in\Sigma_{N,\mathcal{AT}}} \|\mathbf{v} - \mathbf{w}\|_{\ell_2(\Lambda)}$. From this we obtain the approximation space

$$\mathcal{A}^s_{\mathcal{AT}} := \{\mathbf{v} \in \ell_2(\Lambda) :\ \sigma_{N,\mathcal{AT}}(\mathbf{v}) \lesssim N^{-s}\}$$

of vectors in $\ell_2(\Lambda)$ that can be approximated with rate s in aggregated tree structure equipped with the quasi-norm $\|\mathbf{v}\|_{\mathcal{A}^s_{\mathcal{AT}}} := \sup_{N\in\mathbb{N}} N^s \sigma_{N,\mathcal{AT}}(\mathbf{v})$. Later, we show that $\mathbf{u} \in \mathcal{A}^s_{\mathcal{AT}}$, where $\mathbf{u}$ is *some* representation of the solution $u \in V$, implies that the adaptive algorithm we will construct converges with rate s. This property is called *asymptotic optimality.*

For use later, we also set

$$\Sigma_{N,\mathcal{T}_i} := \{\mathbf{v} \in \ell_2(\Lambda_i),\, \#\operatorname{supp}\mathbf{v} \leq N,\, \operatorname{supp}\mathbf{v} \text{ is a tree}\},$$

$\sigma_{N,\mathcal{T}_i}(\mathbf{v}) := \inf_{\mathbf{w}\in\Sigma_{N,\mathcal{T}_i}} \|\mathbf{v} - \mathbf{w}\|_{\ell_2(\Lambda_i)}$, $\mathcal{A}^s_{\mathcal{T}_i} := \{\mathbf{v} \in \ell_2(\Lambda_i) :\ \sigma_{N,\mathcal{T}_i}(\mathbf{v}) \lesssim N^{-s}\}$ equipped with the quasi-norm $\|\mathbf{v}\|_{\mathcal{A}^s_{\mathcal{T}_i}} := \sup_{N\in\mathbb{N}} N^s \sigma_{N,\mathcal{T}_i}(\mathbf{v})$, as well as $\Sigma_{N,i}$, $\sigma_{N,i}$, and $\mathcal{A}^s_i$ equipped with $\|\mathbf{v}\|_{\mathcal{A}^s_i}$, defined by omitting the tree constrained on $\operatorname{supp}\mathbf{v}$.

The question for which s we can find a discrete solution $\mathbf{u} \in \mathcal{A}^s_{\mathcal{AT}}$, and hence obtain convergence of an asymptotically optimal adaptive method with rate s, is linked to the Besov regularity of the continuous solution in an appropriate scale, see [2, 4, 14]. In general Lipschitz domains, to our best knowledge, little is known about regularity of solutions to the incompressible Navier-Stokes equation with respect to this Besov scale. However, for the related Stokes equation and the three-dimensional Navier-Stokes equation in polyhedral cones, it can be shown that in many cases the Besov regularity indeed exceeds the Sobolev regularity, see [5] and [8], respectively. Moreover, numerical experiments from [28] suggest that, even in other cases, the regularity measured in this scale is significantly higher than in the Sobolev scale. Since the Sobolev regularity corresponds to the convergence rate of standard uniform methods, it seems reasonable to use adaptive methods in order to improve the convergence rate.

3 The adaptive algorithm

In this section, we construct an adaptive wavelet Schwarz solver for equation (1) written in divergence-free wavelet frame coordinates. To do so, we briefly present some tools needed for the construction.

3.1 Building Blocks

For the design of an asymptotically optimal adaptive wavelet method, we need to have at hand a couple of elementary building blocks.

First of all, we require a solver for linear subproblems on the subdomains Ω_i. For the construction of such local solvers, we refer, e.g., to [2]. Even though these subproblems are fully linear in the unknowns, we will have to evaluate the nonlinear term $(v \cdot \nabla)v$ in wavelet coordinates to obtain the right-hand side for the subproblems. The evaluation of such nonlinearities is described in [4].

Furthermore, in order to guarantee an optimal balance between degrees of freedom and accuracy, we repeatedly remove very small entries from the discrete iterates. This will be done by the application of a method

$$\mathbf{COARSE}[\mathbf{v}, \varepsilon] : \ell_2(\Lambda) \to \ell_2(\Lambda)$$

that maps a finitely supported $\mathbf{v} \in \ell_2(\Lambda)$ to a near-smallest $\mathbf{v}_\varepsilon \in \ell_2(\Lambda)$ with $\|\mathbf{v} - \mathbf{v}_\varepsilon\|_{\ell_2(\Lambda)} \le \varepsilon$. The construction of such a method involves sorting the entries of $\mathbf{v}$ into buckets by their modulus. For details and further properties, see, for instance, [2, 14]. In particular, it is shown in [14] that there exists a constant $\vartheta \in (0,1)$ such that for $\mathbf{v} \in \mathcal{A}^s_{\mathcal{AT}}$ and a finitely supported $\mathbf{w} \in \ell_2(\Lambda)$ with $\|\mathbf{v} - \mathbf{w}\|_{\ell_2(\Lambda)} \le \vartheta\varepsilon$, it holds that $\tilde{\mathbf{w}} := \mathbf{COARSE}[\mathbf{w}, (1-\vartheta)\varepsilon] \in \mathcal{A}^s_{\mathcal{AT}}$ with $\|\tilde{\mathbf{w}}\|_{\mathcal{A}^s_{\mathcal{AT}}} \lesssim \|\mathbf{v}\|_{\mathcal{A}^s_{\mathcal{AT}}}$ and $\#\operatorname{supp}\tilde{\mathbf{w}} \lesssim \varepsilon^{-1/s}\|\mathbf{v}\|^{1/s}_{\mathcal{A}^s_{\mathcal{AT}}}$.

The second building block that we describe is designed to deal with the redundancy of a frame, i.e., with the fact that $\ker F' \neq \{0\}$. Any vector in $\ker F'$ is in the kernel of $\mathbf{u} \mapsto \mathbf{A}\mathbf{u} + \operatorname{Re}\mathbf{G}\mathbf{u}$, and therefore is not affected by any iterative method to invert this operator. So components in $\ker F'$ which arise in the course of the iteration as a consequence of inexact evaluation of operators, or because of applications of **COARSE** will never be reduced. Assuming u has some representation $u = \mathbf{u}^\top\Psi$ with $\mathbf{u} \in \mathcal{A}^s_{\mathcal{AT}}$, this may have as a consequence that the iterands converge to a representation that is not in $\mathcal{A}^s_{\mathcal{AT}}$, so that consequently an optimal rate s is not realized.

For a frame that is the union of Riesz bases on overlapping subdomains, a way to deal with this problem is, before solving on subdomain i, to remove terms in the expansion of the current iterand that are multiples of wavelets $\psi^{(j)}_\lambda \subset \Psi^{(j)}$ for $j \neq i$ with $\operatorname{supp}\psi^{(j)}_\lambda \subset \Omega_i$. In any case for linear elliptic problems, and assuming a sufficiently large overlap of the subdomains in relation to the maximal diameter of the support of any primal or dual wavelet, in [23] it was shown that for the multiplicative Schwarz method this approach yields an adaptive algorithm that converges with the optimal rate. In [30], it was shown that the same holds true for the additive Schwarz algorithm in case of having two subdomains, whereas numerical experiments indicate that this is also valid for more than two subdomains.

Since no proof of the latter is available, to cope with the redundancy, for completeness here we will resort on the technique introduced in [22]. Under some circumstances, however, the routine **PROJECTION** that will be introduced below can simply be omitted from the adaptive algorithm, whereas nevertheless optimal rates can be observed. We refer to [22, Thm. 3.12, §4.3] for an analysis in a restricted setting, and to [16, 15] for numerical results.

Let Z be a bounded right-inverse of F', i.e., $Z \in B(V, \ell_2(\Lambda))$ with $F'Z = \mathrm{Id}$, and with the projector $\mathbf{Q} := ZF' \in B(\ell_2(\Lambda), \ell_2(\Lambda))$, let $\mathbf{Q}$ be bounded on $\mathcal{A}^s_{\mathcal{AT}}$. A suitable Z will be constructed below. The application of $\mathbf{Q}$ to a vector in $\ell_2(\Lambda)$ does not change the function u it is representing, i.e., $F'\mathbf{Q} = F'$. Yet, if u has *some* representation $\mathbf{u} \in \mathcal{A}^s_{\mathcal{AT}}$, then the application of $\mathbf{Q}$ to any representation $\mathbf{v}$ of u yields a representation in $\mathcal{A}^s_{\mathcal{AT}}$, because $\mathbf{Q}\mathbf{v} = \mathbf{Q}\mathbf{u}$ and $\mathbf{Q}$ is bounded on $\mathcal{A}^s_{\mathcal{AT}}$. In view of this property, we extend our adaptive wavelet method with a recurrent, inexact application of $\mathbf{Q}$ in order to produce a sequence of iterands that is uniformly bounded in $\mathcal{A}^s_{\mathcal{AT}}$.

The routine that approximates the application of $\mathbf{Q}$ within tolerance $\varepsilon > 0$ will be denoted as

$$\mathbf{PROJECTION}[\mathbf{v}, \varepsilon] : \ell_2(\Lambda) \to \ell_2(\Lambda).$$

It maps a $\mathbf{v} \in \ell_2(\Lambda)$ to a $\mathbf{w}_\varepsilon \in \ell_2(\Lambda)$ with $\|\mathbf{w}_\varepsilon - \mathbf{Q}\mathbf{v}\|_{\ell_2(\Lambda)} \leq \varepsilon$.

Now we come to the construction of a suitable Z and thus of $\mathbf{Q}$. For $\delta > 0$ and $0 \leq i \leq m-1$, we set $\Omega_i(-\delta) := \{x \in \Omega_i : B(x;\delta) \cap \Omega \subset \Omega_i\}$. We will assume a sufficiently large overlap of the subdomains in relation to the maximal diameter of any primal or dual wavelet from (8) and (9) in the sense that there exists a $\delta > 0$, that from here on will be fixed, such that

$$\operatorname{diam} \operatorname{supp} \psi_\lambda^{(i)} \cup \operatorname{supp} \tilde{\psi}_\lambda^{(i)} \leq \delta/2, \tag{10}$$

$$\Omega \subset \bigcup_{i=0}^{m-1} \Omega_i(-(1+\tfrac{m-1}{2})\delta). \tag{11}$$

Lemma 3.1 *With $\Lambda_i^\delta := \{\lambda \in \Lambda_i : \operatorname{supp} \psi_\lambda^{(i)} \cap \Omega_i(-\delta) \neq \emptyset\}$, we have $v \mapsto (\tilde{\psi}_\lambda^{(i)}(v))_{\lambda \in \Lambda_i^\delta} \in B(V, \ell_2(\Lambda_i^\delta))$, and, for $v \in V$, $v - \sum_{\lambda \in \Lambda_i^\delta} \tilde{\psi}_\lambda^{(i)}(v)\psi_\lambda^{(i)}$ vanishes on $\Omega_i(-\delta)$.*

Proof. $\Omega = \Omega_i \cup (\Omega \setminus \Omega_i(-\delta/2))$ is an overlapping domain decomposition, and so, as in the proof of Lemma 2.2, for all $v \in V$ there exist $v_i \in V_i$ and $w_i \in \{w \in H_0^1(\Omega \setminus \Omega_i(-\delta/2))^d : \operatorname{div} w = 0\}$ with $v_i + w_i = v$ and $\|v_i\|_{H^1(\Omega)^d}^2 + \|w_i\|_{H^1(\Omega)^d}^2 \eqsim \|v\|_{H^1(\Omega)^d}^2$. Noting that by (10), $(\tilde{\psi}_\lambda^{(i)}(v))_{\lambda \in \Lambda_i^\delta}$ only depends on $v|_{\Omega_i(-\delta/2)}$, both statements follow from $v = v_i$ on $\Omega_i(-\delta/2)$, for the second statement using that $\Psi^{(i)}$ is a Riesz basis for V_i with dual basis $\tilde{\Psi}^{(i)}$, and the definition of Λ_i^δ. □

Next, with $Z_{-1} = H_{-1} := 0$, for $0 \leq i \leq m-1$, we set

$$Z_i : V \to \ell_2(\Lambda_0) \times \cdots \times \ell_2(\Lambda_i) : v \mapsto (Z_{i-1}v, (\tilde{\psi}_\lambda^{(i)}(v - H_{i-1}Z_{i-1}v))_{\lambda \in \Lambda_i^\delta})$$

$$H_i : \ell_2(\Lambda_0) \times \cdots \times \ell_2(\Lambda_i) \to V : (\mathbf{v}_0, \ldots, \mathbf{v}_i) \mapsto \sum_{j=0}^{i} \mathbf{v}_j^\top \Psi^{(j)}.$$

Proposition 3.2 *The mappings Z_i are bounded, and for $v \in V$, $v - H_i Z_i v$ vanishes on $\bigcup_{j=0}^{i} \Omega_j(-(1+\frac{i}{2})\delta)$.*

Proof. The first statement follows easily from Lemma 3.1. For the second statement, we write

$$v - H_i Z_i v = (v - H_{i-1}Z_{i-1}v) - \sum_{\lambda \in \Lambda_i^\delta} \tilde{\psi}_\lambda^{(i)}(v - H_{i-1}Z_{i-1}v)\psi_\lambda^{(i)}.$$

The first term vanishes on $\bigcup_{j=0}^{i-1} \Omega_j(-(1+\frac{i-1}{2})\delta)$, and so $v - H_i Z_i v$ vanishes on $\bigcup_{j=0}^{i-1} \Omega_j(-(1+\frac{i}{2})\delta)$ by (10). By Lemma 3.1, $v - H_i Z_i v$ also vanishes on $\Omega_i(-\delta) \subset \Omega_i(-(1+\frac{i}{2})\delta)$, which completes the proof. □

Setting $Z = Z_{m-1}$, with a slight abuse of notation, we have $Z \in B(V, \ell_2(\Lambda))$, and, using (11), $F'Z = \mathrm{Id}$.

Remark 3.3 *A small refinement of an argument used in the proof of Proposition 3.2 shows that* (11) *can be relaxed to $\Omega \subset \bigcup_{i=0}^{m-1} \Omega_i(-(1+\frac{J}{2})\delta)$, where J is the maximal number of subdomains that have non-empty intersection.*

Finally, to show that $\mathbf{Q}$ is bounded on $\mathcal{A}_{\mathcal{AT}}^s$, it suffices to show that each of the matrices $[\tilde{\psi}_\lambda^{(i)}(\psi_\mu^{(j)})]_{\lambda \in \Lambda_i^\delta, \mu \in \Lambda_j}$ is bounded from $\mathcal{A}_{\mathcal{T}_j}^s$ to $\mathcal{A}_{\mathcal{T}_i}^s$. Fixing $0 \leq i, j \leq m-1$, we know that $\mathbf{B} :=$

$[\tilde{\psi}^{(i)}_\lambda(\psi^{(j)}_\mu)]_{\lambda\in\Lambda_i^\delta,\mu\in\Lambda_j} : \ell_2(\Lambda_j)\to\ell_2(\Lambda_i)$ is bounded. Let $\mathbf{v}\in\mathcal{A}^s_{\mathcal{T}_j}$ and $\varepsilon>0$ be given. Then there exists a $\mathbf{v}_\varepsilon\in\ell_2(\Lambda_j)$ whose support is a tree, with $\|\mathbf{B}\|_{\ell_2(\Lambda_j)\to\ell_2(\Lambda_i)}\|\mathbf{v}-\mathbf{v}_\varepsilon\|_{\ell_2(\Lambda_j)}\le\varepsilon/2$, $\#\operatorname{supp}\mathbf{v}_\varepsilon\lesssim\varepsilon^{-1/s}\|\mathbf{v}\|^{1/s}_{\mathcal{A}^s_{\mathcal{T}_j}}$, and thus $\|\mathbf{v}_\varepsilon\|_{\mathcal{A}^s_{\mathcal{T}_j}}\lesssim\|\mathbf{v}\|_{\mathcal{A}^s_{\mathcal{T}_j}}$, cf. [4, Prop. 6.3].

In [22, §4.5], it was shown that for local $\Psi^{(j)}$ and $\tilde{\Psi}^{(i)}$, where the $\psi^{(j)}_\mu$ are spline wavelets, $\mathbf{B}$ is a so-called s^*-compressible matrix, for a value of s^* that exceeds the best possible rate s for which membership $\mathbf{u}\in\mathcal{A}^s_{\mathcal{AT}}$ can be expected (assuming that this best possible rate is larger than $\frac{1}{2}$). Consequently, see e.g. [2, Corol. 3.10], when $s<s^*$ there exists a $\mathbf{w}_\varepsilon\in\ell_2(\Lambda_i)$ with $\|\mathbf{B}\mathbf{v}_\varepsilon-\mathbf{w}_\varepsilon\|_{\ell_2(\Lambda_i)}\le\varepsilon/2$, thus $\|\mathbf{B}\mathbf{v}-\mathbf{w}_\varepsilon\|_{\ell_2(\Lambda_i)}\le\varepsilon$, and $\#\operatorname{supp}\mathbf{w}_\varepsilon\lesssim\varepsilon^{-1/s}\|\mathbf{v}_\varepsilon\|^{1/s}_{\mathcal{A}^s_j}\le\varepsilon^{-1/s}\|\mathbf{v}_\varepsilon\|^{1/s}_{\mathcal{A}^s_{\mathcal{T}_j}}\lesssim\varepsilon^{-1/s}\|\mathbf{v}\|^{1/s}_{\mathcal{A}^s_{\mathcal{T}_j}}$.

Let T_i be the smallest tree in Λ_i that contains $\operatorname{supp}\mathbf{w}_\varepsilon$. To conclude that $\mathbf{B}:\mathcal{A}^s_{\mathcal{T}_j}\to\mathcal{A}^s_{\mathcal{T}_i}$ is bounded, it remains to show that $\#T_i\lesssim\varepsilon^{-1/s}\|\mathbf{v}\|^{1/s}_{\mathcal{A}^s_{\mathcal{T}_j}}$. Because of $\Psi^{(i)}$ being local, there exists a constant $C>0$ such that

$$\bar{T}_i:=\{\theta\in\Lambda_i:\exists\lambda\in\operatorname{supp}\mathbf{w}_\varepsilon \text{ s.t. } |\lambda|\ge|\theta|\wedge\operatorname{dist}(\operatorname{supp}\psi^{(i)}_\lambda,\operatorname{supp}\psi^{(i)}_\theta)\le C2^{-|\theta|}\}\supset T_i.$$

Obviously, for all $\lambda\in\operatorname{supp}\mathbf{w}_\varepsilon$, there exists a $\mu\in\operatorname{supp}\mathbf{v}_\varepsilon$ with $\operatorname{supp}\tilde{\psi}^{(i)}_\lambda\cap\operatorname{supp}\psi^{(j)}_\mu\ne\emptyset$. By the construction of the sparse approximations for $\mathbf{B}$ in [22], which are used as ingredients of the approximate matrix-vector routine **APPLY** developed in [2], we have that if $|\lambda|\ge|\mu|$, then for all $\nu\in\Lambda_i^\delta$ with $|\mu|\le|\nu|\le|\lambda|$ and $\operatorname{supp}\tilde{\psi}^{(i)}_\nu\cap\operatorname{supp}\psi^{(j)}_\mu\ne\emptyset$ it holds that $\nu\in\operatorname{supp}\mathbf{w}_\varepsilon$. (“Coincidentally” $(\mathbf{w}_\varepsilon)_\nu$ might be zero, in which case formally $\nu\notin\operatorname{supp}\mathbf{w}_\varepsilon$. The point is, however, that when determining the aforementioned upper bound for $\#\operatorname{supp}\mathbf{w}_\varepsilon$, ν has been counted as being part of the support. Related to this, below we will use that for any $\ell\in\mathbb{N}_0$ with $|\mu|\le\ell\le|\lambda|$, there exists a $\nu\in\Lambda_i^\delta$ with $|\nu|=\ell$ and $\operatorname{supp}\tilde{\psi}^{(i)}_\nu\cap\operatorname{supp}\psi^{(j)}_\mu\ne\emptyset$. Although this would be a mild assumption on $\tilde{\Psi}^{(i)}$, it is not needed to impose this, since again by determining the upper bound for $\#\operatorname{supp}\mathbf{w}_\varepsilon$, the existence of such a ν has been taken into account.)

Now considering an arbitrary $\theta\in\bar{T}_i$, let $\lambda\in\operatorname{supp}\mathbf{w}_\varepsilon$ be as in the definition of $\bar{T}_i$, and let $\mu\in\operatorname{supp}\mathbf{v}_\varepsilon$ be as above. If $|\theta|\le|\mu|$, then by definition of $\bar{T}_i$ and $\operatorname{supp}\mathbf{v}_\varepsilon$ being a tree, there exists a $\gamma\in\operatorname{supp}\mathbf{v}_\varepsilon$ with $|\theta|=|\gamma|$ and $\operatorname{dist}(\operatorname{supp}\psi^{(i)}_\theta,\operatorname{supp}\psi^{(j)}_\gamma)\lesssim2^{-|\theta|}$. Otherwise, so when $|\theta|>|\mu|$, there exists a $\nu\in\operatorname{supp}\mathbf{w}_\varepsilon$ with $|\theta|=|\nu|$ and $\operatorname{dist}(\operatorname{supp}\psi^{(i)}_\theta,\operatorname{supp}\tilde{\psi}^{(i)}_\nu)\lesssim2^{-|\theta|}$. From both observations, and the fact that $\Psi^{(i)}$, $\tilde{\Psi}^{(i)}$, $\Psi^{(j)}$ are local, we conclude that $\#T_i\le\#\bar{T}_i\lesssim\#\operatorname{supp}\mathbf{v}_\varepsilon+\#\operatorname{supp}\mathbf{w}_\varepsilon\lesssim\varepsilon^{-1/s}\|\mathbf{v}\|^{1/s}_{\mathcal{A}^s_{\mathcal{T}_j}}$, which completes the proof of $\mathbf{Q}$ being bounded on $\mathcal{A}^s_{\mathcal{AT}}$.

3.2 Construction of the algorithm

We are now ready to define the algorithm we will investigate. This method is the adaptive wavelet version of the algorithm proposed in [20]. Note that bold letters stand for discrete iterates while standard letters stand for their continuous representation, e. g. $v=\mathbf{v}^\top\Psi$. To explicitly formulate the algorithm, we need to fix some constants. Let

$$K:=\sup_{0\ne\mathbf{v}\in\ell_2(\Lambda)}\frac{|\!|\!|\mathbf{v}^\top\Psi|\!|\!|}{\|\mathbf{v}\|_{\ell_2(\Lambda)}},\quad L:=\sup_{0\ne\mathbf{v}\in\ell_2(\Lambda)}\frac{\|\mathbf{Q}\mathbf{v}\|_{\ell_2(\Lambda)}}{|\!|\!|\mathbf{v}^\top\Psi|\!|\!|},$$
$$C:=\sup_{0\ne u,v\in V}\Big\{\frac{|\!|\!|z|\!|\!|}{|\!|\!|u|\!|\!|\,|\!|\!|v|\!|\!|}:a(z,w)=\int_\Omega w\cdot(u\cdot\nabla)v\,,\,(w\in V)\Big\},$$

where $K\le B_\Psi\sup_{0\ne v\in V}\frac{|\!|\!|v|\!|\!|}{\|v\|_{H^1(\Omega)^d}}<\infty$, $L=\sup_{0\ne v\in V}\frac{\|Zv\|_{\ell_2(\Lambda)}}{|\!|\!|v|\!|\!|}<\infty$ by $\mathbf{Q}=ZF'$ and $Z\in B(V,\ell_2(\Lambda))$, and where $C<\infty$ for $d\le4$ has been shown in [20, Lemma 1]. Let M be an upper bound for $|\!|\!|u|\!|\!|$.

By P_i, we denote the $a(\cdot,\cdot)$-orthogonal projector from V onto V_i. It is known that the operator norm

$$\theta := \| I - \omega(P_0 + \ldots + P_{m-1}) \|$$

on $B(V,V)$ is smaller than 1 if $\omega > 0$ is sufficiently small. Then, for sufficiently small Reynolds numbers or sufficiently small datum, and therefore, in the latter case, a sufficiently small solution, we have

$$\rho := \theta + 3\omega \operatorname{Re} mCM < 1. \tag{12}$$

We show a convergence rate $\tilde{\rho} := (1+\rho)/2 < 1$.

Algorithm 1 AddSchw

% Let $l^* \in \mathbb{N}$ be minimal such that $\tilde{\rho}^{l^*} \leq \frac{1}{2KL}\vartheta\tilde{\rho}$.
% Let $\varepsilon_n := \tilde{\rho}^n M$, $n \in \mathbb{N}$.
$u^{(0)} := 0$
for $n = 0, 1, \ldots$ **do**
  $v^{(n,0)} := u^{(n)}$
  for $l = 0, \ldots, l^* - 1$ **do**
    for $i = 0, \ldots, m-1$ **do**
      Compute $\tilde{d}_i^{(n,l)} \in V_i$ as an approximation to the solution $d_i^{(n,l)} \in V_i$ of
        $a(d_i^{(n,l)}, v) = -\operatorname{Re}\int_{\Omega_i} v \cdot (v^{(n,l)} \cdot \nabla) v^{(n,l)} + \operatorname{Re}\int_{\Omega_i} f \cdot v - a(v^{(n,l)}, v)$ for all $v \in V_i$
      with tolerance $\| d_i^{(n,l)} - \tilde{d}_i^{(n,l)} \| \leq \frac{1-\rho}{2m\omega}\varepsilon_n \tilde{\rho}^l$.
    end for
    $v^{(n,l+1)} := v^{(n,l)} + \omega \sum_{i=0}^{m-1} \tilde{d}_i^{(n,l)}$
  end for
  $\tilde{\mathbf{u}}^{(n+1)} :=$ **PROJECTION**$[\mathbf{v}^{(n,l^*)}, \frac{\vartheta}{2K}\varepsilon_{n+1}]$
  $\mathbf{u}^{(n+1)} :=$ **COARSE**$[\tilde{\mathbf{u}}^{(n+1)}, \frac{1-\vartheta}{K}\varepsilon_{n+1}]$
end for

Note that the subproblems on the subdomains are fully linear and the nonlinearity only appears on the right-hand side. Moreover, they are independent so they can be solved in parallel. The first result concerning Algorithm 1 shows convergence given that the Reynolds number is sufficiently small such that (12) holds.

Theorem 3.4 *Assume that $d \leq 4$ and that* (12) *is valid. Then, the iterates from Algorithm 1 fulfill*

$$\| u^{(n)} - u \| \leq \tilde{\rho}^n M.$$

Proof. We show the assertion by induction. The case $n = 0$ is clear by definition. We assume the assertion is true for some $n \in \mathbb{N}$. Denote by $\bar{v}^{(n,1)} := v^{(n,0)} + \sum_{i=0}^{m-1} d_i^{(n,0)}$ the result of the first iteration step with exact subdomain solvers. As in the proof of Theorem 4 in [20], the error after this step can be written as

$$\bar{v}^{(n,1)} - u = (I - \omega(P_0 + \ldots + P_{m-1}))(u^{(n)} - u) + \omega \operatorname{Re} \sum_{i=0}^{m-1} F_i(u^{(n)}, u),$$

where $F_i(u^{(n)}, u)$ is defined as

$$F_i(u^{(n)}, u) = -S_i^{-1}((u^{(n)} \cdot \nabla) u^{(n)} - (u \cdot \nabla) u)$$

and $S_i^{-1} : (H^{-1}(\Omega_i))^d \to V_i$ the solution operator for the Stokes equation on Ω_i. In the proof of Theorem 1 in [20], using $d \le 4$, we see that

$$\|F_i(u^{(n)}, u)\| \le C(\|u^{(n)} - u\| + 2\|u\|)\|u^{(n)} - u\|,$$

hence with $u^{(n)} = v^{(n,0)}$ we obtain the estimate

$$\|\bar{v}^{(n,1)} - u\| \le \theta\|v^{(n,0)} - u\| + \omega \operatorname{Re} mC(\|v^{(n,0)} - u\| + 2\|u\|)\|v^{(n,0)} - u\|.$$

By the induction hypothesis, it is $\|v^{(n,0)} - u\| = \|u^{(n)} - u\| \le \|u^{(0)} - u\| = \|u\| \le M$. From this we obtain

$$\|\bar{v}^{(n,1)} - u\| \le \rho\|v^{(n,0)} - u\| \le \rho\varepsilon_n,$$

where $\varepsilon_n = \tilde{\rho}^n M$. Taking into account the tolerance for the error in the inexact solutions of the local problems, we obtain

$$\|v^{(n,1)} - u\| \le \|\bar{v}^{(n,1)} - u\| + m\omega\frac{1-\rho}{2m\omega}\varepsilon_n \le \left(\rho + \frac{1-\rho}{2}\right)\varepsilon_n = \tilde{\rho}\varepsilon_n.$$

Iterating this argument over l yields $\|v^{(n,l)} - u\| \le \tilde{\rho}^l\varepsilon_n$. In particular, by the choice of l^*, we have $\|v^{(n,l^*)} - u\| \le \frac{1}{2KL}\vartheta\varepsilon_{n+1}$. From this it follows that

$$\|\tilde{\mathbf{u}}^{(n+1)} - \mathbf{Q}\mathbf{u}\|_{\ell_2(\Lambda)} \le \frac{\vartheta}{2K}\varepsilon_{n+1} + \|\mathbf{Q}(\mathbf{v}^{(n,l^*)} - \mathbf{u})\|_{\ell_2(\Lambda)} \le \frac{\vartheta}{2K}\varepsilon_{n+1} + L\frac{1}{2KL}\vartheta\varepsilon_{n+1} = \frac{1}{K}\vartheta\varepsilon_{n+1}. \quad (13)$$

Therefore, we have $\|u^{(n+1)} - u\| \le K\|\mathbf{u}^{(n+1)} - \mathbf{Q}\mathbf{u}\|_{\ell_2(\Lambda)} \le K(\frac{1-\vartheta}{K}\varepsilon_{n+1} + \|\tilde{\mathbf{u}}^{(n+1)} - \mathbf{Q}\mathbf{u}\|_{\ell_2(\Lambda)}) \le \varepsilon_{n+1}$. □

Now, the properties of the coarsening and projection methods allow us to show that the algorithm is asymptotically optimal with respect to the degrees of freedom in the outer iterates $\mathbf{u}^{(n)}$. By this, we mean that we obtain the same rate as the best N-term approximation of any representation $\mathbf{u} \in \ell_2(\Lambda)$ of u that has the aggregated tree structure.

Theorem 3.5 *Assume that the solution u has some representation $\mathbf{u} \in \mathcal{A}^s_{\mathcal{AT}}$. Then, for the iterates $\mathbf{u}^{(n)}$ from Algorithm 1, it holds that*

$$\mathbf{u}^{(n)} \in \mathcal{A}^s_{\mathcal{AT}},$$
$$\#\operatorname{supp}\mathbf{u}^{(n)} \lesssim \varepsilon_n^{-1/s}\|\mathbf{u}\|^{1/s}_{\mathcal{A}^s_{\mathcal{AT}}}.$$

Proof. From (13) and the properties of **COARSE**, for all $n \in \mathbb{N}$, we have $\mathbf{u}^{(n+1)} \in \mathcal{A}^s_{\mathcal{AT}}$ and

$$\#\operatorname{supp}\mathbf{u}^{(n+1)} \lesssim \left(\frac{\varepsilon_{n+1}}{K}\right)^{-1/s}\|\mathbf{Q}\mathbf{u}\|^{1/s}_{\mathcal{A}^s_{\mathcal{AT}}}.$$

Using boundedness of $\mathbf{Q}$ on $\mathcal{A}^s_{\mathcal{AT}}$, we obtain the result. □

3.3 How to solve the local subproblems

Let us now describe how the local subproblems appearing in Algorithm 1 can be solved. This can be done in the same fashion as in [23, 16], based on the Richardson method in [3]. The construction principles from there carry over to the vector-valued setting, compare, for instance, [12]. For convenience, we sketch the algorithm here.

Written in wavelet coordinates, the subproblems amount to solving the equations

$$\mathbf{A}^{(i,i)}\mathbf{d}_i^{(n,l)} = -\operatorname{Re}\mathbf{G}(\mathbf{v}^{(n,l)})|_{\Lambda_i} + \operatorname{Re}\mathbf{f}|_{\Lambda_i} - \sum_{j=0}^{m-1}\mathbf{A}^{(i,j)}\mathbf{v}^{(n,l)},$$

where $\mathbf{A}^{(i,j)} := \{a(\psi_\lambda,\psi_\mu)\}_{\lambda\in\Lambda_i,\mu\in\Lambda_j}$ denotes the (i,j)-th block of the matrix $\mathbf{A}$, and, for any vector $\mathbf{w}\in\ell_2(\Lambda)$, by $\mathbf{w}|_{\Lambda_i}$ we mean the restriction of $\mathbf{w}$ to the index set Λ_i. The matrix $\mathbf{A}^{(i,i)}$ is positive definite. Hence, for a sufficiently small relaxation parameter $\omega > 0$ and with $\mathbf{R}$ being the representation of the residual of the subproblems, the Richardson iteration

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \omega\mathbf{R}(\mathbf{w}^{(k)})$$

converges linearly for any initial vector $\mathbf{w}^{(0)}$. This is still true even if the residual is only approximated up to a given, sufficiently small tolerance. To do so, we have to make use of the vector-valued versions of the methods presented in [2, 4, 14, 12] for approximating the infinite matrix-vector products, the nonlinear term and the right-hand side. Moreover, as in [23, 16] we can see that in each call of the local solver the number of iterations to achieve the prescribed tolerance $\frac{1-\rho}{2m\omega}\varepsilon_n\tilde{\rho}^l$ is a constant independent of n. Therefore, also the computational complexity of the inner iterations can be bounded by a constant multiple of $\varepsilon_n^{-1/s}\|\mathbf{u}\|_{\mathcal{A}^s_{\mathcal{AT}}}^{1/s}$. Hence, by summing up these terms and considering that the support sizes of the iterates is of the same order, it can in principle be shown that the overall algorithm has linear complexity.

A Proofs of Propositions 2.7 and 2.8

We give the missing proofs of two propositions from Sect. 2.4. The proofs apply under conditions (a)–(i) on the wavelets and scaling functions.

Proof of Proposition 2.7. An application of a basis transform shows that

$$\{\phi_{\ell,1},\phi_{\ell,2}-\phi_{\ell,1},\ldots,\phi_{\ell,N_\ell}-\phi_{\ell,N_\ell-1}\}, \quad \Big\{\sum_{k=1}^{N_\ell}\tilde{\phi}_{\ell,k},\sum_{k=2}^{N_\ell}\tilde{\phi}_{\ell,k},\ldots,\tilde{\phi}_{\ell,N_\ell}\Big\} \tag{14}$$

are biorthogonal bases for $\operatorname{span}\Phi_\ell$, $\operatorname{span}\tilde{\Phi}_\ell$.

Because of $\int_{\mathcal{I}}\phi_{\ell,k+1}-\phi_{\ell,k}=0$ by (i), and so $\overset{+}{\Phi}_\ell\subset H_0^1(\mathcal{I})$, integration by parts shows that $\overset{+}{\Phi}_\ell$, $\overset{=}{\tilde{\Phi}}_\ell$ are biorthogonal.

Again (i) and $\mathbb{1}\in\operatorname{span}\tilde{\Phi}_\ell$, by (e), show that $\sum_{k=1}^{N_\ell}\tilde{\phi}_{\ell,k}\in\operatorname{span}\{\mathbb{1}\}$. So for $\lambda\in\mathring{J}$, $|\lambda|\le\ell$, we have $\psi_\lambda\in\operatorname{span}\{\phi_{\ell,2}-\phi_{\ell,1},\ldots,\phi_{\ell,N_\ell}-\phi_{\ell,N_\ell-1}\}$, and so $\overset{+}{\psi}_\lambda\in\operatorname{span}\overset{+}{\Phi}_\ell$. Since $\sum_{k=1}^{N_\ell}\tilde{\phi}'_{\ell,k}=0$, for $\lambda\in\mathring{J}$, $|\lambda|\le\ell$, we have $\overset{=}{\tilde{\psi}}_\lambda\in\operatorname{span}\overset{=}{\tilde{\Phi}}_\ell$.

By $\operatorname{supp}\overset{+}{\phi}_{\ell,k}\subset\operatorname{convhull}(\operatorname{supp}\phi_{\ell,k+1}\cup\operatorname{supp}\phi_{\ell,k})$ and Φ_ℓ being uniformly local ((h)), we have that $\overset{+}{\Phi}_\ell$ is uniformly local. This property together with $\|\phi_{\ell,k}\|_{L_2(\mathcal{I})}\lesssim 1$ shows that $\|\overset{+}{\phi}_{\ell,k}\|_{L_2(\mathcal{I})}\lesssim 1$, and so, again by Φ_ℓ being uniformly local, that $\|\sum_{k=1}^{N_\ell-1}c_k\overset{+}{\phi}_{\ell,k}\|^2_{L_2(\mathcal{I})}\lesssim\sum_{k=1}^{N_\ell-1}c_k^2$.

From $\sum_{k=1}^{N_\ell}\tilde{\phi}'_{\ell,k}=0$, the ordering of the $\tilde{\phi}'_{\ell,k}$ by (i) and $\operatorname{supp}\tilde{\phi}_{\ell,k}\cap\operatorname{supp}\phi_{\ell,k}\neq\emptyset$, and $\tilde{\Phi}_\ell$ being uniformly local ((h)), it follows that $\overset{=}{\tilde{\Phi}}_\ell$ is uniformly local.

Any $u\in\operatorname{span}\tilde{\Phi}_\ell$ can be written as $\sum_{\{\lambda\in J:|\lambda|\le\ell\}}c_\lambda 2^{-|\lambda|}\tilde{\psi}_\lambda$. From $\{2^{-|\lambda|}\tilde{\psi}_\lambda:\lambda\in J\}$ and $\tilde{\Psi}$ being Riesz bases for $H^1(\mathcal{I})$ and $L_2(\mathcal{I})$, respectively, by (b) and Remark 2.4, we have $\|u\|^2_{H^1(\mathcal{I})}\eqsim\sum_{\{\lambda\in J:|\lambda|\le\ell\}}|c_\lambda|^2\le 4^\ell\sum_{\{\lambda\in J:|\lambda|\le\ell\}}|c_\lambda 2^{-|\lambda|}|^2\lesssim 4^\ell\|u\|^2_{L_2(\mathcal{I})}$.

From this so-called inverse inequality, $\|\tilde{\phi}_{\ell,k}\|_{L_2(\mathcal{I})} \lesssim 1$, and $\bar{\tilde{\Phi}}_\ell$ being uniformly local, it follows that $\|\bar{\tilde{\phi}}_{\ell,k}\|_{L_2(\mathcal{I})} \lesssim 1$, and so again from $\bar{\tilde{\Phi}}_\ell$ being local, that $\|\sum_{k=1}^{N_\ell-1} c_k \bar{\tilde{\phi}}_{\ell,k}\|^2_{L_2(\mathcal{I})} \lesssim \sum_{k=1}^{N_\ell-1} c_k^2$, which by biorthogonality is equivalent to $\|\sum_{k=1}^{N_\ell-1} c_k \overset{+}{\phi}_{\ell,k}\|^2_{L_2(\mathcal{I})} \gtrsim \sum_{k=1}^{N_\ell-1} c_k^2$. We conclude that $\overset{+}{\bar{\Phi}}_\ell$, and so $\bar{\tilde{\Phi}}_\ell$, are uniform $L_2(\mathcal{I})$-Riesz bases for their spans. $\square$

Proof of Proposition 2.8. From $\overset{+}{\bar{\Psi}}$ and $\{4^{-|\lambda|}\overset{+}{\psi}_\lambda : \lambda \in \mathring{J}\}$ being Riesz bases for $L_2(\mathcal{I})$ and $H_0^2(\mathcal{I})$, respectively, by Proposition 2.5, and $H_0^2(\mathcal{I}^2) \simeq H_0^2(\mathcal{I}) \otimes L_2(\mathcal{I}) \cap L_2(\mathcal{I}) \otimes H_0^2(\mathcal{I})$, we have that

$$\{(\sum_{j=1}^{2} 16^{|\lambda_j|})^{-\frac{1}{2}} \overset{+}{\psi}_{\lambda_1} \otimes \overset{+}{\psi}_{\lambda_2} : (\lambda_1, \lambda_2) \in \mathring{J} \times \mathring{J}\} \text{ is a Riesz basis for } H_0^2(\mathcal{I}^2). \tag{15}$$

By using that $\{\overset{+}{\psi}_{\lambda_1} \otimes \overset{+}{\psi}_{\lambda_2} : (\lambda_1, \lambda_2) \in \mathring{J} \times \mathring{J}\}$ is a Riesz basis for $L_2(\mathcal{I}^2)$, we infer that for $\ell \in \mathbb{N}_0$,

$$\|\cdot\|_{H^2(\mathcal{I}^2)} \eqsim 4^\ell \|\cdot\|_{L_2(\mathcal{I}^2)} \quad \text{on } \operatorname{span}\{\overset{+}{\psi}_{\lambda_1} \otimes \overset{+}{\psi}_{\lambda_2} : (\lambda_1, \lambda_2) \in \mathring{J} \times \mathring{J},\, \max(|\lambda_1|, |\lambda_2|) = \ell\}. \tag{16}$$

With $\overset{+}{\bar{\Psi}}_{[\ell]} := \{\overset{+}{\psi}_\lambda : \lambda \in \mathring{J},\, |\lambda| = \ell\}$, for $\ell > 0$ an alternative, uniform $L_2(\mathcal{I}^2)$-basis for the space from (16) is given by

$$\overset{+}{\bar{\Psi}}_{[\ell]} \otimes \overset{+}{\bar{\Phi}}_{\ell-1} \cup \overset{+}{\bar{\Phi}}_{\ell-1} \otimes \overset{+}{\bar{\Psi}}_{[\ell]} \cup \overset{+}{\bar{\Psi}}_\ell \otimes \overset{+}{\bar{\Psi}}_{[\ell]}.$$

The latter result, (16), and (15) show that

$$\bigcup_{\ell \in \mathbb{N}_0} 2^{-\ell}\big(2^{-(\ell+1)} \overset{+}{\bar{\Psi}}_{[\ell+1]} \otimes \overset{+}{\bar{\Phi}}_\ell \cup 2^{-(\ell+1)} \overset{+}{\bar{\Phi}}_\ell \otimes \overset{+}{\bar{\Psi}}_{[\ell+1]} \cup 2^{-\ell} \overset{+}{\bar{\Psi}}_{[\ell]} \otimes \overset{+}{\bar{\Psi}}_{[\ell]}\big)$$

is a Riesz basis for $H_0^2(\mathcal{I}^2)$.

By applying minus **curl** to this collection, the collection in the statement of the proposition is obtained. Since, as follows from [11, §I.3.1], $\mathbf{curl} : H_0^2(\mathcal{I}^2) \to \{v \in H_0^1(\mathcal{I}^2)^2 : \operatorname{div} v = 0\}$ is boundedly invertible, the first statement is proven.

The biorthogonality of the collections from (8) and (9) follows from the biorthogonality of $(\Psi, \tilde{\Psi})$, $(\overset{+}{\bar{\Psi}}, \bar{\tilde{\Psi}})$, $(\Phi_\ell, \tilde{\Phi}_\ell)$, $(\overset{+}{\bar{\Phi}}_\ell, \bar{\tilde{\Phi}}_\ell)$, and $\operatorname{span}\{\overset{+}{\psi}_\lambda : \lambda \in J,\, |\lambda| \leq \ell\} = \operatorname{span} \overset{+}{\bar{\Phi}}_\ell$, $\operatorname{span}\{\bar{\tilde{\psi}}_\lambda : \lambda \in J,\, |\lambda| \leq \ell\} = \operatorname{span} \bar{\tilde{\Phi}}_\ell$, and (f).

The locality of both the primal and dual collections follows directly from the (uniform) locality of the primal and dual scaling functions and wavelets from both biorthogonal multiresolution analyses.

What remains to show is the property (7) for the dual collection (9). From Ψ and $\overset{+}{\bar{\Psi}}$, and $\{2^{-|\lambda|}\psi_\lambda : \lambda \in J\}$ and $\{4^{-|\lambda|}\overset{+}{\psi}_\lambda : \lambda \in \mathring{J}\}$ being Riesz bases for $L_2(\mathcal{I})$ and $H_0^1(\mathcal{I})$, respectively, by (a), (c), (ii), and Remark 2.6, and $H_0^1(\mathcal{I}^2) \simeq H_0^1(\mathcal{I}) \otimes L_2(\mathcal{I}) \cap L_2(\mathcal{I}) \otimes H_0^1(\mathcal{I})$, we have that

$$\{(\sum_{j=1}^{2} 4^{|\lambda_j|})^{-\frac{1}{2}} \overset{+}{\psi}_{\lambda_1} \otimes \psi_{\lambda_2} : (\lambda_1, \lambda_2) \in \mathring{J} \times J\} \text{ is a Riesz basis for } H_0^1(\mathcal{I}^2). \tag{17}$$

By using that $\{\overset{+}{\psi}_{\lambda_1} \otimes \psi_{\lambda_2} : (\lambda_1, \lambda_2) \in \mathring{J} \times J\}$ is a Riesz basis for $L_2(\mathcal{I}^2)$, we infer that for $\ell \in \mathbb{N}_0$,

$$\|\cdot\|_{H^1(\mathcal{I}^2)} \eqsim 2^\ell \|\cdot\|_{L_2(\mathcal{I}^2)} \quad \text{on } \operatorname{span}\{\overset{+}{\psi}_{\lambda_1} \otimes \psi_{\lambda_2} : (\lambda_1, \lambda_2) \in \mathring{J} \times J,\, \max(|\lambda_1|, |\lambda_2|) = \ell\}. \tag{18}$$

With $\Psi_{[\ell]} := \{\psi_\lambda : \lambda \in J,\, |\lambda| = \ell\}$, for $\ell > 0$ an alternative, uniform $L_2(\mathcal{I}^2)$-basis for the space from (8) is given by

$$\overset{+}{\bar{\Psi}}_{[\ell]} \otimes \Phi_{\ell-1} \cup \overset{+}{\bar{\Phi}}_{\ell-1} \otimes \Psi_{[\ell]} \cup \overset{+}{\bar{\Psi}}_\ell \otimes \Psi_{[\ell]}.$$

The latter result, (18), and (17) show that

$$\bigcup_{\ell\in\mathbb{N}_0} 2^{-\ell}\big(\overset{\ddagger}{\Psi}_{[\ell+1]}\otimes\Phi_\ell\cup\overset{\ddagger}{\Phi}_\ell\otimes\Psi_{[\ell+1]}\cup\overset{\ddagger}{\Psi}_{[\ell]}\otimes\Psi_{[\ell]}\big)$$

is a Riesz basis for $H_0^1(\mathcal{I}^2)$. Its (unique) dual basis in $H^{-1}(\mathcal{I}^2)$ reads as

$$\bigcup_{\ell\in\mathbb{N}_0} 2^{\ell}\big(\bar{\bar{\tilde{\Psi}}}_{[\ell+1]}\otimes\tilde{\Phi}_\ell\cup\bar{\bar{\tilde{\Phi}}}_\ell\otimes\tilde{\Psi}_{[\ell+1]}\cup\bar{\bar{\tilde{\Psi}}}_{[\ell]}\otimes\tilde{\Psi}_{[\ell]}\big),$$

with the obvious definitions of $\bar{\bar{\tilde{\Psi}}}_{[\ell]}$ and $\tilde{\Psi}_{[\ell]}$.

We conclude that

$$v\mapsto\big(\langle 2^\ell\bar{\bar{\tilde{\phi}}}_{\ell,k}\otimes\tilde{\psi}_{\ell+1,m},v\rangle_{H^{-1}(\mathcal{I}^2)\times H_0^1(\mathcal{I}^2)}\big)_{k\in I_\ell,\,m\in J_{\ell+1},\,\ell\in\mathbb{N}_0},$$
$$v\mapsto\big(\langle 2^\ell\bar{\bar{\tilde{\psi}}}_{\ell,k}\otimes\tilde{\psi}_{\ell,m},v\rangle_{H^{-1}(\mathcal{I}^2)\times H_0^1(\mathcal{I}^2)}\big)_{k\in J_\ell,\,m\in J_\ell,\,\ell\in\mathbb{N}_0}$$

and, analogously,

$$v\mapsto\big(\langle 2^\ell\tilde{\psi}_{\ell+1,k}\otimes\bar{\bar{\tilde{\phi}}}_{\ell,m},v\rangle_{H^{-1}(\mathcal{I}^2)\times H_0^1(\mathcal{I}^2)}\big)_{k\in J_{\ell+1},\,m\in I_\ell,\,\ell\in\mathbb{N}_0},$$
$$v\mapsto\big(\langle 2^\ell\tilde{\psi}_{\ell,k}\otimes\bar{\bar{\tilde{\psi}}}_{\ell,m},v\rangle_{H^{-1}(\mathcal{I}^2)\times H_0^1(\mathcal{I}^2)}\big)_{k\in J_\ell,\,m\in J_\ell,\,\ell\in\mathbb{N}_0}$$

are bounded mappings from $H_0^1(\mathcal{I}^2)$ to the corresponding ℓ_2-spaces, which proves (7). $\square$

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